CONTACT CR-WARPED PRODUCT
SUBMANIFOLDS IN KENMOTSU SPACE FORMS

KADRI ARSLAN, RIDVAN EZENTAS, ION MIHAI,
AND CENGIZHAN MURATHAN

ABSTRACT. Recently, Chen studied warped products which are
CR-submanifolds in Kaehler manifolds and established general sharp
inequalities for CR-warped products in Kaehler manifolds. In the
present paper, we obtain sharp estimates for the squared norm of
the second fundamental form (an extrinsic invariant) in terms of the
warping function for contact CR-warped products isometrically im-
mersed in Kenmotsu space forms. The equality case is considered.
Some applications are derived.

1. Introduction

Let $\widetilde{M}$ be a Hermitian manifold and denote by J the canonical almost
complex structure on $\widetilde{M}$. According to the behavior of the tangent bun-
dle TM with respect to the action of J, we may distinguish two special
classes of submanifolds M in $\widetilde{M}$.

a) complex submanifolds, i.e., $J(T_pM) = T_pM, \forall p \in M$.

b) totally real submanifolds, i.e., $J(T_pM) \subset T_p^\perp M, \forall p \in M$, where
T_pM (resp. $T_p^\perp M$) is the tangent (resp. the normal) space of M at p.
Such submanifolds were defined and studied by Chen and Ogiue[10].

On the other hand, Yano and Ishihara (see [16]) considered a sub-
manifold M whose tangent bundle TM splits into a complex subbun-
dle $\mathcal{D}$ and a totally real subbundle $\mathcal{D}^\perp$. Later, such a submanifold
was called a CR-submanifold ([3, 4]). Blair and Chen[3] proved that

a CR-submanifold of a locally conformal Kaehler manifold is a Cauchy-Riemann manifold in the sense of Greenfield.

The first main result on CR-submanifolds was obtained by Chen[4]: any CR-submanifold of a Kaehler manifold is foliated by totally real submanifolds (i.e., the totally real subbundle is involutive).

As non-trivial examples of CR-submanifolds, we can mention the (real) hypersurfaces of Hermitian manifolds.

Recently, Chen[7] introduced the notion of a CR-warped product submanifold in a Kaehler manifold. In a series of papers ([5, 7]), he investigated such submanifolds. In particular, he established a sharp relationship between the warping function f of a warped product CR-submanifold $M_1 \times_f M_2$ of a Kaehler manifold $\widetilde{M}$ and the squared norm of the second fundamental form $\|h\|^2$ (see [7]).

For other results on warped product submanifolds in complex space forms we refer to [13].

In the present paper, we study contact CR-warped product submanifolds in Kenmotsu space forms. We prove estimates of the squared norm of the second fundamental form in terms of the warping function. Equality cases are investigated. Obstructions to the existence of contact CR-warped product submanifolds in Kenmotsu space forms are derived.

2. Kenmotsu manifolds and their submanifolds

Tanno[15] has classified, into 3 classes, the connected almost contact Riemannian manifolds whose automorphisms groups have the maximum dimensions:

1. homogeneous normal contact Riemannian manifolds with constant ϕ-holomorphic sectional curvature;

2. global Riemannian products of a line or circle and a Kaehlerian space form;

3. warped product spaces $L \times_f F$, where L is a line and F a Kaehlerian manifold.

Kenmotsu [12] studied the third class and characterized it by tensor equations. Later, such a manifold was called a Kenmotsu manifold.

A $(2m + 1)$-dimensional Riemannian manifold $(\widetilde{M}, g)$ is said to be a Kenmotsu manifold if it admits an endomorphism ϕ of its tangent

bundle $T\tilde{M}$, a vector field ξ and a 1-form η, which satisfy:

$$(1) \quad \begin{cases} \phi^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \\ (\tilde{\nabla}_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \\ \tilde{\nabla}_X \xi = X - \eta(X)\xi, \end{cases}$$

for any vector fields X, Y on $\tilde{M}$, where $\tilde{\nabla}$ denotes the Riemannian connection with respect to g.

We denote by ω the fundamental 2-form of $\tilde{M}$, i.e.,

$$\omega(X, Y) = g(\phi X, Y), \quad \forall X, Y \in \Gamma(T\tilde{M}).$$

It was proved that the pairing (ω, η) defines a locally conformal cosymplectic structure, i.e.,

$$d\omega = 2\omega \wedge \eta, \quad d\eta = 0.$$

A Kenmotsu manifold with constant ϕ-holomorphic sectional curvature c is called a Kenmotsu space form and is denoted by $\tilde{M}(c)$. Then its curvature tensor $\tilde{R}$ is expressed by [12]

$$(2) \quad \begin{aligned} 4\tilde{R}(X, Y)Z = {} & (c-3)\{g(Y, Z)X - g(X, Z)Y\} \\ & + (c+1)[\{\eta(X)Y - \eta(Y)X\}\eta(Z) \\ & + \{g(X, Z)\eta(Y) - g(Y, Z)\eta(X)\}\xi \\ & + \omega(Y, Z)\phi X - \omega(X, Z)\phi Y - 2\omega(X, Y)\phi Z]. \end{aligned}$$

Let $\tilde{M}$ be a Kenmotsu manifold and M an n-dimensional submanifold tangent to ξ. For any vector field X tangent to M, we put

$$(3) \quad \phi X = PX + FX,$$

where PX (resp. FX) denotes the tangential (resp. normal) component of ϕX. Then P is an endomorphism of tangent bundle TM and F is a normal bundle valued 1-form on TM.

The equation of Gauss is given by

$$(4) \quad \begin{aligned} \tilde{R}(X, Y, Z, W) = {} & R(X, Y, Z, W) \\ & + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \end{aligned}$$

for any vectors X, Y, Z, W tangent to M.

We denote by H the mean curvature vector, i.e.,

$$H(p) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i),$$

where $\{e_1, ..., e_n\}$ is an orthonormal basis of the tangent space T_pM, $p \in M$.

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r), \quad i, j = 1, ..., n; \ r = n + 1, ..., 2m + 1,$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

By analogy with submanifolds in a Kaehler manifold, different classes of submanifolds in a Kenmotsu manifold were considered (see, for example, [16]).

A submanifold M tangent to ξ is said to be invariant (resp. anti-invariant) if $\phi(T_pM) \subset T_pM, \forall p \in M$ (resp. $\phi(T_pM) \subset T_p^\perp M, \forall p \in M$).

A submanifold M tangent to ξ is called a contact CR-submanifold [16] if there exists a pair of orthogonal differentiable distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on M, such that:

i) $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \{\xi\}$, where $\{\xi\}$ is the 1-dimensional distribution spanned by ξ;

ii) $\mathcal{D}$ is invariant by ϕ, i.e., $\phi(\mathcal{D}_p) \subset \mathcal{D}_p, \forall p \in M$;

iii) $\mathcal{D}^\perp$ is anti-invariant by ϕ, i.e., $\phi(\mathcal{D}_p^\perp) \subset T_p^\perp M, \forall p \in M$.

In particular, if $\mathcal{D}^\perp = \{0\}$ (resp. $\mathcal{D} = \{0\}$), M is an invariant (resp. anti-invariant) submanifold.

3. Contact CR-warped product submanifolds

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f a positive differentiable function on M_1. The warped product of M_1 and M_2 is the Riemannian manifold

$$M_1 \times_f M_2 = (M_1 \times M_2, g),$$

where $g = g_1 + f^2 g_2$ (see, for instance, [8]).

It is well-known that the notion of warped products plays some important role in Differential Geometry as well as in Physics. For a recent survey on warped products as Riemannian submanifolds, we refer to [6].

We recall the following general formulae on a warped product (see [8])

$$(5) \qquad \nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)Z,$$

for any vector fields X, Z tangent to M_1, M_2, respectively.

If X and Z are unit vector fields, it follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$(6) \quad K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}\{(\nabla_X X)f - X^2 f\}.$$

By reference to [11], a warped product submanifold $M_1 \times_f M_2$ of a Kenmotsu manifold $\widetilde{M}$, with M_1 a $(2\alpha + 1)$-dimensional invariant submanifold tangent to ξ and M_2 a β-dimensional anti-invariant submanifold of $\widetilde{M}$, is said to be a *contact CR-warped product submanifold.*

We state the following estimate of the squared norm of the second fundamental form for contact CR-warped products in Kenmotsu manifolds.

THEOREM 3.1. *Let $\widetilde{M}$ be a $(2m+1)$-dimensional Kenmotsu manifold and $M = M_1 \times_f M_2$ an n-dimensional contact CR-warped product submanifold, such that M_1 is a $(2\alpha + 1)$-dimensional invariant submanifold tangent to ξ and M_2 a β-dimensional anti-invariant submanifold of $\widetilde{M}$. Then*

(i) *The squared norm of the second fundamental form of M satisfies*

$$(7) \qquad\qquad \|h\|^2 \geq 2\beta[\|\nabla(\ln f)\|^2 - 1],$$

where $\nabla(\ln f)$ is the gradient of $\ln f$.

(ii) *If the equality sign of (7) holds identically, then M_1 is a totally geodesic submanifold and M_2 is a totally umbilical submanifold of $\widetilde{M}$.*

Moreover, M is a minimal submanifold of $\widetilde{M}$.

Proof. Let $M = M_1 \times_f M_2$ be a warped product submanifold of a Kenmotsu manifold $\widetilde{M}$, such that M_1 is an invariant submanifold tangent to ξ and M_2 an anti-invariant submanifold of $\widetilde{M}$.

For any unit vector fields X tangent to M_1 and Z, W tangent to M_2 respectively, we have:

$$
\begin{aligned}
g(h(\phi X, Z), \phi Z) &= g(\widetilde{\nabla}_Z \phi X, \phi Z) \\
&= g(\phi \widetilde{\nabla}_Z X, \phi Z) \\
&= g(\widetilde{\nabla}_Z X, Z) \\
&= g(\nabla_Z X, Z) \\
&= X \ln f,
\end{aligned}
$$

$$g(h(\phi X, Z), \phi W) = (X \ln f) g(Z, W).$$

(8)

On the other hand, since the ambient manifold $\widetilde{M}$ is a Kenmotsu manifold, it is easily seen that

$$(9) \qquad\qquad h(\xi, Z) = 0.$$

Obviously, (5) implies $\xi \ln f = 1$. Therefore, by (8) and (9), the inequality (7) follows immediately.

Denote by h'' the second fundamental form of M_2 in M. Then, we get

$$g(h''(Z, W), X) = g(\nabla_Z W, X) = -(X \ln f) g(Z, W),$$

or equivalently

$$(10) \qquad\qquad h''(Z, W) = -g(Z, W)\nabla(\ln f).$$

If the equality sign of (7) holds identically, then we obtain

$$(11) \qquad h(\mathcal{D}, \mathcal{D}) = 0, \quad h(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0, \quad h(\mathcal{D}, \mathcal{D}^\perp) \subset \phi\mathcal{D}^\perp.$$

The first condition (11) implies that M_1 is totally geodesic in M. On the other hand, one has

$$g(h(X, \phi Y), \phi Z) = g(\widetilde{\nabla}_X \phi Y, \phi Z) = g(\nabla_X Y, Z) = 0.$$

Thus M_1 is totally geodesic in $\widetilde{M}$.

The second condition (11) and (10) imply that M_2 is totally umbilical in $\widetilde{M}$. Moreover, by (11), it follows that M is a minimal submanifold of $\widetilde{M}$.

In particular, if the ambient space is a Kenmotsu space form, one has the following.

COROLLARY 3.2. *Let $\widetilde{M}(c)$ be a $(2m + 1)$-dimensional Kenmotsu space form of constant ϕ-sectional curvature c and $M = M_1 \times_f M_2$ an n-dimensional non-trivial contact CR-warped product submanifold, satifying*

$$\|h\|^2 = 2\beta[\|\nabla(\ln f)\|^2 - 1].$$

Then, we have

   (a) *M_1 is a totally geodesic invariant submanifold of $\widetilde{M}(c)$. Hence M_1 is a Kenmotsu space form of constant ϕ-sectional curvature c.*

   (b) *M_2 is a totally umbilical anti-invariant submanifold of $\widetilde{M}(c)$. Hence M_2 is a real space form of sectional curvature $\varepsilon > \frac{c-3}{4}$.*

   (c) *If $\beta > 1$, then the warping function f satisfies $\|\nabla f\|^2 = (\varepsilon - \frac{c-3}{4})f^2$.*

Proof. Statement (a) follows from Theorem 3.1.

Also, we know that M_2 is a totally umbilical submanifold of $\widetilde{M}(c)$. Gauss equation implies that M_2 is a real space form of constant sectional curvature $\varepsilon \geq \frac{c-3}{4}$. Moreover, by (5), we see that $\varepsilon = \frac{c-3}{4}$ if and only if the warping function f is constant.

Let R'' denote the Riemann curvature tensor of M_2. Then, we have (see, for instance, [8])

$$R(Z,W)V = R''(Z,W)V - \|\nabla(\ln f)\|^2[g(W,V)Z - g(Z,V)W],$$

for any vectors Z, W, V tangent to M_2.

By applying Gauss equation, we obtain statement (c).

4. Another inequality

In the present section, we will improve the inequality (7) for contact CR-warped product submanifolds in Kenmotsu space forms. Equality case is characterized.

THEOREM 4.1. *Let $\widetilde{M}$ be a $(2m+1)$-dimensional Kenmotsu space form and $M = M_1 \times_f M_2$ an n-dimensional contact CR-warped product submanifold, such that M_1 is a $(2\alpha + 1)$-dimensional invariant submanifold tangent to ξ and M_2 a β-dimensional anti-invariant submanifold of $\widetilde{M}(c)$. Then*

(i) *The squared norm of the second fundamental form of M satisfies*

(12) $$\|h\|^2 \geq 2\beta[\|\nabla(\ln f)\|^2 - \Delta(\ln f) - 1] + \alpha\beta(c+1),$$

where Δ is the Laplacian operator on M_1.

(ii) *The equality sign of (12) holds identically if and only if we have:*

(a) *M_1 is a totally geodesic invariant submanifold of $\widetilde{M}(c)$. Hence M_1 is a Kenmotsu space form of constant ϕ-sectional curvature c.*

(b) *M_2 is a totally umbilical anti-invariant submanifold of $\widetilde{M}(c)$. Hence M_2 is a real space form of sectional curvature $\varepsilon \geq \frac{c-3}{4}$.*

Proof. Let $M = M_1 \times_f M_2$ be a warped product submanifold of a $(2m+1)$-dimensional Kenmotsu space form $\widetilde{M}(c)$, such that M_1 is an invariant submanifold tangent to ξ and M_2 an anti-invariant submanifold of $\widetilde{M}(c)$.

We denote by ν the normal subbundle orthogonal to $\phi(TM_2)$. Obviously, we have

$$T^\perp M = \phi(TM_2) \oplus \nu, \quad \phi\nu = \nu.$$

For any vector fields X tangent to M_1 and orthogonal to ξ and Z tangent to M_2, equation (2) gives

$$\widetilde{R}(X, \phi X, Z, \phi Z) = \frac{c+1}{2} g(X, X) g(Z, Z).$$

On the other hand, by Codazzi equation, we have

$$\widetilde{R}(X, \phi X, Z, \phi Z)$$

$$(13) \qquad = -g(\nabla_X^\perp h(\phi X, Z) - h(\nabla_X \phi X, Z) - h(\phi X, \nabla_X Z), \phi Z)$$

$$+ g(\nabla_{\phi X}^\perp h(X, Z) - h(\nabla_{\phi X} X, Z) - h(X, \nabla_{\phi X} Z), \phi Z).$$

By using the equation (5) and structure equations of a Kenmotsu manifold, we get

$$g(\nabla_X^\perp h(\phi X, Z), \phi Z)$$

$$= X g(h(\phi X, Z), \phi Z) - g(h(\phi X, Z), \nabla_X^\perp \phi Z)$$

$$= X g(\nabla_Z X, Z) - g(h(\phi X, Z), \phi \widetilde{\nabla}_X Z)$$

$$= X((X \ln f) g(Z, Z)) - (X \ln f) g(h(\phi X, Z), \phi Z)$$

$$- g(h(\phi X, Z), \phi h_\nu(X, Z))$$

$$= (X^2 \ln f) g(Z, Z) + (X \ln f)^2 g(Z, Z) - \|h_\nu(X, Z)\|^2,$$

where we denote by $h_\nu(X, Z)$ the ν-component of $h(X, Z)$.

Also, by (8) and (5), we obtain respectively

$$g(h(\nabla_X \phi X, Z), \phi Z) = ((\nabla_X X) \ln f) g(Z, Z),$$

$$g(h(\phi X, \nabla_X Z), \phi Z) = (X \ln f) g(h(\phi X, Z), \phi Z)$$

$$= (X \ln f)^2 g(Z, Z).$$

Substituting the above relations in (13), we find

$$\widetilde{R}(X, \phi X, Z, \phi Z)$$

$$(14) \qquad = 2\|h_\nu(X, Z)\|^2 - (X^2 \ln f) g(Z, Z) + ((\nabla_X X) \ln f) g(Z, Z)$$

$$- ((\phi X)^2 \ln f) g(Z, Z) + ((\nabla_{\phi X} \phi X) \ln f) g(Z, Z).$$

Then the equation (14) becomes

$$(15) \qquad 2\|h_\nu(X, Z)\|^2 = [\frac{c+1}{2} g(X, X) + (X^2 \ln f) - ((\nabla_X X) \ln f)$$

$$+ ((\phi X)^2 \ln f) - ((\nabla_{\phi X} \phi X) \ln f)] g(Z, Z).$$

Let $\{X_0 = \xi, X_1, ..., X_\alpha, X_{\alpha+1} = \phi X_1, ..., X_{2\alpha} = \phi X_\alpha, Z_1, ..., Z_\beta\}$ be a local orthonormal frame on M such that $X_0, ..., X_{2\alpha}$ are tangent to

M_1 and $Z_1, ..., Z_\beta$ are tangent to M_2. Therefore

$$(16) \qquad 2\sum_{j=1}^{2\alpha}\sum_{t=1}^{\beta} \|h_\nu(X_j, Z_t)\|^2 = \alpha\beta(c+1) - 2\beta\Delta(\ln f).$$

Combining (7) and (16), we obtain the inequality (12). The equality case can be solved similarly to the Corollary 3.2.

COROLLARY 4.2. *Let $\widetilde{M}(c)$ be a Kenmotsu space form with $c < -1$. Then there do not exist contact CR-warped product submanifolds $M_1 \times_f M_2$ in $\widetilde{M}(c)$ such that $\ln f$ is a harmonic function on M_1.*

Proof. Assume there exists a contact CR-warped product submanifold $M_1 \times_f M_2$ in a Kenmotsu space form $\widetilde{M}(c)$ such that $\ln f$ is a harmonic function on M_1. Then (16) implies $c \geq -1$.

COROLLARY 4.3. *Let $\widetilde{M}(c)$ be a Kenmotsu space form with $c \leq -1$. Then there do not exist contact CR-warped product submanifolds $M_1 \times_f M_2$ in $\widetilde{M}(c)$ such that $\ln f$ is a non-negative eigenfunction of the Laplacian on M_1 corresponding to an eigenvalue $\lambda > 0$.*

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Kadri Arslan, Ridvan Ezentas, and Cengizhan Murathan
Uludag University
Faculty of Arts and Sciences
Department of Mathematics
Görükle 16059, Bursa, Turkey
E-mail: arslan@uludag.edu.tr

Ion Mihai
Faculty of Mathematics
University of Bucharest
Str. Academiei 14010014 Bucharest, Romania
E-mail: imihai@math.math.unibuc.ro