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THE VECTORS WHICH FORM CONSTANT ANGLES WITH THE FRENET VECTORS

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SUMMARY

In this work, we first, give the following proposition; if the first Frenet Vectors of a curve in E^5 form a constant angle with the direction of a vector E, then

$$\left[\frac{1}{t_{45}}\left[\frac{t_{12}t_{34}}{t_{23}}+\left[\frac{1}{t_{34}}\left(\frac{t_{12}}{t_{23}}\right)'\right]'\right]\right]'+\frac{t_{45}}{t_{34}}\left(\frac{t_{12}}{t_{23}}\right)'=0$$

and conversly, if this relation is fulfilled, then the first Frenet Vectors of the curve form a constant angle with the direction of some vector, where t_{ij}, $1 \leqslant i \leqslant 4$, $2 \leqslant j \leqslant 5$, are the higher curvatures of the curve. Further, we may write this vector and the angle as the following;

$$E = X_1 + \frac{t_{12}}{t_{23}} X_3 + \frac{1}{t_{34}}\left(\frac{t_{12}}{t_{23}}\right)' X_4 + \frac{1}{t_{45}}\left[\frac{t_{12}t_{34}}{t_{23}} + \left[\frac{1}{t_{34}}\left(\frac{t_{12}}{t_{23}}\right)'\right]'\right] X_5$$

$$Cos\theta = \frac{1}{|E|} = constant$$

where θ is the agnle between X_1 and E.

Using the fifth Frenet vectors, we give a similar proposition.

In the special case we present some useful examples.

ÖZET

Bu makalede ilk olarak, aşağıdaki önermeyi verdik.

E^5 de bir eğrinin birinci Frenet Vectörleri bir E vectörü ile sabit bir açı yapıyorsa

$$\left[\frac{1}{t_{45}}\left[\frac{t_{12}t_{34}}{t_{23}}+\left[\frac{1}{t_{34}}\left(\frac{t_{12}}{t_{34}}\right)'\right]\right]\right]'+\frac{t_{45}}{t_{34}}\left(\frac{t_{12}}{t_{23}}\right)'=0 \text{ olur.}$$

Karşıt olarak, bu bağıntı gerçeklendiğinde, bu eğrinin birinci Frenet Vectörleri bir vektör yönü ile sabit bir açı yapar. Bundan ziyade, bu vektörü ve açıyı,

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$$E = X_1 + \frac{t_{12}}{t_{23}} X_3 + \frac{1}{t_{34}} (\frac{t_{12}}{t_{23}})' X_4 + \frac{1}{t_{45}} [\frac{t_{12} t_{34}}{t_{23}} + \frac{1}{t_{34}} (\frac{t_{12}}{t_{23}})']'] X_5$$

$$Cos\ \theta = \frac{1}{|E|} = constant$$

biçiminde yazabiliriz. Burada θ, X ve E arasındaki açıdır.

Beşinci Frenet vektörlerini kullanarak benzer bir önerme sunduk.

Özel durumda yararlı bazı örnekler sunduk.

0. INTRODUCTION

We first, give a proposition of the expression of a tanget vector to E^n. Our notation and terminology may be found in[1] and[2]

In the theory of Differential Geometry, the concept of higher curvatures of curves in Euclidean Space was given by GLUCK[3] and[4]. Recently, we use the Higher curvatures in our studies of many branches of Differential Geometry[4] and[5].

The purpose of this manuscript, is to express some preliminaries about Diferential Geometry, and show the basic properties of the vector which forms a constant angle with the direction of a Frenet Vector.

1. PRELIMINARIES

PROPOSITION 1.1. Let $e_1, e_2, ..., e_n$ be a frame at a point P of E^n. If V is any tangent vector to E^n at P, then

$$V = \sum_{i=1}^{n} \langle V, e_i \rangle e_i$$

where $\langle , \rangle$ denotes the inner product (dot product). A more detailed discussion of this proposition may be found in[1] and[2].

PROPOSITION 1.2. Let X_1, X_2, X_3, X_4, X_5 be the positive oriented orthonormal frame at each point of a curve a in E^5, where

$$X_1 = \alpha_\star (\frac{\partial}{\partial s}), \text{ and } \frac{dX_1}{ds} = dX_1 (\frac{\partial}{\partial s}) \neq 0.$$

Then, we have the Frenet Formulas

$$X'_i (s) = - t_{i-1} (s) X_{i-1} (s) + t_i (s) X_{i+1} (s), \quad 2 \leqslant i \leqslant 4$$

$$X'_5 (s) = - t_{45} (s) X_4 (s)$$

or

$$\begin{bmatrix} X'_1 \\ X'_2 \\ X'_3 \\ X'_4 \\ X'_5 \end{bmatrix} = \begin{bmatrix} 0 & t_{12} & 0 & 0 & 0 \\ -t_{12} & 0 & t_{23} & 0 & 0 \\ 0 & -t_{23} & 0 & t_{34} & 0 \\ 0 & 0 & -t_{34} & 0 & t_{45} \\ 0 & 0 & 0 & -t_{45} & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix}$$

where $t_{ij} : S \rightarrow IR$. A detailed knowledge of this proposition may be found in[3].

DEFINITION 1.3. Using the above notation, the coefficients t_{ij} are called the higher curvatures of the curve α in E^5 [3]..

2. THE MAIN RESULTS

PROPOSITION 2.1. If the first principal vectors of a curve form a constant angle with the direction of a vector E, then

$$[\frac{1}{t_{45}}[\frac{t_{12}t_{34}}{t_{23}}+[\frac{1}{t_{34}}(\frac{t_{12}}{t_{23}})']']]'+\frac{t_{45}}{t_{34}}(\frac{t_{12}}{t_{23}})'=0$$

and vonversly, if this relation is fulfilled, then the first principal vectors of the curve form a constant angle with the direction of some vector. Further, we may write this vector and the angle as the following,

$$E=X_1+\frac{t_{12}}{t_{23}}X_3+\frac{1}{t_{34}}(\frac{t_{12}}{t_{23}})'X_4+\frac{1}{t_{45}}[\frac{t_{12}t_{34}}{t_{23}}+[\frac{1}{t_{34}}(\frac{t_{12}}{t_{23}})']']X_5$$

$$\text{Cos}\,\theta=\frac{1}{|E|}=\text{Constant}$$

where θ is the angle between X_1 and E.

PROOF. We may write

$$<E, X_1>=C$$

where C is a real number. By differentiating, we have

$$t_{12}<E, X_2>=0$$

or

$$<E, X_2>=0$$

In the same pay, we obtain

$$-t_{12}<E, X_1>+t_{23}<E, X_3>=0$$

or

$$<E, X_3>=C\,.\frac{t_{12}}{t_{23}}$$

Differentiating again, we have

$$-t_{23}<E, X_2>+t_{34}<E, X_4>=C.(\frac{t_{12}}{t_{23}})'$$

or

$$<E, X_4>=C\,.\frac{1}{t_{34}}.(\frac{t_{12}}{t_{23}})'$$

Differentiating once again, we have

$$-t_{23}<E, X_3>+t_{45}<E, X_5>=C.[\frac{1}{t_{34}}(\frac{t_{12}}{t_{23}})']'$$

or

$$< E, X_5 > = C \cdot \frac{1}{t_{45}} [\frac{t_{12}t_{34}}{t_{23}} + [\frac{1}{t_{34}} (\frac{t_{12}}{t_{23}})']']$$

Finally, in the same way, we obtain

$$[\frac{1}{t_{45}} [\frac{t_{12}t_{34}}{t_{23}} + [\frac{1}{t_{34}} (\frac{t_{12}}{t_{23}})']']]' + \frac{t_{45}}{t_{34}} (\frac{t_{12}}{t_{23}})' = 0$$

Conversly, if the relation is held, then this vector is constant. The constant vector E forms with the vector X_1 an angle whose cosine equals $1/|E|$ = constant. Without loss generality, we may assume that $C = 1$. Hence, we have proved the proposition.

PROPOSITION 2.2. If the fifth Frenet Vectors of a curve in E^5 form a constant angle with the direction of a vector $\widetilde{E}$, then

$$[\frac{1}{t_{12}} [\frac{t_{23}t_{45}}{t_{34}} + [\frac{1}{t_{23}} (\frac{t_{45}}{t_{34}})']']]' + \frac{t_{12}}{t_{23}} (\frac{t_{45}}{t_{34}})' = 0$$

and conversly, if this relation is fulfilled, then the fifth Frenet Vectors of the curve form a constant angle with the direction of some vector. Further, we may express this vector and the angle as the following,

$$E = \frac{1}{t_{12}} [\frac{t_{23}t_{45}}{t_{34}} + [\frac{1}{t_{23}} (\frac{t_{45}}{t_{34}})']'] X_1 - \frac{1}{t_{23}} (\frac{t_{45}}{t_{34}})' X_2 + \frac{t_{45}}{t_{34}} X_3 + X_5$$

$$\text{Cos}\, \widetilde{\theta} = \frac{1}{|\widetilde{E}|} = \text{constant}$$

where $\widetilde{\theta}$ is the angle between X_5 and $\widetilde{E}$.

PROOF. Consider,

$$< \widetilde{E}, X_5 > = C$$

where C is a real number. Hence, we have

$$-t_{45} < \widetilde{E}, X_5 > = 0$$

or

$$< \widetilde{E}, X_4 > = 0$$

Differentiating, we have

$$-t_{34} < \widetilde{E}, X_3 > + t_{45} < \widetilde{E}, X_5 > = 0$$

$$< \widetilde{E}, X_3 > = C \cdot \frac{t_{45}}{t_{34}}$$

Differentiating again, we obtain

$$-t_{23} < E, X_2 > + t_{34} < \widetilde{E}, X_4 > = C (\frac{t_{45}}{t_{34}})'$$

or

$$<\widetilde{E}, X_2> = -C \cdot \frac{1}{t_{23}} \left(\frac{t_{45}}{t_{34}}\right)'$$

Differentiating again, we obtain

$$-t_{12} <\widetilde{E}, X_1> + t_{23} <\widetilde{E}, X_3> = -C \cdot \left[\frac{1}{t_{23}} \left(\frac{t_{45}}{t_{34}}\right)'\right]'$$

$$<\widetilde{E}, X_1> = \frac{C}{t_{12}} \left[\frac{t_{23} t_{45}}{t_{34}} + \left[\frac{1}{t_{23}} \left(\frac{t_{45}}{t_{34}}\right)'\right]'\right]$$

Differentiating once again, we obtain

$$t_{12} <\widetilde{E}, X_2> = C \left[\frac{1}{t_{12}} \left[\frac{t_{23} t_{45}}{t_{34}} + \left[\frac{1}{t_{23}} \left(\frac{t_{45}}{t_{34}}\right)'\right]'\right]\right]'$$

or

$$\left[\frac{1}{t_{12}} \left[\frac{t_{23} t_{45}}{t_{34}} + \left[\frac{1}{t_{23}} \left(\frac{t_{45}}{t_{34}}\right)'\right]'\right]\right]' + \frac{t_{12}}{t_{23}} \left(\frac{t_{45}}{t_{34}}\right)' = 0$$

Conversly, if the relation is held, then this vector is constant. This constant vector E forms with the vector X_5 an angle whose cosine equals $1/|E| =$ constant. Without loss generality, we may assume that $C = 1$. These results complete the proof of our proposition.

We can say similar results for the Frenet Vectors X_2, X_3, X_4.

It is clear that the results which we have found above may be written again using the higher curvatures of curves in Euclidean Space E^n.

In the special case we have the following results:

If the first Frenet Vectors of a curve in E^3 form a constant angle with the direction of a vector e, then

$$\left(\frac{\tau}{\kappa}\right)' = 0$$

and conversly, if this relation is fulfilled, then the first Frenet Vectors of the curve form a constant angle with the direction of some vector. Further, we may express this vector and the angle as following,

$$e = X_1 + \frac{\kappa}{\tau} X_3$$

$$\mathrm{Cos}\,\theta = \frac{1}{|e|}$$

where θ is the angle between X_1 and e

If the principal normals of a curve form a constant angle with the direction of a vector $\widetilde{e}$, then

$$\left[\frac{\kappa^2 + \tau^2}{\kappa \left(\frac{\tau}{\kappa}\right)'}\right]' + \tau = 0$$

conversly, if this relation is fulfilled, then the principal normals of the curve form a constant angle with the direction of some vector. We can express this vector by

$$\tilde{e} = \frac{\tau}{\kappa^2} \frac{\kappa^2 + \tau^2}{(\frac{\tau}{\kappa})'} X_1 + X_2 + \frac{1}{\kappa} \frac{\kappa^2 + \tau^2}{(\frac{\tau}{\kappa})'} X_3$$

This constant vector $\tilde{e}$ forms with the vector X_2 an angle whose cosine equals $1/|\tilde{e}| =$ constant.

If the binormals of a curve form a constant angle with the direction of a vector $\approx{e}$, then

$$(\frac{\tau}{\kappa})' = 0$$

conversly, if this relation is fulfilled, then the binormals of the curve form a constant angle with the direction of some vector. We can express this vector by

$$\approx{e} = \frac{\tau}{\kappa} X_1 + X_3$$

This constant vector $\approx{e}$ forms with the vector X_3 an angle whose cosine equals $1/|\approx{e}| =$ constant.

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