APPROXIMATE EXPRESSIONS FOR WORST AND BEST CASE CRAMÉR-RAO BOUNDS FOR ESTIMATING THE PARAMETERS OF CLOSE CISOIDS

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Abstract: We present accurate approximations to the worst and the best case Cramér-Rao bounds for estimating the parameters of two closely spaced cisoids observed in additive complex white Gaussian noise. The approximations are valid in the sub-Rayleigh region where the difference between the critical values of the bounds becomes important.

Key Words: Cramér-Rao bound, Sinusoidal parameter estimation, Close sinusoids.

Yakın Sinüslerin Parametrelerinin Kestirimine İlişkin Cramér-Rao Sınırlarının Kritik Değerleri İçin Yaklaşık İfadeler

Özet: Kompleks beyaz Gauss gürültü içinde gözlemlenmiş iki yakın kompleks sinüsün parametrelerinin kestirimine ilişkin Cramér-Rao sınırlarının kritik değerleri için hassas yaklaşık ifadeler sunulmuştur. Bu yaklaşık ifadeler sınırların kritik değerleri arasındaki farkın önemli olduğu alt Rayleigh bölgesinde geçerlidir.

Anahtar Kelimeler: Cramér-Rao sınırı, Sinüzoidal parametre kestirimi, Yakın sinüsler.

1. INTRODUCTION

It is known that the Cramér-Rao bounds (CRBs) for the frequencies, the amplitudes and the phases of two cisoids observed in additive complex white Gaussian noise strongly depend on the phase difference between the cisoids in the sub-Rayleigh region, where the frequency separation between the cisoids is smaller than the Rayleigh limit (the resolution limit of the periodogram). Therefore, in this region it becomes important to determine the largest and the smallest values of the CRBs and the corresponding critical values of the phase difference.

Recent papers Dilaveroğlu (1998) and Dilaveroğlu (1999) provided simple approximate expressions for calculating the critical values of the CRBs for the case of small frequency separations. However, these expressions, being the one-term Taylor approximations, fail to be accurate when the frequency separation is not very small, say, between 0.1 and one Rayleigh limit, which is probably the range of separations of most interest in practice. In this paper, we improve the approximations in Dilaveroğlu (1998) and Dilaveroğlu (1999) by considering further terms of the Taylor expansions of the critical bounds. These improved approximations are very accurate in the whole of the sub-Rayleigh region.

2. MODEL DESCRIPTION

The data model is described by

$$y(t)=\sum_{i=1}^{2}\alpha_i \exp\{j(\omega_i t+\varphi_i)\}+e(t),\ t=n,\ldots,n+N-1 \tag{1}$$

where $j=\sqrt{-1}$, α_i is the amplitude, ω_i is the frequency and φ_i is the phase of the ith cisoid, $i=1,2$, $e(t)$ is a zero-mean complex white Gaussian noise with variance σ^2, N is the number of data samples, and n is the first value of the sampling index t.

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3. APPROXIMATE CRITICAL BOUND EXPRESSIONS

Exact expressions for the critical CRBs and for the critical values of the phase difference for estimating the frequencies in the model in eqn. 1 were derived in Dilaveroğlu (1998), and those for estimating the amplitudes and the phases can be derived, e.g. from the results in Dilaveroğlu (1998). It turns out that the symmetric sampling case where the number of data samples N is odd and the first value of the sampling index $n = -(N-1)/2$ greatly simplifies the expressions. Thus, in this work we assume the symmetric sampling case.

The critical values of the phase difference for estimating the frequencies, the amplitudes and the phases are collected in Table 1. The values are given in the interval $[0, \pi/2]$ since the CRBs are even and periodic functions of the phase difference with period π. Note that the worst and the best case phase differences for frequency estimation coincide with those for amplitude estimation whereas they need to be reversed for phase estimation.

Table 1. Critical values of the phase difference

	Worst-case	Best-case
Frequency estimation	0	$\pi/2$
Amplitude estimation	0	$\pi/2$
Phase estimation	$\pi/2$	0

We next present approximate expressions for the critical CRBs valid in the sub-Rayleigh region. The approximations were obtained by expressing the critical bounds in terms of Taylor series at $\delta\omega = 0$, where $\delta\omega$ denotes the frequency separation, and truncating the series such that the errors in the truncated series were less than about 5% in magnitude (an acceptable level) for all values of $\delta\omega$ in the sub-Rayleigh interval and for all permissible values of N. The results are

$$[\text{CRB}]_{\text{worst}} \cong \frac{1}{\text{SNR}_i \cdot N^3}\left[50\,400 \cdot G_{F,-4} \cdot \lambda^{-4} + 160 \cdot G_{F,-2} \cdot \lambda^{-2}\right] \tag{2}$$

$$[\text{CRB}]_{\text{best}} \cong \frac{1}{\text{SNR}_i \cdot N^3}\left[360 \cdot G'_{F,-2} \cdot \lambda^{-2} - \tfrac{12}{7} \cdot G'_{F,0} + \tfrac{11}{490} \cdot G'_{F,2} \cdot \lambda^{2}\right] \tag{3}$$

for estimating the frequency ω_i,

$$[\text{CRB}]_{\text{worst}} \cong \frac{\sigma^2}{N}\Big[201\,600 \cdot G_{A,-6} \cdot \lambda^{-6} - 4760 \cdot G_{A,-4} \cdot \lambda^{-4} + \tfrac{2680}{99} \cdot G_{A,-2} \cdot \lambda^{-2} + \tfrac{5471}{11\,583} \cdot G_{A,0}\Big] \tag{4}$$

$$[\text{CRB}]_{\text{best}} \cong \frac{\sigma^2}{N}\left[360 \cdot G'_{A,-4} \cdot \lambda^{-4} - \tfrac{138}{7} \cdot G'_{A,-2} \cdot \lambda^{-2} + \tfrac{389}{490} \cdot G'_{A,0}\right] \tag{5}$$

for estimating the amplitude α_i, and

$$[\text{CRB}]_{\text{worst}} \cong \frac{1}{\text{SNR}_i \cdot N}\Big[360 \cdot G_{P,-4} \cdot \lambda^{-4} + \tfrac{72}{7} \cdot G_{P,-2} \cdot \lambda^{-2} + \tfrac{37}{245} \cdot G_{P,0} + \tfrac{416}{282\,975} \cdot G_{P,2} \cdot \lambda^{2}\Big] \tag{6}$$

$$[\text{CRB}]_{\text{best}} \cong \frac{1}{\text{SNR}_i \cdot N}\left[6 \cdot G'_{P,-2} \cdot \lambda^{-2} + \tfrac{1}{5} \cdot G'_{P,0} + \tfrac{13}{4200} \cdot G'_{P,2} \cdot \lambda^{2}\right] \tag{7}$$

for estimating the phase φ_i where $\lambda = N \cdot \delta\omega$, SNR_i denotes the signal-to-noise ratio for the ith cisoid, $SNR_i = \alpha_i^2 / \sigma^2$, and the coefficients G and G' are given in Table 2. Note that these coefficients (very rapidly) approach one as N increases.

Table 2. Coefficient values

$G_{F,-4}$	N^6 / Q_6
$G_{F,-2}$	$\left(N^6 - \frac{53}{2} N^4\right) / Q_6$
$G'_{F,-2}$	N^4 / Q_4
$G'_{F,0}$	$\left(N^4 + \frac{17}{2} N^2\right) / Q_4$
$G'_{F,2}$	$\left(N^4 - \frac{23}{11} N^2 + \frac{51}{11}\right) / Q_4$
$G_{A,-6}$	N^6 / Q_6
$G_{A,-4}$	$\left(N^6 + \frac{7}{17} N^4\right) / Q_6$
$G_{A,-2}$	$\left(N^6 + \frac{281}{134} N^4 + \frac{701}{134} N^2\right) / Q_6$
$G_{A,0}$	$\left(N^6 - \frac{72\,984}{5471} N^4 + \frac{368\,235}{10\,942} N^2 - \frac{296\,065}{10\,942}\right) / Q_6$
$G'_{A,-4}$	N^4 / Q_4
$G'_{A,-2}$	$\left(N^4 + \frac{3}{23} N^2\right) / Q_4$
$G'_{A,0}$	$\left(N^4 - \frac{1122}{389} N^2 + \frac{884}{389}\right) / Q_4$
$G_{P,-4}$	N^4 / Q_4
$G_{P,-2}$	$\left(N^4 - \frac{19}{6} N^2\right) / Q_4$
$G_{P,0}$	$\left(N^4 - \frac{211}{37} N^2 + \frac{499}{74}\right) / Q_4$
$G_{P,2}$	$\left(N^6 - \frac{389}{52} N^4 + \frac{12\,511}{832} N^2 - \frac{12\,241}{1664}\right) / \left(N^2 \cdot Q_4\right)$
$G'_{P,-2}$	N^2 / Q_2
$G'_{P,0}$	$\left(N^2 - \frac{3}{2}\right) / Q_2$
$G'_{P,2}$	$\left(N^4 - \frac{29}{13} N^2 + 1\right) / \left(N^2 \cdot Q_2\right)$
	$Q_6 = N^6 - 14N^4 + 49N^2 - 36$ $Q_4 = N^4 - 5N^2 + 4$ $Q_2 = N^2 - 1$

A recent paper, Swingler (1999), developed, in an entirely empirical way, approximate expressions for the CRBs for the model considered herein (eqn. 1 with symmetrical sampling). We compared the errors in our approximations with the errors in the expressions in Swingler (1999) for the cases of N between 11 and 1001 and $\delta\omega$ between 0.1 and one Rayleigh limit which appear to be of practical interest. The results showed that our approximations have significantly smaller errors than those in Swingler (1999) for almost all the cases considered. Table 3 illustrates the results for the case of $N = 11$ samples.

Table 3. Maximum relative errors (in percent) in the approximations for the case of eleven samples

$\delta\omega$ [1]	Freq. estimation		Amp. estimation		Phase estimation	
	Worst CRB	Best CRB	Worst CRB	Best CRB	Worst CRB	Best CRB
[0.1-0.3)	0.0097 (1.2)	0.0021 (5.1)	8.2e-5 (17.7)	0.0067 (29.8)	3.7e-4 (4.2)	0.015 (6.8)
[0.3-0.5)	0.068 (2.7)	0.049 (10.6)	0.0057 (19.5)	0.19 (17.8)	0.019 (4.2)	0.27 (14.4)
[0.5-0.8)	0.30 (5.9)	0.92 (12.9)	0.35 (19.5)	2.7 (24.8)	0.57 (32.7)	2.5 (24.6)
[0.8-1.0]	0.36 (9.0)	3.7 (11.8)	2.5 (24.5)	5.6 (11.3)	2.4 (44.5)	5.3 (5.6)

[1] in units of Rayleigh limit

Errors in the expressions in Swingler (1999) are shown in parentheses.

4. CONCLUSIONS

Accurate approximations have been presented for the largest and the smallest CRBs for estimating the frequencies, the amplitudes and the phases of two close cisoids in complex white Gaussian noise. These approximations, together with the simple critical phase differences given in Table 1, can safely be used for a quick construction of worst and best case scenarios in testing the performance of practical spectral estimators designed for the sub-Rayleigh regime.

5. REFERENCES

1. Dilaveroğlu, E. (1998) Nonmatrix Cramér-Rao Bound Expressions for High-resolution Frequency Estimators, *IEEE Transactions on Signal Processing*, 46(2), 463-474.
2. Dilaveroğlu, E. (1999) Simple Expressions for Worst and Best Case Cramér-Rao Bounds for Amplitude and Phase Estimation of Closely Spaced Cisoids, *Electronics Letters*, 35(18), 1522-1523.
3. Swingler, D. N. (1999) Approximate Expressions for the Cramer-Rao Lower Bounds on Parameter Estimates for a Pair of Closely Spaced Cisoids, *Signal Processing*, 76(1), 99-104.