THE ROLE OF SAMPLING IN ESTIMATING THE PARAMETERS OF CISOIDS

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ABSTRACT

For a single complex sinusoid (cisoid) in complex white Gaussian noise the dependence of the Cramér-Rao (C-R) bounds on the first sampling time is known. In this paper the first-sampling-time dependence of the bounds is examined for the two-cisoid case. For two cisoids in complex white Gaussian noise it is shown that the largest and the smallest values of the C-R frequency and amplitude bounds do not depend on the first sampling time and that the critical values of the C-R phase bounds are smallest when the sampling times are symmetrical.

ÖZET

Kompleks Sinüslerin Parametrelerinin Kestiriminde Örneklemenin Rolü

Kompleks beyaz Gauss gürültü içindeki bir kompleks sinüs için Cramér-Rao (C-R) sınırlarının ilk örnekleme zamanına bağlılığı bilinmektedir. Bu makalede sınırların ilk örnekleme zamanına bağlılığı iki kompleks sinüs durumu için incelenmiştir. Kompleks beyaz Gauss gürültü içindeki iki kompleks sinüs için C-R frekans ve genlik sınırlarının en büyük ve en küçük değerlerinin ilk örnekleme zamanına bağlı olmadığı ve C-R faz sınırlarının kritik değerlerinin örnekleme zamanları simetrik olduğunda en az olacağı gösterilmiştir.

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1. INTRODUCTION

It is well known that the Cramér-Rao (C-R) bound gives a lower bound on the variance of any unbiased estimator[1]. Therefore, the C-R bound is frequently used for testing the performance of parameter estimators for time-series data models.

The C-R bound expressions for the amplitudes, the phases and the frequencies of superimposed cisoids in complex white Gaussian noise were derived by Rife and Boorstyn[2,3]. For the single cisoid case, it is known that the C-R frequency bound and the C-R amplitude bound are independent of the first sampling time while the C-R phase bound attains its smallest value when the first sampling time is such that the sampling instants are symmetrical[2]. For the multiple cisoid case, due to the complexity of the C-R bound expressions, the dependence of the bounds on several parameters of interest including the first sampling time typically is studied numerically rather than analytically[3].

Recently, the author has derived explicit expressions for the C-R bounds for the two-cisoid case[4]. This paper builds upon the results of Ref. 4 to study the dependence of the C-R bounds on the first sampling time for the two-cisoid case. It is shown that while the bounds depend on the first sampling time in general, the maximum and the minimum values of the frequency and the amplitude bounds do not and that the critical values of the phase bounds are smallest for the symmetric sampling case.

2. THE C-R BOUNDS

A. Single Cisoid Case: We first consider the case in which the data consist of a single cisoid in noise:

$$y(t) = \alpha_0 \exp [j(\omega_0 t + \varphi_0)] + e(t),\ t = n,\dots,n+N-1. \qquad (1)$$

Here α_0 is the amplitude, ω_0 is the frequency, φ_0 is the phase of the cisoid, e(t) represents a zero-mean complex white Gaussian noise with variance σ^2, n is the first value of the sampling time index t, and N is the total number of available data samples. If the number of data samples N is odd and the first sampling time index, $n = -(N-1)/2$, then the sampling is symmetric.

Let $\hat{\alpha}_0$, $\hat{\varphi}_0$, and $\hat{\omega}_0$ be unbiased estimators of the cisoid parameters α_0, φ_0, and ω_0, respectively. Then the variance of the estimators satisfies the C-R theorem[1]:

$$var(\hat{\alpha}_0) \geq B_{\alpha_0},\ var(\hat{\varphi}_0) \geq B_{\varphi_0} \text{ and } var(\hat{\omega}_0) \geq B_{\omega_0}$$

where the bounds B_{α_0}, B_{φ_0} and B_{ω_0} are given by[2]

$$B_{\alpha_0} = \frac{1}{2\cdot(1/\sigma^2)\cdot N} \qquad (2)$$

$$B_{\varphi_0} = \frac{1}{2 \cdot SNR_0 \cdot N} \frac{\Gamma_2}{\Gamma_0 \Gamma_2 - \Gamma_1^2} \quad (3)$$

$$B_{\omega_0} = \frac{1}{2 \cdot SNR_0 \cdot N^3} \frac{\Gamma_0}{\Gamma_0 \Gamma_2 - \Gamma_1^2} \quad (4)$$

Here SNR_0 denotes the signal-to-noise ratio

$$SNR_0 = \frac{\alpha_0^2}{\sigma^2}$$

and, for r = 0, 1, 2,

$$\Gamma_r = \frac{1}{N^{r+1}} \sum_{t=n}^{n+N-1} t^r \quad (5)$$

B. Two Cisoid Case: In this case the data consist of two cisoids in a zero-mean complex white Gaussian noise:

$$y(t) = \sum_{i=1}^{2} \alpha_i \exp[j(\omega_i t + \varphi_i)] + e(t), \; t=n, \ldots, n+N-1. \quad (6)$$

If $\hat{\alpha}_i$, $\hat{\varphi}_i$ and $\hat{\omega}_i$ are unbiased estimators of the signal parameters, respectively, α_i, φ_i and ω_i, $i = 1, 2$, then the variance of the estimates satisfies

$$var(\hat{\alpha}_i) \geq B_{\alpha_i}, \; var(\hat{\varphi}_i) \geq B_{\varphi_i} \text{ and } var(\hat{\omega}_i) \geq B_{\omega_i}$$

The bounds B_{α_i}, B_{φ_i} and B_{ω_i} depend on the frequencies and the phases of the cisoids through the frequency difference $\delta\omega_i = \omega_1 - \omega_2$ and the phase difference $\delta\varphi_i = \varphi_1 - \varphi_2$. They are given by

$$B_{\alpha_i} = \frac{1}{2 . (1/\sigma^2) \cdot N} \frac{1}{\Delta_1} \left(1 + \frac{Y_0 + Y_C \cos(2\delta\varphi) + Y_S \sin(2\delta\varphi)}{X_0 + X_C \cos(2\delta\varphi) + X_S \sin(2\delta\varphi)}\right) \quad (7)$$

$$B_{\varphi_i} = \frac{1}{2 \cdot SNR_i \cdot N} \frac{1}{\Delta_1} \left(1 + \frac{Z_0 + Z_C \cos(2\delta\varphi) + Z_S \sin(2\delta\varphi)}{X_0 + X_C \cos(2\delta\varphi) + X_S \sin(2\delta\varphi)}\right) \quad (8)$$

$$B_{\omega_i} = \frac{1}{2 \cdot SNR_i \cdot N^3} \frac{2 \,.\, \Delta_1 \,.\, K_0}{X_0 + X_C \cos(2\delta\varphi) + X_S \sin(2\delta\varphi)} \tag{9}$$

Here SNR_i denotes the signal-to-noise ratio for the ith cisoid

$$\mathrm{SNR}_i = \frac{\alpha_i^2}{\sigma^2}$$

and

$$X_0 = 2K_0^2 - K_C^2 - K_S^2 \tag{10}$$

$$X_\mathrm{C} = K_S^2 - K_C^2 \tag{11}$$

$$X_\mathrm{S} = -2K_C K_S \tag{12}$$

$$Y_0 = (\mathrm{U}^2 + V^2 + 2Y^2)K_0 + (2\mathrm{UY})K_C + (2\mathrm{VY})K_S$$

$$Y_\mathrm{C} = (\mathrm{U}^2 - V^2)K_0 + (2\mathrm{UY})K_C - (2\mathrm{VY})K_S$$

$$Y_\mathrm{S} = (2\mathrm{UV})K_0 + (2\mathrm{VY})K_C + (2\mathrm{UY})K_S$$

$$Z_0 = (\mathrm{U}^2 + \mathrm{V}^2 + 2\mathrm{Z}^2)K_0 + (2\mathrm{VZ})K_C - (2\mathrm{UZ})K_S$$

$$Z_\mathrm{C} = -(\mathrm{U}^2 - \mathrm{V}^2)K_0 + (2\mathrm{VZ})K_C + (2\mathrm{UZ})K_S$$

$$Z_\mathrm{S} = -(2\mathrm{UV})K_0 - (2\mathrm{UZ})K_C + (2\mathrm{VZ})K_S$$

$$K_0 = \Gamma_2 \,.\, \Delta_1 - \Gamma_0 \,.\, (\Gamma_1^2 + \mathrm{C}_1^2 + \mathrm{S}_1^2) + 2\Gamma_1(C_0C_1 + S_0S_1)$$

$$K_\mathrm{C} = -\mathrm{C}_2 \,.\, \Delta_1 - \mathrm{C}_0 \,.\, (\Gamma_1^2 + \mathrm{C}_1^2 - \mathrm{S}_1^2) + 2\mathrm{C}_1(\Gamma_0\Gamma_1 - S_0S_1)$$

$$K_\mathrm{S} = S_2 \,.\, \Delta_1 + \mathrm{S}_0(\Gamma_1^2 - \mathrm{C}_1^2 + \mathrm{S}_1^2) - 2\mathrm{S}_1(\Gamma_0\Gamma_1 - C_0C_1)$$

$$\Delta_1 = \Gamma_0^2 - \mathrm{C}_0^2 - \mathrm{S}_0^2$$

$$U = \Gamma_0\mathrm{S}_1 - \Gamma_1\mathrm{S}_0, \qquad V = \Gamma_0\mathrm{C}_1 - \Gamma_1\mathrm{C}_0$$

$$Y = C_0\mathrm{S}_1 - C_1\mathrm{S}_0, \qquad Z = \Gamma_0\Gamma_1 - C_0C_1 - S_0S_1$$

where, for $r = 0, 1, 2$, the Γ_r are given by (5), and

$$C_r = \frac{1}{N^{r+1}} \sum_{t=n}^{n+N-1} t^r \cos(\delta\omega \,.\, t)$$

$$S_r = \frac{1}{N^{r+1}} \sum_{t=n}^{n+N-1} t^r \sin(\delta\omega \,.\, t)$$

The expression in (9) was derived in Ref. 4 and the expressions in (7) and (8) follow from the results in Ref. 4.

Using the explicit bound expressions given in (7)-(9) one can get the following expressions for the maximum and the minimum values of the bounds with respect to the phase difference $\delta\varphi$:

$$(B_{\omega_i})_{\max} = \frac{1}{2 \cdot SNR_i . N^3} \frac{\Delta_1 . K_0}{K_0^2 - K_C^2 - K_S^2} \tag{13}$$

$$(B_{\omega_i})_{\min} = \frac{1}{2 \cdot SNR_i \cdot N^3} \frac{\Delta_1}{K_0} \tag{14}$$

$$(B_{\alpha_i})_{\max} = \frac{1}{2 . (1/\sigma^2) . N} \frac{1}{\Delta_1} \left(1 + \frac{Y_0 \cdot M_0 + Y_C \cdot M_C + Y_S \cdot M_S}{X_0 \cdot M_0 + X_C \cdot M_C + X_S \cdot M_S}\right) \tag{15}$$

$$(B_{\alpha_i})_{\min} = \frac{1}{2 . (1/\sigma^2) . N} \frac{1}{\Delta_1} \left(1 + \frac{Y_0 \cdot m_0 + Y_C \cdot m_C + Y_S \cdot m_S}{X_0 \cdot m_0 + X_C \cdot m_C + X_S \cdot m_S}\right) \tag{16}$$

$$(B_{\varphi_i})_{\max} = \frac{1}{2 . SNR_i . N} \frac{1}{\Delta_1} \left(1 + \frac{Z_0 \cdot M'_0 + Z_C \cdot M'_C + Z_S \cdot M'_S}{X_0 \cdot M'_0 + X_C \cdot M'_C + X_S \cdot M'_S}\right) \tag{17}$$

$$(B_{\varphi_i})_{\min} = \frac{1}{2 . SNR_i . N} \frac{1}{\Delta_1} \left(1 + \frac{Z_0 \cdot m'_0 + Z_C \cdot m'_C + Z_S \cdot m'_S}{X_0 \cdot m'_0 + X_C \cdot m'_C + X_S \cdot m'_S}\right) \tag{18}$$

where

$M_0 = A^2 + B^2, M_C = BC - A\sqrt{A^2 + B^2 - C^2},\ M_S = AC + B\sqrt{A^2 + B^2 - C^2}$

$m_0 = A^2 + B^2, m_C = BC + A\sqrt{A^2 + B^2 - C^2},\ m_S = AC - B\sqrt{A^2 + B^2 - C^2}$

$A = X_C Y_O - X_O Y_C,\quad B = X_O Y_S - X_S Y_O,\quad C = X_S Y_C - X_C Y_S,$

and

$$M'_0 = A'^2 + B'^2, M'_C = B'C' - A'\sqrt{A'^2 + B'^2 - C'^2}, \quad M'_S = A'C' + B'\sqrt{A'^2 + B'^2 - C'^2}$$
$$m'_0 = A'^2 + B'^2, m'_C = B'C' + A'\sqrt{A'^2 + B'^2 - C'^2}, \quad m'_S = A'C' - B'\sqrt{A'^2 + B'^2 - C'^2}$$
$$A' = X_C Z_O - X_O Z_C, \quad B' = X_O Z_S - X_S Z_O, \quad C' = X_S Z_C - X_C Z_S,$$

3. THE DEPENDENCE OF THE BOUNDS ON THE FIRST SAMPLING TIME

A. Single Cisoid Case: For this case we will show that the C-R amplitude bound B_{α_0} and the C-R frequency bound B_{ω_0} do not depend on the first sampling time index n and that the C-R phase bound B_{φ_0} is smallest when the sampling is symmetric.

Proposition 1: The C-R amplitude bound B_{α_0} is independent of the first sampling time index n.

Proof: The proof is immediate from (2).

Lemma 1: For $n_1 > n_2$, let the notation $(\cdot)_q$ denote the quantity in parentheses for $n = n_q$, , q = 1, 2. Let

$$v = \frac{(n_1 - n_2)}{N}, \; c = \cos(\delta\omega(n_1 - n_2)) \text{ and } s = \sin(\delta\omega(n_1 - n_2)).$$

We then have

$$\Gamma_0 = 1 \quad \text{for all n} \tag{19}$$
$$(\Gamma_1)_1 = (\Gamma_1)_2 + v \tag{20}$$
$$(\Gamma_2)_1 = (\Gamma_2)_2 + 2v(\Gamma_1)_2 + v^2 \tag{21}$$
$$(C_0)_1 = c(C_0)_2 + s(S_0)_2$$
$$(C_1)_1 = c(C_1)_2 + s(S_1)_2 - vc(C_0)_2 - vs(S_0)_2$$
$$(C_2)_1 = c(C_2)_2 + s(S_2)_2 - 2vc(C_1)_2 - 2vs(S_1)_2 + v^2c(C_0)_2 + v^2s(S_0)_2$$
$$(S_0)_1 = c(S_0)_2 - s(C_0)_2$$
$$(S_1)_1 = c(S_1)_2 - s(C_1)_2 - vc(S_0)_2 + vs(C_0)_2$$
$$(S_2)_1 = c(S_2)_2 - s(C_2)_2 - 2vc(S_1)_2 + 2vs(C_1)_2 + v^2c(S_0)_2 - v^2s(C_0)_2$$

Proof: The lemma follows from straight forward calculations.

Proposition 2: The C-R frequency bound B_{ω_0} is independent of the first sampling time index n.

Proof: The proposition follows from substitution of (19)-(21) into (4).

Proposition 3: The C-R phase bound B_{φ_0} is smallest for the symmetric sampling case.

Proof: For the symmetric sampling case it follows from (5) that $\Gamma_1 = 0$ so that (3) becomes.

$$\left(B_{\varphi_0}\right)_{sym} = \frac{1}{2 \cdot SNR_0 \cdot N}$$

Comparing this with (3) and noting that $\Gamma_2 / \left(\Gamma_0 \Gamma_2 - \Gamma_1^2\right) \geq 1$ with the equality holding if and only if $\Gamma_1 = 0$, the proposition follows.

B. Two Cisoid Case: For this case we will show that the maximum and the minimum values of the C-R amplitude and frequency bounds B_{α_i} and B_{ω_i} do not change with n and that the maximum and minimum values of the C-R phase bound B_{φ_i} are smallest for the symmetric sampling case.

Proposition 4: The critical values of the C-R frequency bound $\left(B_{\omega_i}\right)_{max}$ and $\left(B_{\omega_i}\right)_{min}$ are independent of the first sampling time index n.

Proof: From Lemma 1 we get

$$\left(\Delta_1\right)_1 = \left(\Delta_1\right)_2 \tag{22}$$

$$\left(K_0\right)_1 = \left(K_0\right)_2 \tag{23}$$

$$\left(K_C\right)_1 = c\left(K_C\right)_2 + s\left(K_S\right)_2 \tag{24}$$

$$\left(K_S\right)_1 = c\left(K_S\right)_2 - s\left(K_C\right)_2 \tag{25}$$

The proposition now follows from substitution of (22)-(25) into (13) and (14).

Proposition 5: The critical values of the C-R amplitude bounds $\left(B_{\alpha_i}\right)_{max}$ and $\left(B_{\alpha_i}\right)_{min}$ are independent of the first sampling time index n.

Proof: From (23)-(25) we have

$$\left(X_0\right)_1 = \left(X_0\right)_2 \tag{26}$$

$$\left(X_C\right)_1 = \left(c^2 - s^2\right)\left(X_C\right)_2 + 2cs\left(X_S\right)_2 \tag{27}$$

$$\left(X_S\right)_1 = \left(c^2 - s^2\right)\left(X_S\right)_2 - 2cs\left(X_C\right)_2 \tag{28}$$

From Lemma 1 and (23)-(25) we have

$$(Y_0)_1 = (Y_0)_2 \tag{29}$$

$$(Y_C)_1 = (c^2 - s^2)(Y_C)_2 + 2cs(Y_S)_2 \tag{30}$$

$$(Y_S)_1 = (c^2 - s^2)(Y_S)_2 - 2cs(Y_C)_2 \tag{31}$$

Now, (26)-(31) give

$$(M_0)_1 = (M_0)_2 \tag{32}$$

$$(M_C)_1 = (c^2 - s^2)(M_C)_2 + 2cs(M_S)_2 \tag{33}$$

$$(M_S)_1 = (c^2 - s^2)(M_S)_2 - 2cs(M_C)_2 \tag{34}$$

$$(m_0)_1 = (m_0)_2 \tag{35}$$

$$(m_C)_1 = (c^2 - s^2)(m_C)_2 + 2cs(m_S)_2 \tag{36}$$

$$(m_S)_1 = (c^2 - s^2)(m_S)_2 - 2cs(m_C)_2 \tag{37}$$

The proposition now follows from substitution of (22) and (26)-(37) into (15) and (16).

Proposition 6: The critical values of the C-R phase bound $(B_{\varphi_i})_{\max}$ and $(B_{\varphi_i})_{\min}$ are smallest for the symmetric sampling case.

Proof: It is easy to show that for the symmetric sampling case $(B_{\varphi_i})_{\min}$ is given by

$$(B_{\varphi_i})_{\min,sym} = \frac{1}{2 \cdot SNR_i \cdot N} \frac{1}{\Delta_1} \tag{38}$$

Note from (38) that

$$\Delta_1 > 0 \tag{39}$$

Now (39) and (14) imply that

$$K_0 > 0 \tag{40}$$

From (39), (40) and (13) we have

$$K_0^2 - K_C^2 - K_S^2 > 0 \tag{41}$$

and from (39), (40) and (9) we have

$$X_0 + X_C \cos(2\delta\varphi) + X_S \sin(2\delta\varphi) > 0 \tag{42}$$

Comparison of (8) with (38) and use of (39) and (42) show that

$$B_{\varphi_i} \geq \left(B_{\varphi_i}\right)_{\min,sym}$$

if we can prove that

$$Z_0 + Z_C \cos(2\delta\varphi) + Z_S \sin(2\delta\varphi) \geq 0 \tag{43}$$

Now note that

$$Z_0 + Z_C \cos(2\delta\varphi) + Z_S \sin(2\delta\varphi) \geq Z_0 - \sqrt{Z_C^2 + Z_S^2}$$

From (10)-(12) we have

$$Z_0^2 - Z_C^2 - Z_S^2 = 4Z^2\left[\left(U^2 + V^2\right)\left(K_0^2 - K_C^2 - K_S^2\right) + \left(ZK_0 + VK_C - UK_S\right)^2\right] \tag{44}$$

Use of (41) in (44) shows that $Z_0^2 - Z_C^2 - Z_S^2 \geq 0$. This and the fact (which is not shown here) that $Z_0 \geq 0$ give (43).

To prove

$$\left(B_{\varphi_i}\right)_{\max} \geq \left(B_{\varphi_i}\right)_{\max,sym}$$

we show that

$$\left(B_{\varphi_i}\right)_{\max} - \left(B_{\varphi_i}\right)_{\min} \geq \left(B_{\varphi_i}\right)_{\max,sym} - \left(B_{\varphi_i}\right)_{\min,sym}$$

From (17) and (18) we get

$$\left(B_{\varphi_i}\right)_{\max} - \left(B_{\varphi_i}\right)_{\min} = \frac{1}{2 \cdot SNR_i \cdot N} \frac{1}{\Delta_1} \frac{2\sqrt{A'^2 + B'^2 - C'^2}}{\left(X_0^2 - X_C^2 - X_S^2\right)}$$

We have from (39) and (42) that $\Delta_1 > 0$ and $\left(X_0^2 - X_C^2 - X_S^2\right) > 0$ and from (22) and (26)-(28) that Δ_1 and $\left(X_0^2 - X_C^2 - X_S^2\right)$ do not depend on *n*. Thus, it remains to show that $\sqrt{A'^2 + B'^2 - C'^2}$ is smallest for the symmetric sampling case.

Assume now that *N* is odd and $n_2 = -(N-1)/2$. Straight forward calculation gives

$$\sqrt{(A')_1^2 + (B')_1^2 - (C')_1^2} = (A')_2 + 2v^2\left(\Delta_1\right)_2^2\left(K_0\right)_2\left(K_C\right)_2^2 \tag{45}$$

Since $(A')_2 \geq 0$, $(B')_2 = 0$, $(C')_2 = 0$; $(A')_2$ is just $\sqrt{(A')_2^2 + (B')_2^2 - (C')_2^2}$ and (40) shows that the second term on the right-hand-side of (45) is nonnegative. This completes the proof.

4. CONCLUSIONS

We have studied the dependence of the C-R bounds on the first sampling time index *n* for the time-series data models consisting of one or two cisoids in complex white Gaussian noise. For the single cisod case we have shown that the frequency and the amplitude bounds do not depend on *n* and the phase bound is smallest for the symmetric sampling case. For the two cisoid case we have shown that while the bounds depend on *n* in general, the critical values of the frequency and amplitude bounds do not and the critical values of the phase bounds are smallest for the symmetric sampling case.

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