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Immanuel Kant's Philosophy of Mathematics in Terms of His Theory of Space and Time

Abstract

At the beginning of the modern age, mathematics had a great importance for the study of Nature. Galileo claimed that 'the book of nature was written in a kind of mathematical code, and that if we could only crack that code, we could uncover her ultimate secrets'. But, how can mathematics, consisting of necessary tautological truths that are infallible and non-informative, be regarded as the language of natural sciences, while the knowledge of natural sciences is informative, empirical and fallible? Or, is there another alternative: as Hume claimed, modern sciences only depend on empirical data deriving from our perceptions, rather than having the necessity of mathematics. Many philosophers have tried to find an adequate answer for the problem of the relationship between mathematical necessity and contingent perceptions, but the difficulty remained unsolved until Kant's construction of his original philosophy of the nature as well as the limits of human reason. The main purpose of this study is to show how Kant overcomes this difficulty by making use of the examples of Euclidean geometry and of arithmetic: there are synthetic a priori (a priori, universal, necessary, but at the same time informative) judgments, and indeed mathematical propositions are of this kind.

Key Terms

a priori, a posteriori, analytic, synthetic, synthetic a priori, mathematics, geometry, arithmetic, space, time and imagination.

Uzay ve Zaman Teorisi Açısından Immanuel Kant'ın Matematik Felsefesi

ÖZET

Modern çağın başlangıcında, matematik doğa çalışması için büyük bir öneme sahip olmuştur. Galileo 'doğa kitabının bir çeşit matematiksel kodla yazıldığını,

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bizim ancak bu kodu kırabilmemiz durumunda, onun nihai sırlarını açığa çıkarabileceğimizi' iddia etmiştir. Ancak, doğal bilimlerin bilgisi bilgimizi genişleten, deneysel ve yanılabilir nitelikteyken, nasıl oluyor da kesin ve bilgimizi genişletmeyen zorunlu totolojik doğruları içeren matematik, doğal bilimlerin dili olabiliyor? Ya da, Hume'un iddia ettiği gibi farklı bir seçenek mi var: modern bilimler matematiğin zorunluluğuna sahip olmaktan çok, algılarımızdan türeyen deneysel bilgiye dayanırlar. Pek çok filozof rastlantısal algılar ve matematiksel zorunluluk arasındaki ilişki problemine yeterli cevaplar bulmaya çalıştılar, ancak zorluk Kant'ın kendi orijinal tabiat ve insan bilgisinin sınırları felsefesinin oluşumuna kadar çözülmeden kaldı. Bu çalışmanın amacı Kant'ın aritmetik ve Euclid geometrisi örneklerini kullanarak bu problemin üstesinden nasıl geldiğini göstermektir: *synthetic a priori* (*a priori*, evrensel, zorunlu fakat aynı zamanda bilgimizi genişleten) yargılar vardır ve gerçekten matematiksel önermeler bu türdendir.

Anahtar Sözcükler

a priori, *a posteriori*, analitik, sentetik, sentetik *a priori*, matematik, geometri, aritmatik, uzay ve zaman, hayal gücü (muhayyile).

1 The Modern Background

By tradition, René Descartes, Gottfried Leibniz, Baruch Spinoza and Christian Wolff are on the one side classified as rationalists, whereas John Locke, George Berkeley and David Hume are on the opposite side named as empiricists. Despite this classification, so far as the ontological and epistemological questions concerning the nature of mathematics are concerned, they all appear to share more or less the same idea that mathematics is a theory concerned with our *ideas*, and that it is known a *priori*. To be sure, among them there are differences about the initial acquirement of these ideas. Rationalists like Descartes consider mathematical ideas as 'innate', implanted in the mind by an external force, that is, by God. Those ideas are so clear and distinct, if one establishes simple propositions out of them, one could never doubt their truth. In deductive reasoning the mathematician can use these propositions or premises, and thereby deduce further consequences which are again immune from any reasonable doubt. The rationalism of Leibniz divides human knowledge into two categories by arguing that: 'There are also two kinds of truth, those of reasoning and those of fact. Truths of reasoning are necessary and their opposite is impossible. Truths of fact are contingent and their opposite is possible. When a truth is necessary, its reason can be found by analysis, resolving it into more simple ideas and truths, until we come to those which are primary' (note. 33).[1] All true propositions of mathematics are considered as truths of reason, and their truths are justified on the basis of 'Principle of Contradiction'. According to this principle, it would simply be to contradict ourselves, once we call something a triangle, not to accept it to be a three-sided figure. For this reason, 'all triangles are three-sided' is a necessary truth of reason. On the basis of the definitional equivalence or identicalness of the terms 'triangle = three-sided figure', the proposition 'all triangles are three-sided' can be turned into the proposition 'all three-sided figures

[1] G. W. Leibniz, *The Monadology*, trans. Robert Lata (eBooks@Adelaide, 2010).

are three-sided’.[2] This means that propositions of mathematics are necessarily true in so far as they can be reduced to tautology via analysis.[3]

The empiricists such as Locke hold that all human knowledge is nothing more than ‘the perception of the connexion and agreement, or disagreement and repugnancy of any of our Ideas’.[4] Surely, this also applies to our mathematical knowledge. Mathematical truths are considered by him as general truths that can be acquired only by the mind’s act of ‘abstraction’ on the relation of ideas acquired. So the truths of mathematics do not consist in what is directly perceived from observation and experiment (i.e. empirical data), but rather in the relationship between ideas.[5] In the same way as Locke, Hume divides all the objects of human reason into two kinds, namely ‘relations of ideas’ and ‘matters of fact’. Our knowledge of ‘relations of ideas’ constitute the science of mathematics (geometry, algebra, and arithmetic). Every affirmation made in this realm is intuitively or demonstrably certain. The mere operation of thought discovers the truth of mathematical propositions without depending on anything existing in the external world.[6] Hume goes on to argue that ‘Matters of fact, which are the second objects of human reason, are not ascertained in the same manner; nor is our evidence of their truth, however great, of a like nature with foregoing. The contrary of every matter of fact is still possible; because it can never imply a contradiction. *That the sun will not raise tomorrow* is no less intelligible a proposition and implies no more contradiction, than the affirmation, *that it will rise*’.[7]

The clear implication here is that any meaningful proposition must either designate some kind of relationship between ideas, or represent an acknowledged fact. Although the propositions of the first kind are *necessarily* true or false due to the meanings of their terms, propositions of the second kind are only *contingently* true. This distinction is usually named to be ‘Hume’s Fork’. Propositions, which are not obtained through the mere operation of thought on relationship between ideas and through sense impressions, cannot be placed on either side of this category. They must therefore be rejected as meaningless. Propositions of this kind are those of metaphysics and theology. The idea that every meaningless idea must be thrown into flames is well expressed by Hume at the end of the *Enquiry* as follows: ‘If we take in our hand any volume; of divinity or school metaphysics, for instance; let us ask, *Does it contain any abstract reasoning concerning quantity or number?* No. *Does it contain any*

2 J. Cottingham, *Rationalism* (Granada Publishing, 1984), 59.

3 Leibniz, *The Monadology*, trans. R. Latte, note 35.

4 J. Locke, *An Essay concerning Human Understanding*, ed. P. H. Nidditch (Oxford: Oxford University Press, 1975), 525.

5 J. Locke, *An Essay concerning Human Understanding*, ed. P. H. Nidditch, 643.

6 D. Hume, *An Enquiry concerning Human Understanding*, ed. Peter Millican (Oxford: Oxford University Press, 2007), Section IV, Part I, 18.

7 D. Hume, *An Enquiry concerning Human Understanding*, ed. Peter Millican, Section IV, Part I, 18.

experimental reasoning concerning matter of fact and existence? No. Commit it then to the flames: For it can contain nothing but sophistry and illusion'.[8]

In his *Critique of Pure Reason* of 1781, Immanuel Kant (1724-1804) tried to find a clear resolution of the clash between rationalist orthodoxy and the empiricist scepticism of Hume regarding causal necessity and *a priori* knowledge by reconsidering the relationship between mathematical necessity and contingent perceptions. In the preface to the first edition of the *Critique*, Kant remarks that metaphysics is an area of enquiry that has been much denigrated by some of his modern predecessors: 'Time was when metaphysics was entitled the Queen of all the sciences; and if the will be taken for the deed, the preeminent importance of her accepted tasks gives her every right to this title of honour. Now, however, the changed fashion of the time brings her only scorn; a matron outcast and forsaken, she mourns like Hecuba' (CPR AVİİİ). In his highly original account of knowledge, this philosopher intends to avoid falling into scepticism (exemplified by Hume) as well as dogmatism which represents the belief in unquestioned metaphysical principles. His project surely involves the idea that metaphysics is possible, and it has to be composed of *synthetic a priori* judgements. As an area of enquiry, metaphysics shows us how to formulate or originate true judgements on experience. Metaphysics is possible because the metaphysical condition of our minds gives rise to the existence of our knowledge of experience. In the formation of our knowledge, Kant does not deny the role of experience. But its role is limited: 'There can be no doubt that all our knowledge begins with experience. But though all our knowledge begins with experience, it does not follow that it all arises out of experience' (CPR B1). Our knowledge indeed begins with the experience, but this does not mean that experience is the *only* source of all knowledge. Categories (or forms) of understanding (i.e. *a priori* notions) are presupposed by experience. Besides, space and time, called *forms* of intuition, are pre-condition of our having any experience. The truths of mathematics are assigned by him a special status intermediate between the empirical and the purely logical. Within his epistemological framework, Kant attempts to give an adequate answer to the problem of the application of mathematics to the natural sciences by insisting that mathematical judgements are not analytic as held by tradition. Rather he believed that the truths of mathematics are not only known *a priori* but also *synthetic*. In order to get a clear understanding of the theory, let us begin the examination with the four fundamental concepts, namely *a priori*, *a posteriori*, *analytic*, and *synthetic*.

2 The *a priori* and a *posteriori* Distinction

In the history of western thought, many philosophers either empiricist or rationalist generally held the view that there are two kinds of knowledge which come from different sources. Accordingly, it was generally believed that there are two kinds of propositions, namely *a priori* propositions and a *posteriori* propositions. This distinction actually goes back to Aristotle. Kant also believes in this traditional

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[8] D. Hume, *An Enquiry concerning Human Understanding*, ed. Peter Millican, Section XII, Part III, 120.

differentiation of human knowledge. He gives the definition of *a priori* knowledge in the expected manner to be the knowledge that is 'independent of experience' as follows: 'Such knowledge is entitled *a priori,* and distinguished from the *empirical,* which has its sources *a posteriori*, that is, in experience' (CPR B2/A2). Earlier philosophers like Leibniz usually argued that all *a priori* truths, in particular all mathematical truths, are analytic. In a word, the universality and necessity of *a priori* truths emerges merely from the fact that they are tautologies. And the truth of those propositions can only be determined or justified by reason alone via the analysis of the concepts involved. So, by the term *a priori* (Lat. *from before*), it is meant that 'something can be known *a priori* to be true (or false), if it can be known before, or independently of, sense-experience of the fact in question'.[9] Alternatively, by the term *a posteriori* (Lat. *from after*), it is meant that 'something can be known *a posteriori* if it can be known on the basis of, after, sense-experience of the fact'.[10] Propositions like 'all physical objects are under the influence of gravity' or 'all bachelors are American' or 'the dog is sitting on the sofa' are known to be true (or false) *a posterior,* provided that one appeals to observation or sense experience of the fact in question. On the other hand, a metaphysical proposition like 'God exists', a mathematical proposition like '7+5=12, or a proposition like 'all bachelors are unmarried' is known to be true (or false) *a priori* on the basis of the definitional equivalence or identicalness of the concepts involved. One does not need to appeal to observation or experiment of the matter of fact to determine their truth. In this case, whatever is known as *a priori* knowledge is a conceptual, analytical or logical truth. Sometimes the terms '*a priori* truths' and 'truths of reason' are used alternatively like the phrases '*a priori* reasoning' and 'pure reason'.

However, differing from the traditional view Kant argues that not all *a priori* truths are simply analytic or conceptual. In this connection he states: 'Knowledge *a priori* is either *pure* or *impure*. Pure knowledge *a priori* is that with which no empirical element is mixed up. For example, 'the proposition, 'every alteration has its cause', while an *a priori* proposition, is not a pure proposition, because alteration is a concept which can be derived only from experience' (CPR B3).

As a matter of fact, the notions of 'necessity', 'analyticity' and 'a priority' were not clearly distinguished before the publication of Kant's first *Critique*. In his discussion of *synthetic a priori* truths, Kant makes a clear distinction between the notion of *analytic* and that of *a priori*. This philosopher is convinced that human beings have an important class of truths which can be known *a priori* but not by means of analysis. In other words, there are some synthetic truths which can also be known *a priori*. Every truth known *a priori* is a universal truth in the grounds that it has both necessity and universality. An *a posteriori* judgement or knowledge deals with contingent truths, but *a priori* judgements are concerned with truths both necessary and universal. In this regard, Kant states that 'Necessity and strict universality are thus sure criteria of *a priori* knowledge, and are inseparable from one another' (CPR B4).[11] This suggests that there are two certain criteria by means of which one is able to separate an *a priori* knowledge from the empirical one:

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9 R. M. Martin, *The Philosopher's Dictionary* (Broadview Press, 2003), 29.

10 R. M. Martin, *The Philosopher's Dictionary*, 29.

11 Peter Sedgwick, *Descartes to Derrida*, (Blackwell, 2001), 27.

> First, then, if we have a proposition which in being thought is thought as *necessary*, it as an *a priori* judgement; and if, besides, it is not derived from any proposition except one which also has the validity of a necessary judgement, it is an absolutely *a priori* judgement. Secondly, experience never confers on its judgements true or strict, but only assumed and comparative *universality*, through induction. We can properly only say, therefore, that, so far as we have hitherto observed, there is no exception to this or that rule. If, then, a judgement is thought with strict universality, that is, in such a manner that no exception is allowed as possible, it is not derived from experience, but is valid absolutely *a priori* (CPR B4).

It is clear that the popular understanding of the distinction between *a priori* knowledge and *a posteriori* knowledge as a distinction between empirical and non-empirical emerges from Kant's ideas in his first *Critique*. *A priori* knowledge signifies a kind of knowledge or justification that does not require the existence of evidence from sensory experience. In contrast with *a priori* knowledge, *a posteriori* knowledge refers to a kind of knowledge or justification that requires the existence of evidence from sensory experience. The truth of an *a posteriori* judgement can be known or justified by appealing to the existence of evidence form sensory experience. Similarly, *a posteriori* concepts can only be known or understood by reference to sensory experience. Thus, knowledge which is empirical or based on experience is considered to be *a posteriori* knowledge as distinct from the *a priori* or non-empirical. An *a posteriori* judgement can be based on either ordinary observation, and thereby deal with common contingent truths like 'the dog is running in the park', or it can be based on scientific observation, and thereby concerns itself with contingent truths like under normal conditions of temperature and pressure, 'water boils at 100 Celsius'. So, the truths of ordinary perceptual experience and the natural sciences provide examples of *a posteriori* truths, whereas the truths of logic and mathematics provide examples of *a priori* truths.

Despite the fact that the account of knowledge based on direct mental apprehension and sensory experience is found in the writings of Descartes, Leibniz and Hume, the modern apprehension of the separation of *a priori* from *a posteriori* to be the separation of empirical from the non-empirical basically has its source in Kant's work in his first *Critique*. *A posteriori* knowledge is empirical, based on the *content* of experience, but *a priori* knowledge is transcendental, or concerned with the *forms* of all possible experience. According to Kant, the concept of *a priori* knowledge is neither identical with that of what is logically, metaphysically or necessarily true, nor with that of what is analytically true. The epistemological distinction between *a priori* and *a posteriori* is considered to be different from the logical and semantical distinction. The logical or metaphysical distinction qualifies truths as necessary and contingent, while the semantical distinction characterizes truths as analytic and synthetic. For Kant, the epistemological qualification of what is *a priori* consists only in *a priori* 'modes of knowledge' that human minds possess. The empiricists had claimed that *a priori* knowledge is dependent on the content of experience, whereas, on the other hand, the rationalists believed that *a priori* knowledge is immune from the mixture of any empirical content whatsoever. Differing from his predecessors, Kant originally claimed that *a priori* notions deriving from certain categories are necessary preconditions of our ability to experience the world. These *a priori* concepts are essential for the objective validity of our subjective knowledge. They are 'transcendental' that is, non-empirical, or pure, and therefore do not have their source in experience. They are 'necessary conditions' which are present in us to be constituent parts of our mind. All 'necessary conditions' or the concepts of the understanding are derived from categories. These categories of the understanding such as the category of substance, the category of causality, or the category of necessity are considered by Kant to be *a priori* notions. By means of them, the human mind imposes certain kinds of necessity and objectivity upon the acquired subjective sensory data or experience. In this connection, Kant states:

The objective validity of the categories as *a priori* concepts rests, therefore, on the fact that, so far as the form of thought is concerned, through them alone does experience become possible. They relate of necessity and *a priori* to objects of experience, for the reason that only by means of them can any object whatsoever of experience be thought (CPR B126).

3 Analytic and Synthetic Judgements

In addition to the *a priori* and a *posteriori* distinction, Kant makes another distinction of his own between judgements, as analytic and synthetic.[12] Let us now turn our attention to this division so as to get a better understanding concerning human knowledge, in particular *synthetic a priori* knowledge. Analytic judgements are judgements which only take us into the realm of tautologies and therefore they are non-informative about the world. Analytic judgements such as 'all bachelors are unmarried' or 'all triangles are three sided' are judgements in which the concept of the predicate is already contained within the concept of the subject. So, the concept of being 'unmarried' is contained within the concept of being a 'bachelor'. The definition of a 'bachelor' (an unmarried person) already contains the predicate 'unmarried'. Similarly, the concept of being 'three-sided' is contained within the concept of being a 'triangle'. The definition of a triangle (a three-sided figure) already contains the predicate 'three-sided'. That being the case, these kinds of judgements are true analytically. On the other hand, synthetic judgements are judgements where, as Kant puts it, the concept of the predicate is not contained in the concept of the subject, but rather is added to it in a manner that expands human knowledge. A synthetic judgement provides us substantive information about the real world, and therefore takes us beyond the realm of tautologies. Synthetic judgements such as 'bodies are heavy' or 'all bachelors are American' are judgements in which the property of 'heavy' or 'American' are not contained in the concept of being a 'body' or a 'bachelor'. In regard to analytic and synthetic judgements Kant writes:

> If I say, for instance, 'All bodies are extended,' this is an analytical judgement. For I do not require to go beyond the concept which I connect with body in order to find extension as bound up with it. To meet with this predicate, I have merely to analyse the concept, that is, to become conscious to myself of the manifold which I always think in that concept. The judgement is therefore analytic. But when I say, 'All bodies are heavy' the predicate is something quite different from

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[12] Peter Sedgwick, *Descartes to Derrida*, 27-28.

anything that I think in the mere concept of body in general; and the addition of such a predicate therefore yields a synthetic judgement' (CPR B11).

Before Kant, the empiricist theory of knowledge clearly consisted in this pair of distinctions. It was held by the empiricist that all *a priori* and necessary truths are analytic, in Hume's terms 'relations between ideas'. Analytic judgements were *a priori* in the sense that they are independent of any particular experience. A pure analysis of the concept involved enables one to form such claims. In this sense, analytic judgements, that is to say, *a priori* truths are always universal and necessary by reason of their being tautologies. Thus, they are true under all circumstances. In Leibniz's word, a necessary truth is true in all possible worlds. Alternatively, it was also held by the empiricist that all synthetic truths are attained *a posteriori* (via observation or experiment). Such synthetic truths or propositions give us real information about the world, and for this reason they are always informative. Unlike analytic truths, they do not have the attribute of necessity. They are at all times utterly contingent in so far as they are made true by empirical facts given certain circumstances. A contingent truth is one that is true, but could be false at any given moment.

Notwithstanding, Kant makes a momentous breakaway from the empiricist position when he claims that not all *a priori* judgements are analytic: there are *a priori* truths which are synthetic rather than analytic. He calls them to be *synthetic a priori* judgements. Indeed, he believes that there are genuine *synthetic a priori* judgements or propositions which provide real information about the world in which we live. Despite this qualification, their truth is still *a priori*, universal and necessary. Mathematical propositions (i.e. the truths of geometry and arithmetic) are *synthetic a priori*. In fact, the philosopher's main problem in the first *Critique* was to answer to the question 'How are *synthetic a priori* judgments possible'. Now it is high time to turn our attention to the nature and possibility of *synthetic a priori* judgments, especially in mathematics.

4 The nature of *synthetic a priori* judgments

First of all, let us consider the nature of *a synthetic a priori* judgment.[13] After combining the *a priori-a posteriori* distinction with the *analytic-synthetic* distinction, Kant points out four possible kinds of judgment:[1] *analytic a priori*: all analytic judgments are *a priori* owing to the fact that they are necessarily true. And the necessity always requires apriority.[2] *Analytic a posteriori:* Kant rules out for obvious reasons the possibility of this kind of judgement.

On the other hand, synthetic judgments can be either *a priori* or *a posteriori*:[3] *synthetic a priori:* these sorts of judgments are neither empirical nor contingent judgments. Particularly, knowledge in mathematics, science and metaphysics is dependent upon *synthetic a priori* judgements. Judgements or propositions of this kind are informative. In other words, they provide us real information about the world, but their truths are nonetheless *a priori*, universal and necessary[4] *synthetic a posteriori:*

[13] For a discussion of Kant's account of the *synthetic a priori*, see Greg Restall, 'A Priori Truths', in *Central Issues of Philosophy*, ed. John Shand (Wiley-Blackwell, 2009), 40-42.

judgments of this kind are empirical and thus contingent judgments. Their degree of generality may vary widely.

According to Kant, there are *synthetic a priori* judgements, that is, judgements that are independent of experience but whose truth cannot be determined just by analyzing the terms. Particularly, knowledge in mathematics, science and metaphysics has to be composed of *synthetic a priori* judgements. So, in his first *Critique* he needs to show that there are *synthetic a priori* judgements and to explain how we can know whether they are true. For his purpose here, Kant argues that mathematical propositions are of this kind. In his view, *synthetic a priori* judgments or propositions have three important characteristics, namely *a priori*ty, syntheticity, and intuition-based. First of all, a *synthetic a priori* judgment is *a priori* on the grounds that its meaning and truth is not established by sensory impressions. And whatever is *a priori*, it is also necessarily true. Secondly, a *synthetic a priori* judgment is not analytic but rather synthetic owing to the fact that concepts involved alone do not determine its truth. And therefore its denial is not logically inconsistent. Thirdly, the meaning and truth of *synthetic a priori* judgments is based on pure or *a priori* intuitions. 'Space' and 'time' are considered to be 'pure forms of sensibility', or 'pure intuitions', that is to say, they are the conditions under which we intuit or sense particular objects.[14] Space and time are not independent of the human mind, therefore they are 'nothing' without the existence of the mind. Kant writes: 'Time is therefore a purely subjective condition of our (human) intuition (which is always sensible, that is, so far as we are affected by objects), and in itself, apart from the subject [mind], is nothing' (CPR A35). Only sensate beings have space and time. Objects which are given in the forms of sensibility (i.e. space and time) always have an absolute connection with the human mind. So, space and time are forms of human sensibility. As discursive finite cognizant, human beings can directly access the world of individual empirical objects and facts under the conditions of space and time. The meaning and truth of synthetic *a posteriori* judgments depends only on *empirical intuitions*. But, on the other hand, the meaning and truth of '*synthetic a priori*' judgments is dependent on *pure intuitions*, that is to say, *a priori* representations of space and time, instead of simply depending on *empirical intuitions.* Kant says: 'For since only by means of such pure forms of sensibility can an object appear to us, and so be an object of empirical intuition, space and time are pure intuitions which contain *a priori* the condition of the possibility of objects as appearances, and the synthesis which takes place in them has objective validity' (CPR A89/B122). *A priori* representations of space and time constitute indispensable foundations not only for the possibility of human experience but also for the objective validity of subjective human knowledge. In an evaluation of Kant's doctrine of the *synthetic a priori*, one distinguished commentator expresses his view as follows:

> Just as the actual, experienced world of individual empirical objects and facts—directly accessible only through empirical intuition—is the fundamental ground of the truth of *synthetic a posteriori* propositions, so in turn the total class of objectively humanly experienceable worlds—directly accessible only through pure intuition and indirectly accessible through the objectively valid schematized

[14] Michael Dummett, 'The Philosophy of Mathematics', in *Philosophy 2, Further Through the Subject*, ed. A. C. Grayling (Oxford: Oxford University Press, 1998), 127.

> pure concepts of the understanding under the original synthetic unity of apperception—is the fundamental ground of *synthetic a priori* propositions.[15]

Hanna goes on to argue that *synthetic a priori* judgments are true in all and only the humanly experienceable possible worlds, and otherwise their truth value is meaningless, whereas analytic judgments, as logical truths, are true in all logically possible worlds (i.e. non-phenomenal worlds):

> This entire class of experienceable worlds functions as the global truth-maker for every *synthetic a priori* proposition. Analytic propositions are true in every logically and conceptually possible world without qualification. By contrast, a proposition is *synthetic a priori* if and only if it is true in all and only the humanly experienceable worlds—or, otherwise put, for all and only objects of experiential cognition. Thus *synthetic a priori* necessity is a strong modality that is essentially restricted by the sensory constitution of creatures minded like us. It is necessity for humans and not for gods.[16]

It is clear that the nature of semantic content of *synthetic* a priori judgments is different from that of analytic judgements. The subject matter of an analytic judgement is conceptual or abstract. They are also named 'concept-based truths'. Judgements of this kind do not provide us with information concerning the nature of physical objects, but rather inform us about the relations between concepts or the nature of abstract entities. Therefore, the truth of an analytic judgement consists in the meaning of the concepts involved as well as in some process of logical analysis. However, *synthetic* a priori judgments are intuition-based truths which are established on *pure intuitions*, namely *a priori* representations of space and time. They supply information about the nature of physical objects but their truth is nevertheless necessary, universal and *a priori*.

In respect to their modality, both *synthetic* a priori judgments and analytic judgements appear to be different from one another. Although analytic judgements are true in all logically possible worlds, *synthetic* a priori judgments are only true in all and only humanly experienceable worlds. The truth value of *synthetic* a priori judgments is meaningless without the presence of humanly experienceable worlds. In another of his writing, Hanna nicely and more clearly explains this point: 'According to Kant *synthetic a priori* propositions are necessary, but unlike analytic propositions they are not absolutely necessary or true in every logically possible world. Rather they restrictedly necessary. This is to say that they are true in all and only the humanly experienceable worlds; in worlds that are not experienceable, they are objectively invalid or truth-valueless'[17] It appears that, roughly speaking, *synthetic a priori* judgments are *necessary truths* which are intimately related with human beings as sensate creatures:

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15 Robert Hanna, *Kant and the Foundations of Analytic Philosophy* (Oxford: Clarendon Press, 2001), 244.

16 R. Hanna, *Kant and the Foundations of Analytic Philosophy*, 245.

17 R. Hanna, 'Mathematics for Humans: Kant's Philosophy of Arithmetic Revisited', *European Journal of Philosophy*, 10 (2002), 331.

'Since *synthetic a priori* propositions are world directed, and also necessarily constrained by the conditions of human sensibility, they are meaningful and true just in so far as they apply to all and only the objects of possible and actual human sense experience. And the strict guarantee of that application is yielded directly by the Objects-Conform-to-Mind Thesis'.[18]

5 The *synthetic a priori* nature of mathematical truths

In Kant's day and long afterwards, mathematics was described as the science of space and that of quantity. And on that account, it was separated into geometry and arithmetic. Geometry is considered to be the mathematical study of shapes, figures, and positions in space. Pretty much all mathematicians and philosophers have agreed that geometry is not about the empirical world. The geometrical ideas such as line, circle, triangle and square are not realised in the physical world at all. Indeed, according to Plato, for example, mathematics, including geometry and arithmetic, is not about this world under any circumstances. Mathematical ideas are envisaged by him to be a separate and eternal realm of metaphysical objects, and thereby they are abstract, have no spatiotemporal or causal properties, and eternal and unchanging in nature. For this reason, mathematics is nothing more than a theory of definite abstract metaphysical objects, which neither exist by depending on any kind of human thought or mind, nor on anything perceptible or perceptible objects. In this understanding, mathematics is wholly *a priori* on the grounds that our knowledge of mathematical objects is in no way dependent on perceptual experience of physical objects. Mathematical knowledge is certain and indubitable, and it is gained by proof which is expected to be immune any reasonable doubt. In mathematics, the proof must have its unproved starting points. In particular, Plato put forth a special effort for the idea of 'proof' and doubted whether these starting points can be known with certainty. His insistence on accurate definitions and clear hypotheses provided the groundwork of Euclid's mathematical system.[19] Through the ascent of reason, one can assess the eternal realm of metaphysical objects. By turning ones back on sense perception and the visible world, one is able to discover the true knowledge (or the mathematical knowledge) that is already hidden within him before his birth into this world.

Along with Plato, Descartes also considers mathematics to be an *a priori* science. But, as distinct from Plato, mathematics is seen by him as a theory of certain mental objects or ideas which 'have their own true and immutable natures'. Let us take Descartes' favourite example of the idea of a triangle to see how he discusses the certainty and indubitablity of mathematical knowledge:

> I find within me countless ideas of things which even though they may not exist anywhere outside me still cannot be called nothing; for although in a sense they can be thought of at will, they are not my invention but have their own true and immutable natures. When, for example, I imagine a triangle, even if perhaps no such figure exists, or has ever existed, anywhere outside my thought, there is still

[18] R. Hanna, *Kant and the Foundations of Analytic Philosophy*, 28.

[19] David Bostock, *Philosophy of Mathematics* (Wiley-Blackwell,2009), 1-6, 37.

> a determinate nature, or essence, or form of the triangle which is immutable and eternal, and not invented by me or dependent on my mind. This is clear from the fact that various properties can be demonstrated of the triangle, for example that its three angles equal two right angles, that its greatest side subtends its greatest angle, and the like; and since these properties are ones which I now clearly recognize whether I want to or not, even if I never thought of them at all when I previously imagined the triangle, it follows that they cannot have been invented by me. [20]

In Descartes, there seems to be a close relationship between innate ideas and the discovery of eternal truths. In a letter to Mersenne, Descartes claims that 'the mathematical truths which you call eternal have been laid down by God' and, a bit onwards, that 'they are all inborn in our minds'.[21] All mathematical truths or objects can only exist in the human mind and their existence is not dependent on the perceptible ideas or anything else that is physical. This being the case, our perceptual experience and the existence of a physical world do not have any value for our knowledge of these eternal objects or truths. The idea here is that our knowledge of mathematics is totally a *priori*, that is, independent of any perceptual experience. Descartes too regards mathematical knowledge to be absolutely certain and indubitable knowledge, which is acquired by proof that consists in unproved starting points (clear and distinct ideas) known with absolute certainty by the intuition of the mind. According to the Cartesian intuitionism, true knowledge, including mathematics, is actually based on intuition of 'simple natures' or essences. The *simplicity* of the starting-points in mathematics enables one to' intuit' the truths in question.[22] Cartesian theory of clear and distinct ideas is particularly designed to give an account of how our basic starting points can be known with certainty, and hence certain and indubitable mathematical knowledge is possible. In this regard, a well-known commentator on the Cartesian philosophy states:

> Just as with 'intuition', then, the crucial point about a 'clear and distinct perception' is the absolute simplicity of its content. A clear and distinct perception is self-evident because it is straightforwardly accessible to my 'mind's eye', and does not contain any extraneous implications which take me beyond that of which I am directly aware. Thus in the case of a simple mathematical proposition such as 'two and two make four', if I focus on the content of this proposition, I have right there, in front of my mind, all I need to be sure that the proposition is true.[23]

Plato and Descartes are just two examples from among many philosophers who believed that mathematics, that is to say, geometry and arithmetic are indeed not about the empirical world. Mathematical ideas or objects do not exist in the physical world. But, do not we get these mathematical ideas or objects, such as lines, circles, triangles,

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[20] R. Descartes, Fifth Meditation (AT VII64: CSM II 44-5).

[21] R. Descartes, *Letter to Mersenne*, 15 April 1630 (AT I 145-6: CSM III 23).

[22] Descartes writes: 'By 'intuition' I do not mean the fluctuating testimony of the senses or the deceptive judgement of the imagination as it botches things together, but the conception of a clear and attentive mind which is so easy and distinct that there can be no room for doubt about what we are understanding. Alternatively, and this comes to the same thing, intuition is the indubitable conception of a clear and attentive mind which proceeds solely from the light of reason' (*AT X*. 368; *CSM I*. 14).

[23] J. Cottingham, *The Rationalists* (Oxford: Oxford University Press, 1988), 35.

and numbers from sense experience? Perhaps, the answer of a psychologist to this question might be positive. He might argue that being out in the world of experience is required for us to get these ideas. One is unable to get the idea of a circle without looking at, for example, the moon or the sun, or the idea of straight line without looking at, for instance, a tree growing straight up. But, although such an idea might give an account of how human beings developed these ideas, the ideas themselves, according to Kant, are independent of any particular experience. In Kantian vision, the matter is not whether one is born with a concept of triangle or whether one develops this concept in the course of his lifetime. He points to the fact that as sensuous beings we have two sources of knowledge, namely time and space, 'from which bodies of *a priori* synthetic knowledge can be derived (Pure mathematics is brilliant example of such knowledge, especially as regards space and its relations)' (CPR B55/A39).

For Kant, mathematical ideas themselves are not dependent of any particular experience, and the truths about these ideas cannot be determined by sense experience. Moreover, first of all, mathematics is general, that is, it is the 'form' into which all experience must fit. At this point, let us consider geometry as one of fundamental branches of mathematics. It is also general, and the judgements of geometry are *a priori*. The nature or form of the triangle is immutable and independent of any particular experience. If we prove or demonstrate various properties of the triangle, for example that its three angles equal two right angles, that its greatest side subtends its greatest angle, and the like, they are true for all triangles, whether one drawn in the diagram on a blackboard or on a computer screen.

Kant insists on the idea that judgements of geometry are *a priori*. But at this point one question arises: 'Are the judgements of geometry analytic or synthetic?' An analytic judgement is claimed to be the one whose truth can be determined just by analysing the terms involved. In other words, an analytic statement or judgement is considered to be *a priori* on the ground that the truth of that statement depends only on the definition of the concepts contained, so any reference to the evidence of experience or observation is omitted for the justification of its truth as totally unnecessary. In contrast, a synthetic judgement is one whose truth can solely be determined by means of something outside its terms. It is rather made or known on the ground of experience or observation via the senses. At first sight, judgements of geometry appear to be analytic. This is because studying geometry seems to begin with definitions. In the course of a proof, by using definitions, the terms are analysed. Let us here remember the philosophy of logical positivism. It is ultimately dependent upon a sharp distinction between analytic and synthetic as well as upon the thesis that all *a priori* propositions must be analytic. The mathematical terms such as number, addition and so on can be defined in purely logical terms. This being the case, one can deduce mathematical theorems from completely logical axioms. For this philosophy, mathematics actually was nothing more than the extension of logic. The truths of arithmetic and geometry are simply comprehensive tautologies. One of the major theoretician of logical positivism, namely Bertrand Russell (1872-1970) strongly contended that mathematical reasoning, that is to say, arithmetical or geometrical reasoning, is purely logical in so far as it proceeds analytically in accordance with concepts. However, Kant makes a crucial departure from such views when he claims that there are genuine *synthetic a priori* judgements or propositions which are informative about the world despite the fact that their truths are still *a priori*, universal and necessary. Indeed, mathematical knowledge is of this kind.

Thus, for Kant, mathematical reasoning, that is, arithmetical and geometrical reasoning cannot proceeds analytically or purely logically, but rather necessitates the activity of construction in pure intuition.

While lecturing on mathematics, Kant was well-trained in Euclidean geometry and reasoning, and drew a deep inspiration from it. Due to his method of reasoning, Euclid has been considered as the founder of the axiomatic deductive method. In order to reach to desired conclusion, in the proof every step must be justified by appealing to common notions, postulates or previously proved propositions. In connection with Euclidean reasoning style, Lisa Shabel writes: '(T)he diagrams *enable* the reasoning of the demonstration by warranting deductive inference. Crucial steps in common Euclidean derivations are taken by virtue of observations made on the bases of a diagram'.[24] It appears that mathematics develops proofs and discoveries by taking into consideration previously proved propositions, postulates or common notions. Although the diagrams are not essential for proving geometrical theorems, the diagrammatic or schematic reasoning is indispensable in the mathematical reasoning-process, and also in justification. According to Shabel, Euclidean reasoning through diagrams 'provides an interpretive model for the function of a transcendental schema'.[25]

Let us here, for instance, consider Euclid's very first proposition (I, 1): 'On a given finite straight line to construct an equilateral triangle'[26] Here Euclid shows us how to construct an equilateral triangle on the specified straight line AB. The proof runs in the following manner: draw a circle (BCD) with centre A and radius AB, and then again draw another circle (ACE) with centre B and radius BA. The point of intersection of these circles will be, say, the point of C. Now, draw line AC and line BC. By construction (by using the definition of a circle (Def.15) and common notion (1), it follows that ABC is the equilateral triangle because AC=AB=BC.[27]

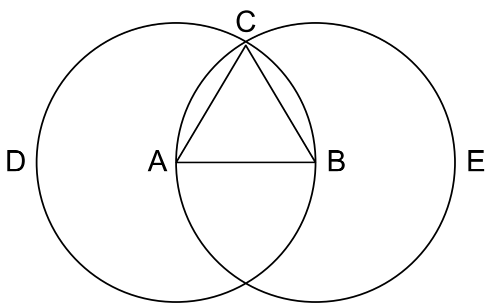

Figure 1.
Euclid and the equilateral triangle.

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[24] L. Shabel, *Mathematics in Kant's Critical Philosophy* (Routledge, 2003a), 38.

[25] L. Shabel, *Mathematics in Kant's Critical Philosophy, 109.*

[26] Euclid, *The Thirteen Books of Euclid's Elements Translated from the Text of Heiberg*, with introduction and commentary by T. L. Heath, 241.

[27] See also, M. Friedman, 'Kant' Theory of Geometry', *The Philosophical Review*, Vol. 94, No.4 (Oct. 1985), 461-462.

But one wonders how we can know that the existence of a point C where the two circles intersect actually follow from the postulates. The modern objector argues that Euclid has neither proved that there is a point C, nor shown that the two circles intersect. Thus, Euclid is not right in claiming the existence of C while not taking into consideration 'the continuity axiom' in his list of Postulates and Common notions. Accordingly, Euclid's proof has been found defective and from his axioms one cannot infer there is a point C. This is because, Euclid's *Elements* has not got axioms of motion, congruence (except for one), order and continuity. Despite this, he uses both motion and continuity in the proofs. Whatever the logical deficiencies in the structure of Euclid's *Elements* may be, for the purpose of our study let's leave this matter aside.

Despite the fact that Euclid's method in *Elements* is deductive and there is some analytic reasoning in geometry as appeared in the above example, for Kant, the judgements of geometry, that is, the proved propositions are synthetic. How one can know the existence of a point C where the two circles intersect? In this connection, David Bostock explicitly states: 'All that one can say is that this is entirely obvious when one constructs a diagram as Euclid says, i.e. we *cannot imagine* this diagram without imagining a point of intersection. But this is a matter of what we can imagine, not a matter of what follows logically from Euclid's axioms.'[28] Accordingly, it is clear that we make such constructions in space, that is to say, our spatial intuitions enable us to make geometrical constructions like an equilateral triangle. The geometrical reasoning requires the concept of *space*. It would be better to say that the activity of *a priori* spatial reasoning necessitates our spatial intuition. Instructions and diagrams held in geometry make it possible for us to engage in an activity of spatial reasoning through spatial intuition. For Kant, geometry is the subject field which concerns itself with 'concrete, but non-empirical concepts by reference to visualizable intuitions',[29] as distinct from other abstract fields such as philosophy. The nature of mathematical or geometrical proof is distinct from other kinds of deductive argumentation, e. g. a philosophical argument. Although philosophy and mathematics, in particular geometry, each seems to be an *a priori* inquiry, so far as their methods and the absolute character of their results are concerned, they are quite different. In this regard, Kant states:

> Suppose a philosopher be given the concept of a triangle and he be left to find out, in his own way, what relation the sum of its angles bears to a right angle. He has nothing but the concept of a figure enclosed by three straight lines, and possessing three angles. However long he meditates on this concept, he will never produce anything new. He can analyse and clarify the concept of a straight line or of an angle or of the number of three, but he can never arrive at any properties not already contained in these concepts. Now let the geometrician take up these questions. He at once begins by construction a triangle. Since he knows that the sum of two right angles is exactly equal to the sum of all the adjacent angles which can be constructed from a single point on a straight line, he prolongs one side of his triangle and obtains two adjacent angles, which together are equal to two right angles. He then divides the external angle by drawing a line parallel to

28 D. Bostock, *Philosophy of Mathematics*, 58.

29 Timothy Gowers (ed.), *The Princeton Companion to Mathematics* (Princeton University Press, 2008),136.

> the opposite side of the triangle, and observes that he has thus obtained an external adjacent angle which is equal to an internal angle – and so on. In this fashion, through a chain of inferences guided throughout by intuition, he arrives at a fully evident and universally valid solution of the problem (CPR A716/B744-A717/B745).

In geometry or geometrical proof, the use of diagrams and the function that they have make it different from other kinds of *a priori* inquiry like philosophy which also makes use of deductive argumentation. The diagrams offered in geometry refer not only to a method for abstract reasoning, but also to an intuition, that is, giving concrete form to an abstract mathematical concept or idea by locating or apprehending it in space and time.[30] Kant states 'I entitle a magnitude extensive when the representation of the parts makes possible, and therefore necessarily precedes, the representation of the whole. I cannot represent to myself a line, however small, without drawing it in thought, that is, generating from a point all its parts one after another (CPR B203/A163). Throughout the *Elements* Euclid used diagrams and therefore he was a main example of the use of diagrammatic reasoning in geometry. Kant actually tries to provide an epistemological account and base for Euclid's diagrammatic reasoning by offering the idea of schematic construction in pure intuition in regard to mathematical proofs in particular or mathematical thinking in general. For Kant, diagrams are actually visualizable intuitions in the mind of the perceiver. 'The schema is in itself always a product of imagination' (CPR A140). The philosopher continues to argue that 'Indeed it is schemata, not images of objects, which underlie our pure sensible concepts. No image could ever be adequate to the concept of a triangle in general. It would never attain that universality of the concept which renders it valid of all triangles, whether right-angled, obtuse-angled, or acute-angled; it would always be limited to a part only of this sphere. The schema of the triangle can exist nowhere but in thought. It is a rule of synthesis of the imagination, in respect to pure figures in space' (CPR B180/A141). Geometrical reasoning necessitates the construction of geometrical figures in our faculty of imagination. That is why geometry is neither a pure empirical science, nor a tautological discipline like deductive logic that is immune from any empirical content. Although a geometrical proof is held tightly by logic or deductive reasoning, it is not simply a logical analysis of the terms used, as indeed envisaged by the logical positivists. Geometry or a geometrical proof involves *synthetic a priori* judgements based on pure or *a priori* intuition (i.e. space as well as time), rather than on the interrelations of pure logical concepts. Besides, it has also *synthetic a priori* reasoning. But, like analytic truths, discoveries of *synthetic a priori* reasoning are equally certain and necessary.

Having made a clear contrast between the reasoning used in mathematics and that of other fields like metaphysics or philosophy in the section named 'The Discipline of Pure Reason in the Sphere of Dogmatism'[31], Kant makes it clear that mathematical

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[30] Leo Corry, 'The Development of the Idea of Proof', in *The Princeton Companion to Mathematics*, ed. Timothy Gowers (Princeton and Oxford: Princeton University Press, 2008), 137.

[31] In this regard, the philosopher states that '*Philosophical* knowledge is the *knowledge gained by reason from concepts*; mathematical knowledge is the knowledge gained by reason from the *construction* of concepts. To *construct* a concept means to exhibit *a priori* the intuition

reasoning has a greater power when compared with the others by reason of its employment of 'intuition'. Here a special emphasis is particularly placed on geometry. In the part called the 'Transcendental Aesthetic', the concept of 'space' is identified with 'an *a priori* intuition' (CPR A24). An *a priori* intuition is not an empirical thing, but rather it is something that is found 'at the root of all our conceptions of space'. With regard to the matter, Kant states: '[G]eometrical propositions, that, for instance, in a triangle two sides together are greater than the third, can never be derived from the general concepts of line and triangle, but only from intuition, and this indeed *a priori*, with a apodeictic certainty' (CPR A25). Spatial concepts like that of a triangle can only be constructed in intuition. What Kant suggests is that once one constructs a particular example of that concept in his imagination or draws it on paper or on computer screen (however imperfect), then he is able to concentrate on its properties. In other words, first of all, one has to imagine or construct a 'particular triangle' in imagination so as to demonstrate a general principle or a truth concerning all triangles. Having discovered those of its properties which are essential to the nature of a triangle, the finding or the reasoning will be generalised and used to be apply to all triangles in geometry. It is this method which actually enables a mathematician to consider general concepts in a state of being concrete. The mathematician can draw an actual and particular figure, e.g. triangle and make it more complete with additional constructions. To do this, into the argument he introduces, for example, new appropriate extra lines, circles. In this way, in order to be proved the proposition is made more obvious, and a general geometrical proposition is displayed. But, on the other hand, when a philosopher is asked to discover, by the philosophical method, 'what relation the sum of its angles bears to a right angle', he only makes an analysis of 'the conception of a right line, of an angle, or of the number three' and so on but, 'but he will not discover any properties not contained in these conceptions' (CPR A716/B744). This actually shows that in fact philosophical reasoning can proceed analytically according to concepts, that is to say, it is in fact purely a logical procedure, but geometrical reasoning goes beyond this. It rather necessitates a further activity, that is, the activity of 'construction in pure intuition'.

Al this point, it would be beneficial to consider the proposition that 'the three interior angles of any triangle always equal two right angles'. This has been also expressed in modern terms as 'the sum of the angles of a triangle equal 180 degrees'. How do we get to know that this proposition is true - how did Euclid prove that the sum of the angles of a triangle equal two right angles? The answer certainly requires a clear understanding of Euclid's proof. If this were an analytic judgment, according to Kant, one could have proved it simply by analyzing the terms involved in the statement, that is to say, the sum of the angles of a triangle is two right angles. If one is a philosopher or metaphysician, he may analyze the terms involved in the judgement such as 'the concept of a triangle', the concept 'sum', the concept 'three' or 'sum of the angles of a triangle' and so on. The meditation on the definition of a triangle is only going to elucidate on analytic truth. By analysing the concept of a triangle, one may get three angels or three sides because it is a triangle. The analysis of the concept of 'sum' would

which corresponds to the concept. For the construction of a concept we therefore need a *non-empirical* intuition' (CPR A713/B741).

give the result of something being added. Via the analysis of the concept of 'three' one might get the sum of one and one and one. Even if one analyses the phrase 'sum of the angles of a triangle', this analysis will not provide him with the value of that sum. The question still will remain whether this sum is two right angles, or three right angles, or something else. But the mathematician does not only pay attention on the definition of a triangle, but also starts the construction of figures (for instance, a sample triangle) in his imagination. This act is not simply meditation on the concept of triangularity, it goes beyond that. Through the intuition of a triangular region of space, the propositions about triangles can be derived and proved with certainty.[32] Kant follows the example of a Euclidean proof (Elements: Book I, Prop. 32), where Euclid proved that 'the sum of the interior angles of a triangle is equal to the sum of two right angles'.

Proposition 32:

In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.

Let ABC be a triangle, and let one side of it BC be produced to D;

I say that the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC, and the three interior angles of the triangle ABC, BCA, and CAB are equal to two right angles.

For let CE be drawn through the point C parallel to the straight line AB. [I.31]

Then, since AB is parallel to CE, and AC has fallen upon them, the alternate angles BAC, ACE are equal to one another. [I. 29]

Again, since AB is parallel to CE, and the straight line BD has fallen upon them, the exterior angle ECD is equal to the interior and opposite angle ABC. [I. 29]

But the angle ACE was also proved equal to the angle BAC; therefore the whole angle ACD is equal to the two interior and opposite angles BAC and ABC.

Let the angle ACB be added to each; therefore the angles ACD and ACB are equal to the three angles ABC, BCA, CAB.

But the angles ACD, ACB are equal to two right angles; [I. 13] therefore the angles ABC, BCA, CAB are also equal to two right angles.

Therefore etc. Q.E.D.[33]

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[32] D. Bostock, *Philosophy of Mathematics*, 56-57.

[33] Euclid, *The Thirteen Books of Euclid's Elements Translated from the Text of Heiberg*, with introduction and commentary by T. L. Heath, 316-17.

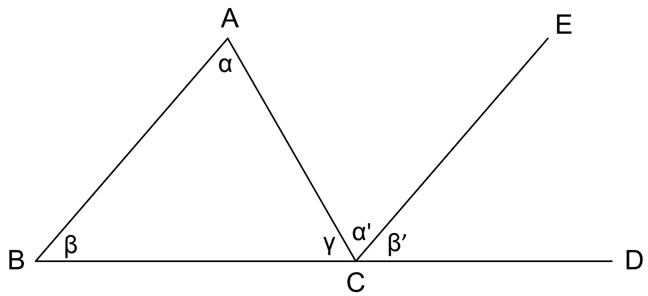

Figure 2:
Euclid and the interior angles of a triangle.

In the proof, the first step is to construct a sample triangle. The next important step is to extend the original figure, that is, the base of the sample triangle by constructing some *further* lines. So as to create an exterior angle, in particular its one side is extended. In the next step, this exterior angle is divided by constructing a line through the vertex of that angle parallel to the opposite side of the triangle. After making those constructions, Euclid argued about the equality of various angles, that is, the opposite interior angles are each equal to one part of the exterior angle, and therefore with certainty concludes that $\alpha = \alpha'$ and $\beta = \beta'$, so $\alpha + \beta + \gamma = \alpha' + \beta' + \gamma = 180°$.

Kant makes his crucial comment about this reasoning by insisting on the fact that construction in pure intuition is crucially significant to this proof. Referring to Euclid's reasoning involved here, he states that 'In this fashion, *through a chain of inferences guided throughout by intuition*, he arrives at a fully evident and universally valid solution of the problem' (emphasis added) (CPR A717/B745). The main feature of this proof is those constructions that mathematician have made 'on the ground of intuition'. Without those constructions, the proof simply cannot proceed. Therefore, the proof requires the extension of one side of the triangle so as to originate an exterior angle. Then, it also requires the division of this exterior angle by means of a line in order to construct a parallel line to the opposite side of the triangle, and so on. Over and above those of (an idealized version of) the figure originally given, the constructions of extra lines AB, BD, CE and so on are necessary components in order for the proof to proceed successfully. Indeed, all geometrical proofs necessitate some sort of construction in intuition. This first of all shows the fact that geometrical proofs are themselves 'spatial' objects. In addition to this fact, what is significant to Kant is that 'the mathematics of extension or geometry, along with its axioms is dependent on the successive synthesis of the productive imagination in the construction of the figures'. The geometrician actively draws or constructs these lines aforementioned in thought in a continuous manner. For this reason, geometrical proofs are also 'spatio-temporal' objects (i.e. the motions of points existing in both space and time). In this regard, Friedman writes 'That construction in pure intuition involves not only spatial objects, but also spatio-temporal objects (the motions of points) explains why intuition is able to supply *a priori* knowledge of (the pure part of) physics'.[34] As a matter of fact, one cannot conceive spatial concepts like lines, circles and triangles unless constructing them in thought, that is to say, actually and continuously drawing them in the inner sense. On the other hand, unless representing temporal concepts in time, one is unable to conceive them. In other words, it is the spatial concepts which render possible pure temporal concepts. Not only do inner sense and outer sense require one another, but the representations of time and space do also.[35] In connection with the matter, Kant himself states:

> We cannot think a line without drawing it in thought, or a circle without describing it. We cannot represent the three dimensions of space save by setting three lines at right angles to one another from the same point. Even time itself we cannot represent, save in so far as we attend, in drawing of a straight line (which has to serve as the outer figurative representation of time), merely to the act of the synthesis of the manifold whereby we successively determine inner sense, and in so doing attend to the succession of this determination in inner sense (CPR B154).

So, despite the gap, there seems to be some kind of symmetrical relation between the sensations of the outer and inner forms of intuition (time and space). Mathematics is synthetic on the ground that geometrical objects are constructible. At this point, Kant asks a momentous question about the source of those constructions. 'Where did one make those constructions?' In a way similar to that of many philosophers and mathematicians, Kant believes that geometry is not something about the physical world. The physical lines, triangles or circles that one draws on paper, blackboard, or computer screen are nothing but rather imperfect pictures of the actual construction. The philosopher replies to the question here by saying that one makes those constructions in 'space'. In regard to the concept of space Kant departs from both Newtonian (or absolute) and Leibnizian (or relational) views. In his own view, space is to be identified with our outer *a priori* intuition; similarly time is to be equated with our inner *a priori* intuition. Beyond anything else, like Newton, Kant claims that space is not empirical concept that has been derived from outer experience; one can neither see, nor touch space. Indeed, it is not something that is an object of sense experience: 'Space is not an empirical concept which has been derived from outer experiences ... The representation of space cannot, therefore, be empirically obtained from the relations of outer appearance. On the contrary, this outer experience is itself possible at all only through that representation' (CPR B38). The Newtonian view suggests that space (and also time) exist independently of being perceived and of any objects (things in themselves) in space and time. Therefore, both space and time are real existences. The idea is that space is 'out there somewhere'. The Leibnizian view, in contrast, is that space (and

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[34] M. Friedman, 'Kant' Theory of Geometry', 460.

[35] In this connection, Robert Hanna states: 'He (Kant) states explicitly that (the representation of) time is the 'mediate condition of outer appearances', which is to say that the perception of objects in space necessarily implements temporal form: 'all appearances whatsoever - that is, all objects of the senses - are in time, and necessarily stand in time relations'. Indeed, the very possibility of representing the motion of material objects in space presupposes the representation of time. Correspondingly, for Kant we always represent our own inner mental states and acts in direct relation to space'. See his book, *Kant and the Foundations of Analytic Philosophy*, 216.

time) exist depending upon the relations holding between the things in themselves, and irrelevant to being perceived or intuited.[36] By rejecting both these views, Kant argues that space does not exist 'out there somewhere', but rather, likewise causality, it is always in our minds. Without space, one is unable to give consideration to the world. Although one can represent space as being empty of objects, he can never represent to himself the absence of space. So, for him, space is indeed in our minds, and these lines that he wrote are very relevant:

> Space is a necessary *a priori* representation, which underlies all outer intuitions. We can never represent to ourselves the absence of space, though we can quite well think it as empty space. It must therefore be regarded as the condition of the possibility of appearances, and not as a determination dependent upon them. It is an *a priori* representation, which necessarily underlies outer appearances (CPR A24).

The clear implication here is that even though one can imagine a space without objects (i.e. empty space), he is unable to imagine objects without space. Space is an *a priori* form of outer perception; respectively time is *a priori* form of inner perception. Through the agency of the space or outer priori intuition that we have in our minds, we put all our sense perceptions in order, and thereby we affirm for instance that this is something that is situated next to that, or behind that, or to the right of that, and so on. Space as the form of our outer sense or pure outer intuition is always one and the same. This characterisation also applies to the status of geometry. A good summary about the nature of space is given by Kant as follows:

> If, therefore, space (and the same is true of time) were not merely a form of your intuition, containing conditions *a priori*, under which alone things can be outer objects to you, and without which subjective conditions outer objects are in themselves nothing, you could not in regard to outer objects determine anything whatsoever in an *a priori* and synthetic manner. It is, therefore, not merely possible or probable, but indubitably certain, that space and time, as the necessary conditions of all outer and inner experience, are merely subjective conditions of all our intuition, and that in relation to these conditions all objects are therefore mere appearances, and not given us as things in themselves which exist in this manner (CPR B66-9/A49).

Within Kant's technical terms, space refers to a pure *a priori* intuition of the intellect as stated 'the original representation of space is an intuition *a priori*, and not a conception' (CPR A25). *Synthetic a priori* judgements or knowledge are only possible on the basis of pure *a priori* intuition. For Kant, *a priori* intuition is something that is unique, and therefore the mathematics of extension or geometry is unique too. 'Space is essentially one; the manifold in it, and therefore the general concept of spaces, depends solely on [the introduction of] limitations (CPR A25). If intuition is something unique, or if there is just one space in our minds, then the mathematics of extension or geometry is unique as well. By reason of the fact that space as 'an *a priori*, and not an empirical, intuition underlies all concepts of space' (CPR A25), all geometrical principles and concepts are themselves derived from pure spatial intuition. Supposing that the space is unique in our minds, and again supposing that the three interior angles of any triangle in

[36] A. Ward, *Kant: Three Critiques* (Cambridge: Polity Press, 2006), 33.

this space are equal to two right angles – or the sum of the angles of a triangle equal 180 degrees, then the space in our minds must be the Euclidian one. That is to say, Kant's space needs to comply with the orders of Euclid's fifth postulate. Therefore, indeed not only for Kant but for many philosophers, there is no other space or geometry but solely the Euclidian one. Euclidian geometry develops a set of propositions which are based on Euclid's five postulates offered in the *Elements*, in particular, Euclidean space in which the fifth postulate, occasionally named the parallel postulate, holds to be true.[37] Geometrical knowledge or judgement is *synthetic a priori* insofar as it is based on constructions of pure or *a priori* intuition. The truth of theorems in geometry cannot be established through just analysing the terms used in them. These theorems are proved with a special reference to something (i.e. our intuition of space) that is outside of the words of the theorems.

So far as arithmetic is concerned, Kant restricts himself to the theory of natural numbers rather than providing an account of any theorems of number theory like the unique-prime-factorization theorem, according to which any integer greater than 1 can be written as a *unique* product (up to ordering of the factors) of prime numbers. He discusses elementary numerical equations by offering the arithmetical example '7+5=12'. The requirement for arithmetic reasoning is not the concept of *space*, but the concept of *time*.[38] As in the case of geometrical proofs, on this occasion we also deal with some kind of rational deduction for the demonstration of arithmetical proofs. Although this rational deduction is pure and *a priori* in the sense of not being empirical, it is again not sufficient. Arithmetical reasoning, like geometrical reasoning, goes beyond what analytical reasoning offers us. The truths of arithmetic are *synthetic a priori*, just as are the truths of geometry. Here the celebrated passage is worth quoting in full:

> We might, indeed, at first suppose that the proposition 7 + 5 = 12 is a merely analytical proposition, and follows by the principle of contradiction from the concept of a sum of 7 and 5. But if we look more closely we find that the concept of the sum of 7 and 5 contains nothing save the union of the two numbers into one, and in this no thought is being taken as to what that single number may be which combines both. The concept of 12 is by no means already thought in merely taking this union of 7 and 5; and I may analyse my concept of such a possible sum as long as I please, still I shall never find the 12 in it (CPR B15).

It is clear from the passage that one might initially believe that the proposition 7+5=12 is only an analytic proposition, since it apparently seems to follow from the conception of a sum of seven and five in accordance with the principle of contradiction. But the situation is not quite like that. On the contrary to the idea of analyticity, Kant states that 'We have to go outside these concepts, and call in the aid of intuition which corresponds to one of them, our five fingers ... five points, adding to the concept of 7, until by unit, the five given in the intuition' (Ibid). And the philosopher continues to argue that 'First starting with the number 7, and for the concept of 5 calling in the aid of

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37 Euclid, Postulate 5: 'That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles'.

38 Michael Dummett, 'The Philosophy of Mathematics', 128.

fingers of my hand as intuition, I now add one by one to the number 7 the units which I previously took together to form the number 5, and with the aid of that figure [the hand] see the number 12 coming into being' (CPR B15-B16). If I understand him correctly, Kant established a special relation between the concept of the totality (the sum of) positive integers and that of a 'sequence'. The positive integers are nothing more than those elements, points or moments. One only can attain or achieve the positive integers through beginning with the number 1 and successively adding 1 (i.e. '1+1=2', '2+1=3', '3+1=4', and so on). Whatever is accomplished successively, that is, a sequence, it is carried out in time. If the sequence proceeds in time, then it is actually temporal. So, in fact, the totality of natural numbers actually derives from an *a priori* intuition of time.

So, according to Kant, 7+5=12 is a synthetic statement, because, the concept '12' is not contained anywhere within the concept '5', or the concept '7', or the concept '+', when analyzed individually and combined. The idea of '12' must be attained through application to objects in the intuition. This point becomes even more evident, when we take large numbers. The judgment that 7+5=12 actually necessitates that the *sum* of 7 and 5 must be constructed, that is to say, through *counting*, either by mental arithmetic (counting points), or e.g. by using fingers. The idea of counting points or fingers does not literally means that we have to count dots in order to represent large numbers, but rather illustrates the fact that arithmetic is in fact constructive in nature.[39] The sum is constructed in accordance with the rules which are defined for addition. In the method involved here, each of the rules is reduced to simpler rules, so that the whole thing is ultimately reduced to the pure intuition of time, that is to say, the counting of successive moments of time, at least in principle. In the construction, one concept cannot literally be a part of the other (e.g. 7+1=8, 7+2=9, 7+3=10, 7+4=many and so on). Thus, the proposition 7+5=12 is not analytic, as far as Kant's definition is concerned. But, in his thought, the technique of analysis seems to apply solely to the subject term of a proposition rather than the predicate-term. But taking both concepts and analysing them both might make a difference, and thereby their coincidence can be seen. Leibniz did this by gave a proof that '2+2= 4' through a process of analysing the terms.

Definitions:	2 = 1+1;	3 = 2+1;	4 = 3+1.
Proof:	2+2 = 2+1+1	(Def.2)	
	= 3+1	(Def.3)	
	= 4	(Def.4)	

Let's put it in another way: 2 by definition 1+1. So, 2+2 can be divided as 2+2 = (1+1) + (1+1). Then, grouping can be disposed of as: (1+1) + (1+1) = 1+1+1+1, then regrouped them in the following manner:

= (1+1) + 1+1	(by Def. 2: '2=1+1', one can write)
= 2+1+1	
= (2+1) + 1	(by Def. 3: '3=2+1', one can write)
= 3+1	(This is the definition of 4)
= 4	

39 D. Bostock, *Philosophy of Mathematics*, 47-48.

In the proposition '2+2' (the subject term) and 4 (the predicate term) are identical, since they can be analysed on the grounds of the same concepts, that is to say, '1' and '+1'. In other words, the proof is achieved through analyzing '2 + 2' into a sum of 1's, and then employing that sum of 1's to construct the number 4. But Kant would be reluctant to claim that the proof in question is analytic. For him, this is actually a construction, because, in the course of putting all those 1's together, one is actually constructing the number 4 out of the pieces 2+2. In arithmetic, this construction is a process which takes place in 'time'. Leibniz's proof is successful by reason of the fact that every step of this construction takes place in order and in time. Like space, time is a pure *a priori* intuition of the intellect, and also is unique. Thus, arithmetic is unique too. Due to this *a priori* intuition we order our perceptions in time. On the ground of it, we claim, for instance, that this case or event is earlier (or later) than the other one. In regard to time Kant writes:

> [T]ime is an *a priori* condition of all appearances whatsoever. It is the immediate condition of inner appearances (of our souls), and thereby the immediate condition of outer appearances. Just as I can say *a priori* that all outer appearances are in space, and are determined *a priori* in conformity with the relations of space, I can also say, from the principle of inner sense, that all appearances whatsoever, that is, all objects of the senses, are in time, and necessarily stand in time-relations (CPR A34-B51).

6 Conclusion

As a result, it is clear that Kant characterizes mathematics to be the science of space and also of quantity, and thereby divides it into geometry and arithmetic. Geometry is built on our a priori intuition of space, and arithmetic on our a priori intuition of time. The truths of geometry and those of arithmetic are synthetic a priori rather than being either solely analytic or synthetic. The geometrical reasoning necessitates the existence of the construction of figures in our imagination. In fact, the geometrical proofs are dependent of what one is or is not capable of imagining. Geometrical knowledge is a priori in so far as one is incapable of imagining things otherwise. In other words, because of *a priori* intuition of space, one is incapable of constructing things or figures in our imagination otherwise. The spatial nature of our outer experience is not owing to the nature of things experienced, but rather to our human way of perceiving things. Our geometrical knowledge is synthetic because not only space, but also time functions as a pre-condition in the course of construction of figures in our imagination.

In the case of arithmetic, the construction in pure intuition also plays an essential role in the discovery of the results. The representation of pure units in pure intuition plays an essential role in our cognition of basic *a priori* arithmetical truths like 7+5=12. Such propositions of arithmetic are not analytic. In order to acknowledge the fact that 5 added to 7 gives rise to 12, one has to go beyond these concepts and appeal to the assistance of intuition. The contingent empirical representations of 5 can be used so as to verify the correctness of 7+5=12. By means of forming an image of five things and adding them (the units contained in the five given in the intuition) to a given image of 7 things one by one, we find out the outcome that 7+5=12. The totality of natural numbers actually derives from the *a priori* intuition of time, because it is actually accomplished successively, that is, a sequence, and carried out in time, and therefore it is actually temporal.

Thus, for Kant, mathematics, either arithmetic or geometry, is *synthetic a priori* rather than being *analytic a priori*. In other words, the truths of mathematics are not only known *a priori* but also *synthetic*. This would mean that they are both *a priori*, that is, necessary and universal, and informative, that is, providing substantive information about the real world. Kant's main problem in the first *Critique* was to explain the possibility of such knowledge, that is to say, mathematics is not a purely logical science possessing the features of a priority, universality and necessity, but it is an informative science giving real information about our world.

Abbreviations

'CPR': Kant, *Critique of Pure Reason*, trans. N. Kemp Smith (New York: St Martin's, 1965).

'CSM': John Cottingham, Robert Stoothoff, and Dugald Murdoch (eds. and trans.), *The Philosophical Writings of Descartes,* vols: I & II (Cambridge: Cambridge University Press, 1985).

'CSMK': John Cottingham, Robert Stoothoff, Dugald Murdoch, and Anthony Kenny (eds. and trans.), vol. III (Cambridge: Cambridge University Press, 1991).

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