Bull. Austral. Math. Soc. 53C25, 53C40
Vol. 64 (2001) [201–212]

CERTAIN INEQUALITIES FOR SUBMANIFOLDS IN (K, μ)-CONTACT SPACE FORMS

Kadri Arslan, Ridvan Ezentas, Ion Mihai, Cengizhan Murathan and Cihan Özgür

Chen (1999) established a sharp relationship between the Ricci curvature and the squared mean curvature for a submanifold in a Riemanian space form with arbitrary codimension. Matsumoto (to appear) dealt with similar problems for submanifolds in complex space forms.

In this article we obtain sharp relationships between the Ricci curvature and the squared mean curvature for submanifolds in (k, μ)-contact space forms.

1. (k, μ)-Contact Space Forms

A differentiable manifold $\widetilde{M}^{2n+1}$ is said to be a *contact manifold* if it admits a global differential 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $\widetilde{M}^{2n+1}$.

Given a contact form η, one has a unique vector field ξ, which is called the *characteristic vector field*, satisfying

$$\eta(\xi) = 1, \; d\eta(\xi, X) = 0, \tag{1.1}$$

for any vector field X.

It is well-known that, there exists a Riemannian metric g and a $(1,1)$-tensor field φ such that

$$\eta(X) = g(X, \xi), \; d\eta(X, Y) = g(X, \varphi Y), \; \varphi^2 X = -X + \eta(X)\xi, \tag{1.2}$$

where X and Y are vector fields on $\widetilde{M}$.

From (1.2) it follows that

$$\varphi\xi = 0, \; \eta \circ \varphi = 0, \; g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{1.3}$$

Received 21st August, 2001
This paper is prepared during the third author's visit to the Uludag University, Bursa, Turkey in July 2000. The third author is supported by the Scientific and Technical Research Council of Turkey (TÜBITAK) for NATO-PC Advanced Fellowships Programme.

A differentiable manifold $\widetilde{M}^{2n+1}$ equipped with structure tensors (φ, ξ, η, g) satisfying (1.2) is said to be a *contact metric manifold* and is denoted by $\widetilde{M} = \left(\widetilde{M}^{2n+1}, \varphi, \xi, \eta, g\right)$.

On a contact metric manifold $\widetilde{M}$, we can define a (1,1)-tensor field h by $h = (L_\xi \varphi)/2$, where L denotes Lie differentiation. Then we may observe that h is symmetric and satisfies

$$h\xi = 0, \quad h\varphi = -\varphi h, \quad \widetilde{\nabla}_X \xi = -\varphi X - \varphi h X, \tag{1.4}$$

where $\widetilde{\nabla}$ is the Levi–Civita connection ([**2**]).

For a contact metric manifold $\widetilde{M}$ one may define naturally an almost complex structure on $\widetilde{M} \times \mathbb{R}$. If this almost complex structure is integrable, $\widetilde{M}$ is said to be a *Sasakian manifold*. A Sasakian manifold is characterised by the condition

$$(\widetilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{1.5}$$

for all vector fields X and Y on the manifold [**1**].

A contact metric manifold $\widetilde{M}$ is Sasakian if and only if

$$\widetilde{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{1.6}$$

for all vector fields X and Y ([**1**]).

Let $\widetilde{M}$ be a contact metric manifold. The (k, μ)-*nullity distribution* of $\widetilde{M}$ for the pair (k, μ) is a distribution

$$\begin{aligned} N(k, \mu) : p \to N_p(k, \mu) = \Big\{ Z \in T_pM \mid \widetilde{R}(X, Y)Z \\ = k\big[g(Y, Z)X - g(X, Z)Y\big] + \mu\big[g(Y, Z)hX - g(X, Z)hY\big] \Big\}, \end{aligned} \tag{1.7}$$

where $k, \mu \in \mathbb{R}$ and $k \leqslant 1$ (see [**7**]).

If $k = 1$, then $h = 0$ and $\widetilde{M}$ is a Sasakian manifold ([**2**]). Also one has $\operatorname{tr} h = 0$, $\operatorname{tr} h\varphi = 0$ and $h^2 = (k - 1)\varphi^2$. So if the characteristic vector field ξ belongs to the (k, μ)-nullity distribution then we have

$$\widetilde{R}(X, Y)\xi = k\big[\eta(Y)X - \eta(X)Y\big] + \mu\big[\eta(Y)hX - \eta(X)hY\big]. \tag{1.8}$$

Moreover, if M has constant φ-sectional curvature c then it is called a (k, μ)-contact space form and is denoted by $\widetilde{M}(c)$.

The curvature tensor of $\widetilde{M}(c)$ is given by [7]:

$$
\begin{aligned}
4\widetilde{R}(X,Y)Z = {} & (c+3)\{g(Y,Z)X - g(X,Z)Y\} \\
& + (c+3-4k)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
& + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \\
& + (c-1)\{2g(X,\varphi Y)\varphi Z + g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X\} \\
& - 2\{g(hX,Z)hY - g(hY,Z)hX + g(X,Z)hY \\
& - 2g(Y,Z)hX - 2\eta(X)\eta(Z)hY + 2\eta(Y)\eta(Z)hX \\
& + 2g(hX,Z)Y - 2g(hY,Z)X + 2g(hY,Z)\eta(X)\xi \\
& - 2g(hX,Z)\eta(Y)\xi - g(\varphi hX,Z)\varphi hY + g(\varphi hY,Z)\varphi hX\} \\
& + 4\mu\{\eta(Y)\eta(Z)hX - \eta(X)\eta(Z)hY \\
& + g(hY,Z)\eta(X)\xi - g(hX,Z)\eta(Y)\xi\}.
\end{aligned} \tag{1.9}
$$

If $k \neq 1$, then $\mu = k+1$ and $c = -2k-1$.

2. Riemannian invariants

The Riemannian invariants of a Riemannian manifold are the intrinsic characteristics of the Riemannian manifold. In this section we recall a string of Riemannian invariants on a Riemannian manifold ([4]).

Let M be a Riemannian manifold. Denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$ and by ∇ the Riemannian connection on M.

For any orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of the tangent space T_pM, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j).$$

We denote by $(\inf K)(p) = \inf\{K(\pi);\ \pi \subset T_pM, \dim \pi = 2\}$, and we introduce the first Chen invariant $\delta_M(p) = \tau(p) - (\inf K)(p)$.

Let L be a subspace of T_pM of dimension $r \geqslant 2$ and $\{e_1, e_2, \dots, e_r\}$ an orthonormal basis of L. We define the scalar curvature $\tau(L)$ of the r-plane section L by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta),\ \alpha, \beta = 1, \dots, r.$$

Given an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of the tangent space T_pM, we simply denote by $\tau_{1\dots r}$ the scalar curvature of r-plane section spanned by $e_1, \dots, e_r$. The scalar curvature $\tau(p)$ of M at p is nothing but the scalar curvature of the tangent space of M at p. And if L is a 2-plane section, $\tau(L)$ is nothing but the sectional curvature $K(L)$ of L.

For an integer $l \geqslant 0$, we denote by $S(n,l)$ the finite set which consists of k-tuples $(n_1, n_2, \dots, n_l)$ of integers $\geqslant 2$ satisfying $n_1 < n$ and $n_1 + \dots + n_l \leqslant n$. Denote by $S(n)$ the set of l-tuples with $l \geqslant 0$ for a fixed n.

For each l-tuples $(n_1, \dots, n_l) \in S(n)$, we introduce a Riemannian invariant defined by

$$\delta(n_1, \dots, n_l) = \tau(p) - S(n_1, \dots, n_l)(p),$$

where

$$S(n_1, \dots, n_l) = \inf\{\tau(L_1) + \dots + \tau(L_l)\};$$

$L_1, \dots, L_l$ run over all l mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j, j = 1, \dots, l$.

We define:

$$d(n_1, \dots, n_l) = \frac{n^2\left(n + l - 1 - \sum_{j=1}^{l} n_j\right)}{2\left(n + l - \sum_{j=1}^{l} n_j\right)},$$

$$b(n_1, \dots, n_l) = \frac{1}{2}\left[n(n-1) - \sum_{j=1}^{l} n_j(n_j - 1)\right].$$

We recall the following

Lemma 2.1. ([**3**]) *Let $a_1, \dots, a_n, b \in \mathbb{R}$ such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right).$$

Then we have $2a_1a_2 \geqslant b$. Moreover, $2a_1a_2 = b$ if and only if $a_1 + a_2 = a_3 = \dots = a_n$.

Let M be an n-dimensional submanifold of $\widetilde{M}(c)$. We denote by $\tilde{h}$ the second fundamental form and by R the Riemann curvature tensor of M. Then the equation of Gauss is given by

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g\big(\tilde{h}(X,W), \tilde{h}(Y,Z)\big) - g\big(\tilde{h}(X,Z), \tilde{h}(Y,W)\big), \tag{2.1}$$

for any vectors X, Y, Z, W tangent to M.

We denote by H the mean curvature vector, that is,

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} \tilde{h}(e_i, e_i),$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis of the tangent space T_pM, $p \in M$.

Also, we set $\widetilde{h}_{ij}^r = g\Big(\widetilde{h}(e_i, e_j), e_r\Big)$ and

$$\|\widetilde{h}\|^2 = \sum_{i,j=1}^{n} g\big(\widetilde{h}(e_i, e_j), \widetilde{h}(e_i, e_j)\big).$$

For any tangent vector field X to M, we put $\varphi X = PX + FX$, where PX and FX are the tangential and normal components of φX respectively.

Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of T_pM. We denote

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j).$$

Suppose L is a k-plane section of T_pM and X a unit vector in L. We choose an orthonormal basis $\{e_1, \dots, e_k\}$ of L such that $e_1 = X$.

Define the Ricci curvature Ric_L of L at X by $\mathrm{Ric}_L(X) = K_{12} + K_{13} + \cdots + K_{1k}$, where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j. We simply called such a curvature a k-Ricci curvature.

Recall that for a submanifold M in a Riemannian manifold, the relative null space of M at a point $p \in M$ is defined by

$$\mathcal{N}_p = \big\{X \in T_pM \mid \widetilde{h}(X, Y) = 0, Y \in T_pM\big\}.$$

3. Ricci curvature and squared mean curvature

Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for submanifolds in real space forms (see [**5**]).

We prove similar inequalities for certain submanifolds of a (k, μ)-contact space form $\widetilde{M}(c)$.

We consider the case in which the submanifold M is normal to ξ in a (k, μ)-contact space form $\widetilde{M}(c)$ with $k < 1$. The case $k = 1$ is the Sasakian case which has been considered by the third author in [**8**].

Theorem 3.1. *Let M be an n-dimensional submanifold normal to ξ of a $(2m+1)$-dimensional (k, μ)-contact space form $\widetilde{M}(c)$. Then:*

(i) *For each unit vector $X \in T_pM$, we have*

$$\mathrm{Ric}(X) \leqslant \frac{1}{4}\big\{n^2\|H\|^2 - \|hX\|^2 - 3(k+1)\|PX\|^2 + 2(1-k)(n-1)\big\}. \tag{3.1}$$

(ii) *If $H(p) = 0$, then a unit tangent vector X at p satisfies the equality case of* (3.1) *if and only if $X \in \mathcal{N}_p$.*

(iii) *The equality case of* (3.1) *holds identically for all unit tangent vectors at p if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

PROOF: Let $X \in T_pM$ be a unit tangent vector at p. We choose an orthonormal basis $\{e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n, e_{2n+1}, \dots, e_{2m+1}\}, e_{2m+1} = \xi$ in $T_p\widetilde{M}(c)$, such that $e_1, \dots, e_n$ are tangent to M at p, with $e_1 = X$.

Then, from the equation of Gauss, we have

$$(3.2)\quad 2\tau = n^2\|H\|^2 - \left\|\widetilde{h}\right\|^2 + \frac{1-k}{2}n(n-1) - \frac{3}{2}(k+1)\|P\|^2 - \frac{1}{2}\|h\|^2 - \sum_{i,j=1}^{n} g^2(he_i, e_j).$$

From (3.2), we get

$$\begin{aligned} n^2\|H\|^2 &= 2\tau + \sum_{r=n+1}^{2m+1}\left[(h_{11}^r)^2 + (h_{22}^r + \dots + h_{nn}^r)^2 + 2\sum_{i<j}\left(h_{ij}^r\right)^2\right] \\ &\quad - 2\sum_{r=n+1}^{2m+1}\sum_{2\leqslant i<j\leqslant n} h_{ii}^r h_{jj}^r - \frac{1-k}{2}n(n-1) + \frac{3}{2}(k+1)\|P\|^2 + \frac{1}{2}\|h\|^2 \\ &\quad + \sum_{i,j=1}^{n} g^2(he_i, e_j) \end{aligned}$$

$$\begin{aligned} (3.3)\qquad &= 2\tau + \frac{1}{2}\sum_{r=n+1}^{2m+1}\left[(h_{11}^r + \dots + h_{nn}^r)^2 + (h_{11}^r - h_{22}^r - \dots - h_{nn}^r)^2\right] \\ &\quad + 2\sum_{r=n+1}^{2m+1}\sum_{i<j}\left(h_{ij}^r\right)^2 \\ &\quad - 2\sum_{r=n+1}^{2m+1}\sum_{2\leqslant i<j\leqslant n} h_{ii}^r h_{jj}^r - \frac{1-k}{2}n(n-1) + \frac{3}{2}(k+1)\|P\|^2 + \frac{1}{2}\|h\|^2 \\ &\quad + \sum_{i,j=1}^{n} g^2(he_i, e_j). \end{aligned}$$

It follows that

$$\begin{aligned} (3.4)\quad \frac{1}{2}n^2\|H\|^2 &\geqslant 2\tau - \frac{1}{2}(1-k)n(n-1) + \frac{3}{2}(k+1)\|P\|^2 + \sum_{i<j} g^2(he_i, e_j) \\ &\quad + \frac{1}{2}\|h\|^2 - 2\sum_{r=n+1}^{2m+1}\sum_{2\leqslant i<j\leqslant n}\left[\widetilde{h}_{ii}^r\widetilde{h}_{jj}^r - \left(\widetilde{h}_{ij}^r\right)^2\right]. \end{aligned}$$

From the equation of Gauss, we find

$$(3.5)\quad 2\sum_{2\leqslant i<j\leqslant n} K_{ij} = \sum_{r=n+1}^{2m+1}\sum_{2\leqslant i<j\leqslant n}\left[\widetilde{h}^r_{ii}\widetilde{h}^r_{jj} - \left(\widetilde{h}^r_{ij}\right)^2\right] + \frac{1}{2}(1-k)(n-1)(n-2)$$
$$- \frac{3}{2}(k+1)\sum_{i=2}^{n}\|Pe_i\|^2 - \frac{1}{2}\sum_{i=2}^{n}\|he_i\|^2 - \sum_{i,j=2}^{n} g^2(he_i, e_j).$$

Substituting (3.5) in (3.4), one gets

$$\frac{1}{2}n^2\|H\|^2 \geqslant 2\,\mathrm{Ric}\,(X) + \frac{1}{2}\|hX\|^2 + \frac{3}{2}(k+1)\|PX\|^2 - (1-k)(n-1),$$

or equivalently (3.1).

(ii) Assume $H(p) = 0$. Equality holds in (3.1) if and only if

$$\begin{cases} h^r_{12} = \cdots = h^r_{1n} = 0, \\ h^r_{11} = h^r_{22} + \cdots + h^r_{nn}, r \in \{n+1, \ldots, 2m\}. \end{cases}$$

Then $h^r_{1j} = 0$, $\forall j \in \{1, \ldots, n\}, r \in \{n+1, \ldots, 2m\}$, that is, $X \in \mathcal{N}_p$.

(iii) The equality case of (3.1) holds for all unit tangent vectors at p if and only if

$$\begin{cases} h^r_{ij} = 0, i \neq j, r \in \{n+1, \ldots, 2m\}, \\ h^r_{11} + \cdots + h^r_{nn} - 2h^r_{ii} = 0, \quad i \in \{1, \ldots, n\}, r \in \{n+1, \ldots, 2m\}. \end{cases}$$

We distinguish two cases:

(a) $n \neq 2$, then p is a totally geodesic point;

(b) $n = 2$, it follows that p is a totally umbilical point.

The converse is trivial. ∎

A submanifold M normal to ξ is said to be invariant (respectively anti-invariant) if $\varphi(T_pM) \subset T_pM$, for all $p \in M$ (respectively $\varphi(T_pM) \subset T_p^{\perp}M$, for all $p \in M$). It is known that an invariant submanifold of a (k, μ)-contact space form is minimal.

COROLLARY 3.2. *Let M be an n-dimensional invariant submanifold of a (k, μ)-contact space form $\widetilde{M}(c)$, $(k < 1)$. Then:*

(i) *For each unit vector $X \in T_pM$ orthogonal to ξ, we have*

$$(3.6)\qquad \mathrm{Ric}\,(X) \leqslant -\frac{1}{4}(5k+2) + 2n(1-k)\}.$$

(ii) *A unit tangent vector $X \in T_pM$ orthogonal to ξ satisfies the equality case of (3.6) if and only if $X \in \mathcal{N}_p$.*

(iii) *The equality case of (3.6) holds identically for all unit tangent vectors orthogonal to ξ at p if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

COROLLARY 3.3. *Let M be an n-dimensional anti-invariant submanifold of a (k,μ)-contact space form $\widetilde{M}(c), (k<1)$. Then:*

(i) *For each unit vector $X \in T_pM$ orthogonal to ξ, we have*

$$\operatorname{Ric}(X) \leqslant \frac{1}{4}\left\{n^2\|H\|^2 - 2(n-1)(k+1) + (k-1)\right\}. \tag{3.7}$$

(ii) *If $H(p)=0$, then a unit tangent vector $X \in T_pM$ orthogonal to ξ satisfies the equality case of (3.7) if and only if $X \in \mathcal{N}_p$.*

(iii) *The equality case of (3.7) holds identically for all unit tangent vectors orthogonal to ξ at p if and only if either p is a totally geodesic point or $n=2$ and p is a totally umbilical point.*

5. B.Y. Chen's inequalities

Chen proved a sharp inequality for submanifolds M in real space forms $\widetilde{M}(c)$ involving the scalar curvature and sectional curvature of M (intrinsic invariants) and the squared mean curvature (extrinsic invariant).

THEOREM 4.1. ([**3**]) *Given an m-dimensional real space form $\widetilde{M}(c)$ and an n-dimensional submanifold M, $n \geqslant 3$, we have:*

$$\inf K \geqslant \tau - \frac{n-2}{2}\left\{\frac{n^2}{n-1}\|H\|^2 + (n+1)c\right\}. \tag{4.1}$$

The equality case of inequality (4.1) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \dots, e_m\}$ of $T_p^{\perp}M$ such that the shape operators of M in $\widetilde{M}(c)$ at p have the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a+b=\mu, \tag{4.2}$$

$$A_r = \begin{pmatrix} \widetilde{h}^r_{11} & \widetilde{h}^r_{12} & 0 & \cdots & 0 \\ \widetilde{h}^r_{12} & -\widetilde{h}^r_{11} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{4.3}$$

where we denote by

$$A_r = A_{e_r}, \quad r = n+1, \dots, m,$$
$$\widetilde{h}_{ij}^r = g\big(\widetilde{h}(e_i, e_j), e_r\big), \quad r = n+1, \dots, m.$$

In an analogous way we prove an inequality for submanifolds M normal to ξ in (k,μ)-contact space forms $\widetilde{M}(c)$. The Sasakian case was studied in [**6**].

THEOREM 4.2. *Given a $(2m+1)$-dimensional (k,μ)-contact space form $\widetilde{M}(c)$ and a submanifold M normal to ξ, $\dim M = n$, $n \geqslant 3$, we have:*

(i) *For any invariant plane section $\pi \subset T_pM$*

$$\text{(4.4)} \quad K(\pi) \geqslant \tau - \frac{(n-2)n^2}{2(n-1)} \|H\|^2 - \frac{1}{4}\{n(1-k)(n-1) + 3(k+1)\|P\|^2\} - (1+2k).$$

(ii) *For any anti-invariant plane section $\pi \subset T_pM$*

$$\text{(4.5)} \quad K(\pi) \geqslant \tau - \frac{(n-2)n^2}{2(n-1)} \|H\|^2 - \frac{1}{4}\{n(1-k)(n-1) + 3(k+1)\|P\|^2\} + \frac{1}{2}(1-k).$$

The equality case of inequalities (4.4) and (4.5) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \dots, e_{2m+1}\}, e_{2m+1} = \xi$, of $T_p^{\perp}M$ such that the shape operators of M in $\widetilde{M}(c)$ at p have the forms (4.2) and (4.3).

PROOF: From equation (2.1) we have

$$\text{(4.6)} \quad 2\tau = n^2 \|H\|^2 - \left\|\widetilde{h}\right\|^2 + \frac{1-k}{2} n(n-1) - \frac{3}{2}(k+1)\|P\|^2 - \frac{1}{2}\|h\|^2 - \sum_{i,j=1}^{n} g^2(he_i, e_j).$$

Putting

$$\text{(4.7)} \quad \varepsilon = 2\tau - \frac{n^2(n-2)}{n-1} \|H\|^2 - \frac{1-k}{2} n(n-1) + \frac{3}{2}(k+1)\|P\|^2 + \sum_{i,j=1}^{n} g^2(he_i, e_j),$$

it follows that

$$\text{(4.8)} \quad n^2 \|H\|^2 = \left(\varepsilon + \left\|\widetilde{h}\right\|^2\right)(n-1)$$

Let $\pi \subset T_pM, \pi = \mathrm{sp}\{e_1, e_2\}, \{e_1, \dots, e_n\} \subset T_pM, e_{n+1} = \big(1/\|H\|\big)H$, $\{e_{n+1}, \dots, e_{2m}, e_{2m+1} = \xi\} \subset T_p^{\perp}M$. The equation (4.8) becomes

$$\left(\sum_{i=1}^{n} \widetilde{h}_{ii}^{n+1}\right)^2 = (n-1)\left(\sum_{i=1}^{n} \big(\widetilde{h}_{ii}^{n+1}\big)^2 + \sum_{i \neq j} \big(\widetilde{h}_{ij}^{n+1}\big)^2 + \sum_{r \geqslant n+2} \sum_{i,j} \big(\widetilde{h}_{ij}^{r}\big)^2 + \varepsilon\right)$$

By Chen's Lemma, one obtains

$$2\widetilde{h}_{11}^{n+1}\widetilde{h}_{22}^{n+1} \geqslant \sum_{i\neq j}\left(\widetilde{h}_{ij}^{n+1}\right)^2 + \sum_{r\geqslant n+2}\sum_{i,j}\left(\widetilde{h}_{ij}^{r}\right)^2 + \varepsilon.$$

Gauss equation gives

$$K(\pi) = \widetilde{R}(e_1,e_2,e_1,e_2) + \sum_{r=n+1}^{2m+1}\left(\widetilde{h}_{11}^{r}\widetilde{h}_{22}^{r} - \left(\widetilde{h}_{12}^{r}\right)^2\right).$$

By using (1.9), we get

$$\begin{aligned} K(\pi) = &\frac{1}{2}(1-k) - \frac{3}{2}(1+k)g^2(e_1,Pe_2) - \frac{1}{2}\{g^2(he_1,e_2) - g(he_2,e_2)g(he_1,e_1) \\ &- 2g(he_1,e_1) - 2g(he_2,e_2) - g^2(\varphi he_1,e_2) + g(\varphi he_2,e_2)g(\varphi he_1,e_1)\} \\ &+ \sum_{r=n+1}^{2m+1}\left(\widetilde{h}_{11}^{r}\widetilde{h}_{22}^{r} - \left(\widetilde{h}_{12}^{r}\right)^2\right) \\ = &\frac{1}{2}(1-k) - \frac{3}{2}(1+k)g^2(e_1,Pe_2) + \widetilde{h}_{11}^{n+1}\widetilde{h}_{22}^{n+1} - \left(\widetilde{h}_{12}^{n+1}\right)^2 \\ &+ \sum_{r=n+1}^{2m}\left(\widetilde{h}_{11}^{r}\widetilde{h}_{22}^{r} - \left(\widetilde{h}_{12}^{r}\right)^2\right) + \frac{1}{2}\left[\widetilde{h}_{11}^{2m+1}\widetilde{h}_{22}^{2m+1} - \left(\widetilde{h}_{12}^{2m+1}\right)^2\right] \\ \geqslant &\frac{1}{2}(1-k) - \frac{3}{2}(1+k)g^2(e_1,Pe_2) + \frac{\varepsilon}{2}. \end{aligned}$$

∎

The equality case follows from the above equations and Chen's Lemma.

THEOREM 4.3. *Given a $(2m+1)$-dimensional (k,μ)-contact space form $\widetilde{M}(c)$ and a submanifold M normal to ξ, $\dim M = n$, $n \geqslant 3$, we have:*

$$(4.9)\quad \delta(n_1,\dots,n_r) \leqslant d(n_1,\dots,n_r)\,\|H\|^2 + \frac{1}{2}\left\{\frac{(c+3)}{4}n(n-1) + \frac{1}{2}(k-1)n\right\} + \frac{(1-k)}{4}\sum_{j=1}^{r} n_j(n_j-1).$$

PROOF: Let $n_1,\dots,n_l \geqslant 2$, $n_1 < n$, $n_1+\dots+n_l \leqslant n$. The equation (4.6) may be written as

$$(4.10)\quad 2\tau = n^2\|H\|^2 - \left\|\widetilde{h}\right\|^2 + \frac{1}{2}(1-k)n(n-1) - \frac{3}{2}(k+1)\|P\|^2 - \frac{1}{2}\|h\|^2 - \sum_{i,j=1}^{n} g^2(he_i,e_j),$$

and denote

$$\varepsilon = 2\tau - 2d(n_1, \dots, n_l)\|H\|^2 - \frac{1}{2}(1-k)n(n-1) - \frac{3}{2}(k+1)\|P\|^2 + \frac{1}{2}\|h\|^2 - \sum_{i,j=1}^{n} g^2(he_i, e_j),$$

and $\gamma = n + l - \sum_{j=1}^{l} n_j$. We have $n^2\|H\|^2 = \left(\varepsilon + \|\widetilde{h}\|^2\right)\gamma$. Let $p \in M$, $\{e_1, \dots, e_n\} \subset T_pM$, $L_1, \dots, L_k \subset T_pM$ such that

$$\begin{aligned} L_1 &= \mathrm{Sp}\{e_1, \dots, e_{n_1}\}, \\ L_2 &= \mathrm{Sp}\{e_{n_1+1}, \dots, e_{n_1+n_2}\}, \dots, \\ L_l &= \mathrm{Sp}\{e_{n_1+\dots+n_{l-1}+1}, \dots, e_{n_1+\dots+n_l}\}. \end{aligned}$$

Then

$$\tau(L_j) = \sum_{\substack{\alpha,\beta\in\Delta_j \\ \alpha<\beta}} K(e_\alpha \wedge e_\beta),$$

where $\Delta_j = (n_1 + \dots + n_{j-1} + 1, \dots, n_1 + \dots + n_j)$. By Gauss equation one has

$$\begin{aligned} K(e_\alpha \wedge e_\beta) &= R(e_\alpha, e_\beta, e_\alpha, e_\beta) \\ &= \widetilde{R}(e_\alpha, e_\beta, e_\alpha, e_\beta) + g\left(\widetilde{h}(e_\alpha, e_\alpha), \widetilde{h}(e_\beta, e_\beta)\right) - g\left(\widetilde{h}(e_\alpha, e_\beta), \widetilde{h}(e_\alpha, e_\beta)\right). \end{aligned}$$

By (1.9), we get

$$\widetilde{R}(e_\alpha, e_\beta, e_\alpha, e_\beta) = 1 - \frac{1}{2}(k+1) - \frac{1}{2}\left[g(\widetilde{h}(e_\alpha, e_\alpha), \xi)g(\widetilde{h}(e_\beta, e_\beta), \xi) - g^2(\widetilde{h}(e_\alpha, e_\beta), \xi)\right],$$

which implies

$$\tau(L_j) = \frac{1}{2}(1-k) - \frac{3}{2}(k+1)\sum_{\substack{\alpha,\beta\in\Delta_j \\ \alpha<\beta}} g^2(Pe_\alpha, e_\beta) + \sum_{\alpha,\beta}\sum_{r=n+1}^{2m+1}\left(\widetilde{h}^r_{\alpha\alpha}\widetilde{h}^r_{\beta\beta} - (\widetilde{h}^r_{\alpha\beta})^2\right).$$

Then

$$\begin{aligned} \sum_{j=1}^{l} \tau(L_j) &\geqslant \frac{1}{4}(1-k)\sum_{j=1}^{l} n_j(n_j - 1) - \frac{3}{2}(k+1)\|P\|^2 + \frac{\varepsilon}{2} \\ &\geqslant \frac{1}{4}(1-k)\sum_{j=1}^{l} n_j(n_j-1) - \frac{3}{4}(k+1)\|P\|^2 \\ &\quad + \tau - d(n_1, \dots, n_l)\|H\|^2 - \frac{(1-k)}{2}n(n-1) + \frac{1}{4}\|h\|^2. \end{aligned}$$

Therefore

$$\delta(n_1,\dots,n_l) \leqslant d(n_1,\dots,n_l)\,\|H\|^2 + \frac{1}{4}(1-k)\Big\{n(n-1) - \sum_{j=1}^{l} n_j(n_j-1)\Big\} + \frac{3}{4}(k+1)\,\|P\|^2 - \frac{1}{4}\,\|h\|^2 . \qquad \square$$

References

[1] D.E. Blair, *Contact manifolds in Riemannian heometry*, Lecture Notes in Math. **509** (Springer-Verlag, Berlin, Heidelberg, New York, 1976).

[2] D.Blair, T.Koufogiorgos and B.J.Papantoniou, 'Contact metric manifolds satisfying a nullity condition', *Israel J.Math.* **91** (1995), 189–214.

[3] B.Y. Chen, 'Some pinching and classification theorems for minimal submanifolds', *Arch. Math.* **60** (1993), 568–578.

[4] B. Y. Chen, 'Some new obstructions to minimal and Lagrangian isometric immersions', *Japan. J. Math.* **26** (2000), 105–127.

[5] B.Y. Chen, 'Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions', *Glasgow Math. J.* **41** (1999), 33–41.

[6] F. Defever, I. Mihai and L. Verstraelen, 'B.Y.Chen's inequality for C-totally real submanifolds in Sasakian space forms', *Boll. Un. Mat. Ital.* **11** (1997), 365–374.

[7] T. Koufogiorgos, 'Contact Riemannian manifolds with constant φ-sectional curvature', in *Geometry and topology of submanifolds* **8** (World Scientific, Singapore, 1995), pp. 195–197.

[8] I. Mihai, 'Ricci curvature of submanifolds in Sasakian space forms', *J. Austral. Math. Soc.* (to appear).

Arslan, Ezentas, Murathan and Özgür
Uludag University
Department of Mathematics
Görükle 16059, Bursa
Turkey

Mihai
Faculty of Mathematics
Str. Academiei 14
70109 Bucharest
Romania