A STUDY ON THE THERMAL-ENTRY LENGTH PROBLEM IN DUCTS WITH CONSTANT SURFACE TEMPERATURE

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Abstract: In this paper a systematic procedure is introduced to deal with thermal entry length problem in tubes with constant surface temperature. Alternative relations are derived to calculate the rate of heat transfer and Nusselt number. An approximate relation between dimensionless temperature and dimensionless axial coordinate is proposed. A correlation is determined for the calculation of mean Nusselt number. Comparisons are made with available infinite series solutions. It is concluded that the proposed equations gives good results especially close the entrance region.

Key Words: Thermal-entry length, Heat transfer in ducts.

Sabit Yüzey Sıcaklığına Sahip Borularda Isıl-Giriş Uzunluğu Problemi Üzerine Bir Çalışma

Özet: Bu çalışmada sabit yüzey sıcaklığına sahip borularda ısıl-giriş uzunluğu probleminin çözümü için sistematik bir yöntem sunulmaktadır. Isı transferini ve Nusselt sayısını hesaplamak için alternatif bağıntılar çıkarılmıştır. Boyutsuz eksenel koordinat ve boyutsuz sıcaklık arasındaki ilişki için bir bağıntı önerilmiştir. Ortalama Nusselt sayısının belirlenmesi için bir korelasyon geliştirilmiştir. Sonuçlar sonsuz seri çözümleri ile karşılaştırılmış ve bu çalışmadaki bağıntıların oldukça iyi sonuç verdiği gösterilmiştir.

Anahtar Kelimeler: Isıl-giriş uzunluğu, Borularda ısı transferi.

Nomenclature

A	area, m^2
c_p	specific heat at constant pressure
D	diameter, m
h_x	local convection coefficient, W/m^2K
h	mean convection coefficient, W/m^2K
i	enthalyp of fluid, J/kg, m^2/s^2
k	thermal conductivity, W/mK
$\dot{m}$	mass flow rate, kg/s
p	perimeter, m
Pr	Prandtl number, ν/α
r_0	radius, m
Re	Reynolds number, $u_m D/\nu$
NTU	number of transfer units, $hA/\dot{m}c_p$
Nu	mean Nusselt number, hD/k
Nu_∞	value of mean Nusselt number when $x^+ \to \infty$
T	temperature, K
T_e	mean fluid temperature at exit, K
T_i	mean fluid temperature at inlet, K
T_m	mean fluid temperature, K
T_s	surface temperature, K
u_m	mean velocity, m/s
x	axial coordinate, m
x^+	dimensionless axial coordinate, $(x/r_0)/RePr$
$\dot{Q}$	rate of heat transfer, W
$\dot{Q}_x$	the rate of heat transfer from inlet to any x location, W

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Greek letters

α thermal diffusivity, $k/\rho c_p$, m^2/s

θ_m nondimensional temperature, $(T_s-T_m) / (T_s-T_i)$

ρ fluid density, kg/m^3

ν kinematic viscosity, m^2/s

Subscripts

e exit

i inlet

m mean

max maximum

s surface

1. INTRODUCTION

Pipes or ducts, in which fluids flow through, are one of the essential elements of the heating and cooling applications. The fluid in such applications is forced to flow by a fan or pump through a tube that is sufficiently long to accomplish the desired heat transfer. Laminar forced convection in circular ducts was investigated by Graetz (1883, 1885) and Nusselt (1910). These independent investigations led to a classical problem usually referred as the Graetz-Nusselt problem in the literature. Graetz-Nusselt problem considers the development of the temperature profile (or thermal-entry length) in the case of a fully developed velocity profile, but with a uniform fluid temperature at the point where heat transfer begins. A more comprehensive literature survey with detailed information on this topic for circular and noncircular ducts has been compiled by Shah and London (1978), and an updated review has been reported by Shah and Bhatti (1987). A complete literature review is not given here for the sake of brevity, instead, a brief account will be given only. For the analytical solution of the Graetz-Nusselt problem, many studies have been presented in the literature. The solutions for the fluid temperature in the duct has been presented in the form of an infinite series in terms of eigenvalues and eigenfunctions. The main feature of these analytical procedures are that only one term needs to be evaluated in the region where the temperature is considered as fully developed. On the contrary, it is unattractive feature is that the number of terms in the series is required for good accuracy increases drastically close to the entrance.

More studies on this subject have been carried out recently and they are outlined in References (Shah and Bhatti (1987), Shah and London (1978), Kays and Crawford (1993)). These investigations employed cumbersome mathematical procedures which required the evaluation of intricate functions for its practical interpretation. While the investigations have contributed greatly to an understanding of the mathematical procedures to the classical problems, the basic issue of providing simple solutions still remains unresolved.

It should be noted that these solutions apply rigorously only when a hydrodynamic starting length is provided so that the velocity profile is fully developed before heat transfer starts, a condition rarely encountered in technical applications. However, these kind of solutions are excellent approximations for fluids whose Prandtl numbers are high relative to unity. If the Prandtl number is bigger than about unity, it must follow that the velocity profile develops more rapidly than the temperature profile. In fact, if the Prandtl number is greater than about five, the velocity profile leads the temperature profile sufficiently that a solution based on an already fully developed velocity profile will apply quite accurately even though there is no hydrodynamic starting length.

The present study is concerned with an approximate solution methodology for the thermal-entry-length problem. The procedure deals with global (or mean) values. In view of this approach, heat transfer parameters are provided for the thermal-entry-length problems.

2. BASIC FORMULATION and ANALYSIS

When a fluid heated or cooled as it flows through a tube, the temperature of a fluid at any cross-section changes from T_s at the surface of the wall at the cross section to some maximum (or minimum in the case of heating) at the tube center. For the internal flow cases, it is convenient to work with an average or mean temperature T_m. The mean temperature T_m will continue to change in the flow direction whenever convection heat transfer exist between the surface and the fluid. In the absence of any work interactions as well as negligible potential and kinetic energy changes, the conservation of the energy equation for the steady flow of a fluid in a tube can be expressed as

$$\dot{Q} = \dot{m}(i_e - i_i) \qquad \text{(W)} \qquad (1)$$

where i_i and i_e are the fluid enthalpies at the inlet and exit of the tube, respectively, and $\dot{Q}$ is the rate of heat transfer to or from the fluid. If the fluid is not undergoing a phase change and constant specific heat is assumed, equation (1) can be expressed as:

$$\dot{Q} = \dot{m}c_p(T_e - T_i) \qquad \text{(W)} \qquad (2)$$

where T_i and T_e are the mean fluid temperatures at the inlet and exit of the tube, respectively.

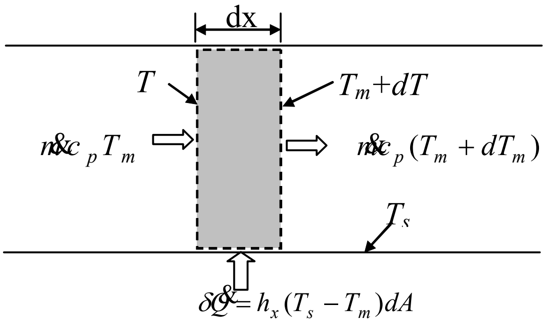

Figure 1:
Differential control volume for internal flow in a tube.

Consider the heating of a fluid in a tube of constant cross-section whose inner surface is maintained at a constant temperature of T_s. It is known that the mean temperature of the fluid T_m will increase in the flow direction as a result of heat transfer. The energy balance on a differential control volume shown in Fig. 1 gives

$$\dot{m}c_p dT_m = h_x(T_s - T_m)dA \qquad \text{(W)} \qquad (3)$$

where h_x is the local heat transfer coefficient and T_s and T_m are the surface and the mean temperatures at that location. That is, the increase in the energy of the fluid (represented by an increase in its mean temperature by dT_m) is equal to the heat transferred to the fluid from the tube surface by convection. Noting that the differential surface area is $dA = pdx$ where p is the perimeter of the tube, and that $dT_m = -d(T_s - T_m)$, since T_s is constant, the relation above can be rearranged as

$$\frac{d(T_s - T_m)}{(T_s - T_m)} = -\frac{h_x p}{\dot{m}c_p}dx \qquad (4)$$

Integrating from $x = 0$ (tube inlet where $T_m = T_i$) to $x = L$ (tube exit where $T_m = T_e$) gives

$$\ln\frac{(T_s - T_e)}{(T_s - T_i)} = -\frac{hA}{\dot{m}c_p} \qquad (5)$$

where $A = pL$ is the surface area of the tube and h is the mean convection heat transfer coefficient and it is defined as:

$$h = \frac{1}{L}\int_0^L h_x dx \qquad (6)$$

Taking the exponential of both sides of the equation (5) and solving for T_e gives the following very useful relation for the determination of the mean fluid temperature at the tube exit:

$$T_e = T_s - (T_s - T_i)e^{-hA/\dot{m}c_p} \tag{7}$$

This relation can also be used to determine the mean fluid temperature T_m (x) at any x by replacing $A = pL$ and T_e by px and $T_m(x)$, respectively.

$$T_m(x) = T_s - (T_s - T_i)e^{-hpx/\dot{m}c_p} \tag{8}$$

Note that the temperature difference between the fluid and the surface decays exponentially in the flow direction, and the rate of decay depends on the magnitude of the exponent $hA/\dot{m}c_p$. This dimensionless parameter is called number of transfer units, denoted by NTU, and is a measure of the effectiveness of the heat transfer systems.

Assuming that in the limit situation, mean temperature will be equal to the surface temperature. In the case of $T_m = T_s$, the rate of heat transfer reaches its maximum value and can be determined from:

$$\dot{Q}_{\max} = \dot{m}c_p(T_s - T_i) \qquad \text{(W)} \tag{9}$$

From inlet to any x location, the rate of heat transfer can be obtained from:

$$\dot{Q}_x = \dot{m}c_p(T_m(x) - T_i) \qquad \text{(W)} \tag{10}$$

Combining equations (9) and (10), a relation can be expressed as:

$$\frac{\dot{Q}_x}{\dot{Q}_{\max}} = \frac{(T_m(x) - T_i)}{(T_s - T_i)} \tag{11}$$

Now, it is convenient to introduce the following dimensionless variables:

$$\theta_m = \frac{(T_s - T_m)}{(T_s - T_i)} \tag{12}$$

$$x^+ = \frac{x/r_o}{\mathrm{Re}\,\mathrm{Pr}} \tag{13}$$

where x is the axial distance from the point where heat transfer starts, r_0 is the radius of the pipe. Now, combining equations (11) and (12) results:

$$\frac{\dot{Q}_x}{\dot{Q}_{\max}} = 1 - \theta_m \tag{14}$$

This results is quiet interesting since the rate of heat transfer can be easily calculated by the knowledge of dimensionless temperature. Rearranging the equation (8) yields

$$\frac{(T_s - T_m(x))}{(T_s - T_i)} = e^{-hpx/\dot{m}c_p} \qquad \text{or} \qquad \theta_m = e^{-hpx/\dot{m}c_p} \tag{15}$$

Equation (15) can be rewritten in terms of the NTU (which is also function of x) as:

$$\theta_m = e^{-NTU} \qquad \text{or} \qquad -\ln\theta_m = NTU \tag{16}$$

Substituting this into equation (14), it is obtained

$$\frac{\dot{Q}_x}{\dot{Q}_{\max}} = 1 - e^{-NTU} \tag{17}$$

For any axial location equation (5) can be rearranged as

$$\ln\frac{(T_s - T_m)}{(T_s - T_i)} = -\frac{hpx}{\dot{m}c_p} \qquad \text{or} \qquad \ln\theta_m = -\frac{h2\pi r_0 x}{\rho\pi r_0^2 u_m c_p} \tag{18}$$

and equation (13) substituted into equation (18), it can be found that

$$-\ln\theta_m = Nu \cdot 2x^+ \qquad \text{or} \qquad \theta_m = e^{-Nu\cdot 2x^+} \tag{19}$$

Comparing equations (16) and (19), it can be shown that

$$NTU = Nu \cdot 2x^+ \tag{20}$$

Solving equation (18) for $\dot{m}c_p$, it can be obtained as

$$\dot{m}c_p = \frac{hpx}{\ln\dfrac{(T_s - T_i)}{(T_s - T_m)}} \tag{21}$$

Substituting this into equation (10), it can be found that

$$\dot{Q}_x = hpx\Delta T_{\ln} \qquad \text{(W)} \tag{22}$$

where

$$\Delta T_{\ln} = \frac{T_m - T_i}{\ln\dfrac{(T_s - T_i)}{(T_s - T_m)}} = \frac{\Delta T_i - \Delta T_m}{\ln\dfrac{\Delta T_i}{\Delta T_m}} \tag{23}$$

where ΔT_{ln} is the logarithmic mean temperature difference. $\Delta T_i = T_s - T_i$ and $\Delta T_m = T_s - T_m$ are the temperature differences between the surface and the fluid at the inlet and the any x location of the tube, respectively.

One point left undetermined is the relation between x^+ and θ_m. This can be obtained from the analytical solution of the differential equation as a series form mentioned in the introduction section. Instead of this way, more practical an easy way will be presented here, an approximate function is proposed for the relation between x^+ and θ_m.For very large values of x^+, the mean value of Nusselt number, Nu_∞, can be easily determined or can be found in the literature. A relation between x^+ and θ_m may be assumed as follow:

$$x^{+} = -\frac{\ln\theta_m}{2}\frac{\sqrt{(1-\theta_m)}}{Nu_\infty} \tag{24}$$

and using equations (19) and (20), and rearranging equation (24), it can be shown that:

$$\frac{Nu}{Nu_\infty} = \frac{1}{\sqrt{(1-\theta_m)}} = \sqrt{\frac{\dot{Q}_{\max}}{\dot{Q}_x}} \tag{25}$$

Note that all the equations given above valid for all the constant surface temperature convection problems in ducts. It means that these equations can be used in both laminar and turbulent flow cases, and they can be also used for hidrodynamically and thermally developing or fully developed flows.

3. RESULTS and DISCUSSION

By the use of equations (16) and (17), the change of θ_m and Q/Q_{max} with NTU is shown in Table I. As it can be seen from the equation (16), θ_m decreases exponentially from unity to zero with increasing NTU. Whereas, Q/Q_{max} increases exponentially from zero to unity with increasing NTU, and it reaches the value of about 0.99 at NTU ≈ 5. Considering equation (20), Table I also presents the change of θ_m and Q/Q_{max} with $Nu2x^+$ (or NTU). It can be concluded that these results are unique, and the value of $Nu2x^+$ can be easily calculated from equations (19) and (20).

Table I. Comparison the present results with the infinite series solution given by Kays and Crawford (1993) on the selected x^+ values.

x^+	NTU ($2Nu_m x^+$)	Q/Q_{max}	θ_m (Eq. 24)	Nu_m (Eq. 25)	θ_m (Ser. Sol.)	Nu_m (Ser. Sol.)
0.00001	0.00175	0.00175	0.99825	87.52	0.99805	97.47
0.00002	0.00278	0.00278	0.99722	69.46	0.99717	70.87
0.00005	0.00512	0.00511	0.99489	51.21	0.99488	51.34
0.0001	0.00814	0.00811	0.99189	40.65	0.99178	41.21
0.001	0.038	0.03729	0.96271	18.95	0.96202	19.31
0.002	0.0605	0.05871	0.94129	15.11	0.94061	15.29
0.004	0.0966	0.09208	0.90792	12.06	0.90763	12.10
0.006	0.1272	0.11944	0.88056	10.59	0.88083	10.56
0.008	0.1547	0.14333	0.85667	9.67	0.85755	9.60
0.01	0.1803	0.16498	0.83502	9.01	0.83649	8.92
0.02	0.2915	0.25286	0.74714	7.28	0.75122	7.14
0.04	0.4760	0.37874	0.62126	5.95	0.62818	5.81
0.06	0.6392	0.47229	0.52771	5.33	0.53505	5.21
0.08	0.7918	0.54698	0.45302	4.95	0.45921	4.86
0.1	0.9400	0.60937	0.39063	4.69	0.39476	4.64
0.2	1.6350	0.80505	0.19495	4.08	0.18924	4.15
0.4	3.0060	0.95051	0.04949	3.75	0.04385	3.91
0.6	4.4200	0.98797	0.01203	3.68	0.01017	3.82
0.8	5.8660	0.99717	0.00283	3.67	0.00236	3.78
1	7.3240	0.99934	0.00066	3.66	0.00055	3.76
2	14.640	0.9999996	$4.4x10^{-7}$	3.66	$3.6x10^{-7}$	3.71

It can be also seen from Table I that for NTU >5, the exit temperature of the fluid becomes almost equal to the surface temperature, $T_m \approx T_s$. Noting that the fluid temperature can approach the surface temperature but cannot cross it, and NTU of about 5 indicates that the limit is reached for heat transfer, and the heat transfer will not increase no matter how much we extend the length of the tube. A small value of NTU, on the other hand, indicates more opportunities for heat transfer, and the heat transfer will continue increasing as the tube length is increased. And all these conclusions can be seen numerically in Table I.

Table I. also shows the comparisons between the present results obtained from the equations (24) and (25) and the infinite series solution given by Kays and Crawford (1993) on the selected x^+ values. The constants and eigenvalues for the infinite series are given in Appendix. Fifty terms are considered in the series calculations. The value of Nu_∞ in the equations (24) and (25) was taken as 3.66 for the present calculations. When the results in the first row are compared, there are about 10% differences between the present and the series solution results. This differences mainly resulted by the series calculations. Table II shows that how θ_m and Nusselt number (especially at small x^+ locations) are affected by the number of the terms considered in the series. To get correct results, they requires more terms involved in the calculations for small x^+ values. It can be easily seen that for $x^+ \geq 0.00002$ the differences between the present results and the infinite series solution are less than 4%. And it is interesting that comparing the results presented in Table I and Table II, it can be seen that the infinite series solutions are approaching the present results when the number of the terms in the series are increased.

Table II. The effect of the number of the terms in the infinite series on the θ_m and Nusselt number.

x^+	θ_m 7 Term	Nu_m 7 Term	θ_m 30 Term	Nu_m 30 Term	θ_m 40 Term	Nu_m 40 Term	θ_m 50 Term	Nu_m 50 Term
0.00001	0.98278	868.38	0.99706	147.10	0.99773	113.68	0.99805	97.47
0.0001	0.98047	98.51	0.99164	41.92	0.99176	41.31	0.99178	41.21
0.001	0.95951	20.62	0.96202	19.31	0.96202	19.31	0.96202	19.31
0.002	0.93987	15.48	0.94061	15.29	0.94061	15.29	0.94061	15.29
0.004	0.90755	12.11	0.90763	12.11	0.90763	12.10	0.90763	12.10
0.006	0.88082	10.57	0.88083	10.56	0.88083	10.56	0.88083	10.56
0.008	0.85754	9.60	0.85755	9.60	0.85755	9.60	0.85755	9.60
0.01	0.83649	8.92	0.83649	8.92	0.83649	8.92	0.83649	8.92

4. CONCLUSIONS

Thermal-entry-length problem with constant surface temperature in tubes has been studied systematically. Different heat transfer parameters have been evaluated for comparative assessment. Alternative relations are derived to calculate the rate of heat transfer and Nusselt number. An approximate function is proposed for the relation between dimensionless axial coordinate, x^+, and dimensionless temperature, θ_m. The proposed function is used to evaluate the mean Nusselt number. The present solution was compared with the available results and the series solution in the literature. As a whole the present method sets forth a systematic procedure of evaluation of heat transfer parameters and yields quiet good results with less effort compare to the other analytic and numerical solution procedures. Although, in the present study only circular cross sectional tubes are considered, the present method can also be used with the tubes which have noncircular cross section. And this may be considered as quiet useful tool for many practical applications.

5. REFERENCES

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Appendix

For circular tube with constant surface temperature, the infinite series solution of the dimensionless mean temperature θ_m is given by Kays and Crawford [6] as

$$\theta_m = 8\sum_{n=0}^{\infty}\frac{G_n}{\lambda_n^2}\exp(-\lambda_n^2 x^+) \qquad (A.1)$$

Where λ_n is the eigenfunction and G_n is the constant. From n = 0 to 4, the values of λ_n^2 and G_n are given in Table A.I.

For n > 2, $\lambda_n = 4n + (8/3),\ G_n = 1.01276\lambda_n^{-1/3}$

Table A.I. Infinite series solution functions for the circular tube with constant surface temperature.

n	λ_n^2	G_n
0	7.313	0.749
1	44.61	0.544
2	113.9	0.463
3	215.2	0.415
4	348.6	0.383