Uludağ Üniversitesi
Eğitim Fakülteleri Dergisi
Cilt: VII, Sayı: 2, 1992

The Solutions of Certain Classes of Fredholm Integral Equations by Means of Taylor Series

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SUMMARY

In this paper, several classes of Fredholm integral equations which have Taylor series solutions are studied. The method used for solving these equations is a generalization of the method described by R.P. Kanwal and K.C. Liu[1]*. The obtained solution involves a Taylor series approximation, the coefficients of which are solutions of a linear algebraic system. The coefficients are computed by means of matrix equation that depends on the kernel. To illustrate the method, it is applied to certain linear Fredholm integral equations. The results are compaired with those already published*[1.2.3]*.*

ÖZET

Belirli Sınıftan Fredholm Integral Denklemlerin Taylor Serisi Yardımıyla Çözümleri

Bu makalede, Taylor seri çözümlerine sahip bazı Fredholm integral denklemler incelenmiştir. Bu denklemlerin çözümü için kullanılan yöntem R.P. Kanwal ve K.C. Liu[1] *tarafından sunulan yöntemin bir genellemesidir. Elde edilen çözüm, bir Taylor seri yaklaşımıdır ki bunun katsayıları bir lineer cebrik sistemin çözümleridir. Katsayılar çekirdeğe bağlı matris denklemi yardımıyla hesaplanır. Yöntem, bazı lineer Fredholm integral denklemlere uygulanarak açıklanır ve sonuçlar yayınlanmış olan sonuçlarla karşılaştırılır*[1.2.3]*.*

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INTRODUCTION

Integral equations occur in the fields of mechanics, mathematical physics, differential equations, pure analysis, functional analysis, engineering and stochastic processes, and form one of the most useful tools. An integral equation is an equation in which an unknown function appears under one or more integral signs and is usually difficult to solve analytically.

In this study, we shall deal only with linear Fredholm integral equations of the first and second kinds. These equations are defined in the forms

$$f(x) + \lambda \int_a^b K(x, y)\, g(y)\, dy = 0 \qquad (1)$$

and

$$g(x) = f(x) + \lambda \int_a^b K(x, y)\, g(y)\, dy \qquad (2)$$

where the limits of integration a and b are constants ($a \leq x,y \leq b$). In both equations, g is the unknown function, while $f(x)$ and $K(x,y)$ are the known functions; λ is a nonzero, real or complex parameter. $K(x,y)$ is called the kernel of the equation. If $K(x,y) = K(y,x)$, the kernel is called symmetric. If however, in (1) or (2) either a or b or both are infinite, or if the kernel $K(x,y)$ becomes infinite in the region of integration, the equation is said to be singular. If, in (2), $f(x) = 0$ the equation is said to be homogenous.

Recently there is an increasing interest towards integral equations. Fredholm, Neumann, Hilbert-Schmidt, Chebyshev and Taylor expansion approaches are well-known[1-5]. In particular, it has been discovered that many integral equations can be solved with the help of the Taylor series[1]. We discuss here a generalized Taylor method for solving the equations (1) and (2).

METHOD OF SOLUTION

Let us consider the Fredholm integral equation (2) and assume that it has a solution in the form

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{n!} g^{(n)}(c)\,(x-c)^n, \quad a \leq c \leq b \qquad (3)$$

which is a Taylor series about $x = c$. Here the coefficients $g^{(n)}(c)$ are constants to be computed.

To solve (2) we differentiate in n times with respect to x and get

$$g^{(n)}(x) = f^{(n)}(x) + \lambda \int_a^b \frac{\partial^{(n)} K(x,y)}{\partial x^n} g(y)dy$$

When we put x = c in this equation we have

$$g^{(n)}(c) = f^{(n)}(c) + \lambda \int_a^b \frac{\partial^{(n)} K(x,y)}{\partial x^n}\Big|_{x=c} g(y)dy \tag{4}$$

Next, we expand g(y) in Taylor series at y = c, that is,

$$g(y) = \sum_{m=0}^{\infty} \frac{1}{m!} g^{(m)}(c)(y-c)^m \tag{5}$$

and substitute it in (4). The result is

$$g^{(n)}(c) = f^{(n)}(c) + \lambda \sum_{m=0}^{\infty} T_{nm}\, g^{(m)}(c) \tag{6}$$

so that

$$T_{nm} = \frac{1}{m!} \int_a^b \frac{\partial^{(n)} K(x,y)}{\partial x^n}\Big|_{x=c} (y-c)^m dy \tag{7}$$

The infinite relations (6) with the unknown Taylor coefficients occur an infinite lineer algabraic system. The system (6) can be solved approximately by a suitable truncation sheme, say n,m = 0, 1, ..., N. In this case, we obtain the matrix equation

$$TG = F \tag{8}$$

where

$$G = \begin{bmatrix} g^{(0)}(c) \\ g^{(1)}(c) \\ \cdot \\ \cdot \\ \cdot \\ g^{(N)}(c) \end{bmatrix} \quad T = \begin{bmatrix} \lambda T_{00}-1 & \lambda T_{01} & \ldots\lambda T_{0N} \\ \lambda T_{10} & \lambda T_{11}-1 & \ldots\lambda T_{1N} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \lambda T_{N0} & \lambda T_{N1} & \ldots\lambda T_{NN}-1 \end{bmatrix} \quad F = \begin{bmatrix} -f^{(0)}(c) \\ -f^{(1)}(c) \\ \cdot \\ \cdot \\ \cdot \\ -f^{(N)}(c) \end{bmatrix}$$

Also, we can use the matrix equation (8) for solving the integral equation (1). In this case, the matrix T becomes

$$T = \lambda\,[T_{nm}],\ n,m = 0,1,\ldots,N \qquad (9)$$

If $D(\lambda) = [T] \neq 0$, the equation (8) may be written in the form

$$G = T^{-1} F \qquad (10)$$

which determines uniquely the unknown coefficients $g^{(n)}(c)$, $n = 0,1,\ldots, N$. When we substitute these values in (3) we obtain the Taylor series solution

$$g(x) \cong \sum_{n=0}^{N} \frac{1}{n!}\, g^{(n)}(c)\,(x-c)^n \qquad (11)$$

ILLUSTRATIONS

Example 1. Let us illustrate the method by means of the inhomogen integral equation

$$g(x) = x + \lambda \int_0^1 (xy^2 + x^2y)\, g(y)dy$$

where $K(x,y) = xy^2 + x^2y$, $f(x) = x$, $a = 0$, $b = 1$, const.

Taking $c = 0$ and $N = 2$, we evaluate the quantities $f^{(n)}(0)$ and T_{nm} as

$$f^{(0)}(0) = 0 \quad f^{(1)}(0) = 1 \quad f^{(2)}(0) = 0$$

$$T_{00} = 0 \qquad T_{01} = 0 \qquad T_{02} = 0$$

$$T_{10} = 1/3 \qquad T_{11} = 1/4 \qquad T_{12} = 1/10$$

$$T_{20} = 1 \qquad T_{21} = 2/3 \qquad T_{22} = 1/4$$

we substitute these values in (8) and obtain

$$\begin{bmatrix} -1 & 0 & 0 \\ \lambda/3 & \lambda/4^{-1} & \lambda/10 \\ \lambda & 2\lambda/3 & \lambda/4^{-1} \end{bmatrix} \begin{bmatrix} g^{(0)}(0) \\ g^{(1)}(0) \\ g^{(2)}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

When we solve this equation, we get

$$g^{(0)}(0) = 0 \quad g^{(1)}(0) = \frac{60\lambda - 240}{\lambda^2 + 120\lambda - 240}, \quad g^{(2)}(0) = \frac{-160\lambda}{\lambda^2 + 120\lambda - 240}$$

Substituting these values in the Taylor expansion (11) we obtain

$$g(x) = [(60\lambda - 240)x - 80\lambda x^2]/(\lambda^2 + 120\lambda - 240)$$

which is the exact solution[2] (p.10).

Example 2. Let us find the approximate solution of the singular integral equation

$$g(x) = \sin x + \int_0^{\infty} e^{-x-y} g(y)\, dy \qquad (12)$$

where $f(x) = \sin x$, $\lambda = 1$, $a = 0$, $b = \infty$, $K(x, y) = \exp(-x-y)$.

If we take $c = 0$ and $N = 5$, we obtain the matrices T and F as

$$T = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & -2 & -1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & -2 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & -2 \end{bmatrix}$$

and

$$F = [0 \quad -1 \quad 0 \quad 1 \quad 0 \quad -1]^t$$

Then we substitute T and F in (8) and solve the result equation and get

$$g^{(0)}(0) = 1,\ g^{(1)}(0) = 0,\ g^{(2)}(0) = 1,\ g^{(3)}(0) = -2,\ g^{(4)}(0) = 1,\ g^{(5)}(0) = 0$$

Substituting these values in (11) we have the approximate solution

$$g(x) \cong 1 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{24}x^4 \qquad (13)$$

It is known from[3] (p. 508) that the exact solution of (12) is

$$g(x) = \sin x + e^{-x} \qquad (14)$$

Consequently, we observe that the obtained solution (13) is the first four terms in the Taylor expansion of the exact solution (14).

Example 3. Let us apply the method to the Fredholm integral equation of the first kind

$$-\frac{1}{3}x - \frac{1}{4} + \lambda \int_0^1 (x + y)\, g(y)\, dy = 0 \qquad (15)$$

Let c = 0 and N = 2. Then if we substitate the matrices T (obtained from (9) and F in (8), we have

$$\lambda \begin{bmatrix} 1/2 & 1/3 & 1/8 \\ 1 & 1/2 & 1/6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} g^{(0)}(0) \\ g^{(1)}(0) \\ g^{(2)}(0) \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/3 \\ 0 \end{bmatrix}$$

The solution of this equation is

$$g^{(0)}(0) = -\frac{1}{6\lambda} + \frac{1}{12}A,\quad g^{(1)}(0) = \frac{1}{\lambda} - \frac{1}{2}A,\quad g^{(2)}(0) = A$$

where A is an arbitrary constant. Thus the solution of (15) becomes

$$g(x) = \left(-\frac{1}{6\lambda} + \frac{1}{12}A\right) + \left(\frac{1}{\lambda} - \frac{1}{2}A\right)x + \frac{1}{2}A\,x^2,\ \lambda \neq 0$$

Example 4. Solve the symmetric integral equation

$$g(x) = x^2 + 1 + \frac{3}{2}\int_{-1}^{1} (xy + x^2y^2)\, g(y)\, dy$$

Taking c = 0, N = 2 and following the similar way, we find the values

$$g^{(0)}(0) = 1,\quad g^{(1)}(0) = A,\quad g^{(2)}(0) = 10$$

where A is an arbitrary constant. Thus the required solution is

$$g(x) = 1 + Ax + 5x^2$$

which is the exact solution[2] (p. 154).

Example 5. Let us consider the homogen integral equation

$$g(x) = \lambda \int_0^1 (1-3xy)\, g(y)\, dy \tag{16}$$

For c = 0 and N = 2, the matrix equation associated with (16) becomes

$$\begin{bmatrix} \lambda-1 & \frac{\lambda}{2} & \frac{\lambda}{6} \\ -\frac{3}{2} & -\lambda-1 & -\frac{3}{8}\lambda \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} g^{(0)}(0) \\ g^{(1)}(0) \\ g^{(2)}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

Here $D(\lambda) = (\lambda^2 - 4)/4$ and the eigenvalues are $\lambda = \pm 2$. Therefore, (16) will have a unique solution if and only if $\lambda \neq \pm 2$. Then (16) has only the trivial solution. For $\lambda = 2$, (17) gives

$$g^{(0)}(0) = k, \quad g^{(1)}(0) = -k, \quad g^{(2)}(0) = 0$$

where k is an arbitrary constant. Then the eigenfunction is $g(x) = k(1-x)$. Similarly, for $\lambda = -2$, the corresponding eigenfunction is $g(x) = k(1-3x)$. These results agree with the results given by[2] (p. 22).

CONCLUSIONS

The method presented in this study is useful in finding approximate and also exact solutions of certain Fredholm integral equations (as demostrated examples).

The method gives higher convergence speed than those of the picard type method, provided that the trancation scheme of series is appropriate.

The Taylor series approximation is more effective for polynomial approximation of a function and differs little from the best polynomial approximation of the degree in many cases of practical interest. Since its derivative or integral can be computed easily, Taylor series seems to be useful for obtaining numerical solutions of integral equations.

Integral equations are usually difficult to solve analytically. In particular, singular integral equations are more difficult, but may be solved approximately. Special techniques are required for its solution (Ex. 2).

The integral (7) is a fundamental relation to establish the matrix equation (8) associated with integral equations. In the case of singular integral equations if (7) is a convergence improper integral, the method may be used (Ex. 2).

In the matrix equation (8). If $D(\lambda) = |T| \neq 0$, the inhomogenous integral equation (2) has one and only one solution (Exs. 1,2); if $D(\lambda) = |T| = 0$, then the integral equation either is insoluble or has an infinite number of solution (Exs. 3, 4, 5).

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