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ON MULTIPOINT NONLOCAL BOUNDARY VALUE PROBLEMS FOR HYPERBOLIC DIFFERENTIAL AND DIFFERENCE EQUATIONS

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Abstract. The nonlocal boundary value problem for differential equation

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = f(t) \quad (0 \le t \le 1), \\ u(0) = \sum\limits_{r=1}^{n} \alpha_r u(\lambda_r) + \varphi, u_t(0) = \sum\limits_{r=1}^{n} \beta_r u_t(\lambda_r) + \psi, \\ 0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_n \le 1 \end{cases}$$

in a Hilbert space H with the self-adjoint positive definite operator A is considered. The stability estimates for the solution of the problem under the assumption

$$\sum_{k=1}^{n} |\alpha_k + \beta_k| + \sum_{k=1}^{n} |\alpha_k| \sum_{\substack{m=1 \\ m \neq k}}^{n} |\beta_m| < |1 + \sum_{k=1}^{n} \alpha_k \beta_k|$$

are established. The first order of accuracy difference schemes for the approximate solutions of the problem are presented. The stability estimates for the solution of these difference schemes under the assumption

$$\sum_{k=1}^{n} |\alpha_k| + \sum_{k=1}^{n} |\beta_k| + \sum_{k=1}^{n} |\alpha_k| \sum_{k=1}^{n} |\beta_k| < 1$$

are established. In practice, the nonlocal boundary value problems for one dimensional hyperbolic equation with nonlocal boundary conditions in space variable and multidimensional hyperbolic equation with Dirichlet condition in space variables are considered. The stability estimates for the solutions of difference schemes for the nonlocal boundary value hyperbolic problems are obtained.

1. Introduction

It is known that most problems in fluid mechanics (dynamics, elasticity) and

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other areas of physics lead to partial differential equations of the hyperbolic type (see, e.g., [1-12] and the references given therein).

In the present paper, the nonlocal boundary value problem

$$\left\{\begin{array}{l}\dfrac{d^2u(t)}{dt^2}+Au(t)=f(t) \quad (0\leq t\leq 1),\\ u(0)=\sum\limits_{j=1}^{n}\alpha_j u(\lambda_j)+\varphi, u_t(0)=\sum\limits_{j=1}^{n}\beta_j u_t(\lambda_j)+\psi,\\ 0<\lambda_1\leq\lambda_2\leq ...\leq\lambda_n\leq 1\end{array}\right. \tag{1.1}$$

for differential equations of hyperbolic type in a Hilbert space H with self-adjoint positive definite operator A is considered.

A function $u(t)$ is called a *solution* of the problem (1.1) if the following conditions are satisfied:

(i) $u(t)$ is twice continuously differentiable on the segment $[0,1]$. The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.

(ii) The element $u(t)$ belongs to $D(A)$ for all $t\in[0,1]$ and the function $Au(t)$ is continuous on the segment $[0,1]$.

(iii) $u(t)$ satisfies the equation and the nonlocal boundary conditions (1.1).

In the paper [6], the nonlocal boundary value problem (1.1) in the cases $\alpha_j=0$, $j=2,\cdots,n$ and $\beta_j=0$, $j=2,\cdots,n$, $\lambda_1=1$ was considered. The following theorem on the stability was proved.

Theorem 1.1. *Suppose that* $\varphi\in D(A)$, $\psi\in D(A^{\frac{1}{2}})$ *and* $f(t)$ *is continuously differentiable function on* $[0,1]$ *and* $|1+\alpha_1\beta_1|>|\alpha_1+\beta_1|$. *Then, there is a unique solution of the problem* (1.1) *and the stability inequalities*

$$\max_{0\leq t\leq 1}\| u(t)\|_H\leq M\left[\|\varphi\|_H+\|A^{-1/2}\psi\|_H+\max_{0\leq t\leq 1}\|A^{-1/2}f(t)\|_H\right],$$

$$\max_{0\leq t\leq 1}\| A^{1/2}u(t)\|_H\leq M\left[\|A^{1/2}\varphi\|_H+\|\psi\|_H+\max_{0\leq t\leq 1}\|f(t)\|_H\right],$$

$$\max_{0\leq t\leq 1}\left\|\frac{d^2u(t)}{dt^2}\right\|_H+\max_{0\leq t\leq 1}\|Au(t)\|_H\leq M[\|A\varphi\|_H+\|A^{1/2}\psi\|_H$$
$$+\|f(0)\|_H+\int_0^1\|f'(t)\|_H\,dt]$$

hold, where M *does not depend on* φ,ψ *and* $f(t)$, $t\in[0,1]$.

Moreover, the first and second orders of accuracy difference schemes for the approximate solutions of this problem were presented. The stability estimates for the solution of these difference schemes under the assumption $1 > |\alpha_1||\beta_1| + |\alpha_1| + |\beta_1|$ were established. The stability estimates for the solutions of difference schemes for the approximate solutions of the nonlocal boundary value hyperbolic problems were obtained.

We are interested in studying the stability of solutions of the problem (1.1) under the assumption

$$\sum_{k=1}^{n} |\alpha_k + \beta_k| + \sum_{m=1}^{n} |\alpha_m| \sum_{\substack{k=1 \\ k \neq m}}^{n} |\beta_k| < |1 + \sum_{k=1}^{n} \alpha_k \beta_k| \ . \tag{1.2}$$

In the present paper, the stability estimates for the solution of the problem (1.1) are established. The first order of accuracy difference schemes for approximately solving the boundary value problem (1.1) are presented. The stability estimates for the solution of these difference schemes and its first and second order difference derivatives are established. In practice, the stability estimates for the solutions of the difference schemes of nonlocal boundary value problems for one dimensional hyperbolic equation with nonlocal boundary conditions in space variable and the multidimensional hyperbolic equation with Dirichlet condition in space variables are obtained.

Finally, note that nonlocal boundary value problems for parabolic, elliptic equations and equations of mixed types have been studied extensively (see for instance [14-42] and the references therein).

2. The Differential Hyperboli Equation. The Main Theorem

Let H be a Hilbert space, A be a positive definite self-adjoint operator with $A \geq \delta I$, where $\delta > \delta_0 > 0$. Throughout this paper, $\{c(t), t \geq 0\}$ is a strongly continuous cosine operator-function defined by the formula

$$c\,(t) = \frac{e^{itA^{1/2}} + e^{-itA^{1/2}}}{2}.$$

Then, from the definition of the sine operator-function $s\,(t)$

$$s(t)u = \int_0^t c(s)u\ ds$$

it follows that

$$s\,(t) = A^{-1/2}\frac{e^{itA^{1/2}} - e^{-itA^{1/2}}}{2i}.$$

For the theory of cosine operator-function we refer to [1] and [13].

Throughout this section for simplicity we put

$$B_n=\sum_{k=1}^{n}\beta_k c(\lambda_k)+\sum_{m=1}^{n}\alpha_m c(\lambda_m)-\sum_{m=1}^{n}\sum_{k=1}^{n}\alpha_m\beta_k\left(c(\lambda_m)c(\lambda_k)+As(\lambda_m)s(\lambda_k)\right).$$

Now, let us give some lemmas that will be needed below.

Lemma 2.1. *The estimates hold:*

$$\|c(t)\|_{H\to H}\le 1,\ \left\|A^{1/2}s(t)\right\|_{H\to H}\le 1. \tag{2.1}$$

Lemma 2.2. *Suppose that the assumption* (1.2) *holds. Then, the operator* $I-B_n$ *has an inverse* $T=(I-B_n)^{-1}$*and the following estimate is satisfied:*

$$||T||_{H\to H}\le\frac{1}{|1+\sum_{k=1}^{n}\alpha_k\beta_k|-\sum_{k=1}^{n}|\alpha_k+\beta_k|-\sum_{m=1}^{n}|\alpha_m|\sum_{\substack{k=1\\k\neq m}}^{n}|\beta_k|}. \tag{2.2}$$

Proof. Using assumption (1.2), we obtain $1+\sum_{k=1}^{n}\alpha_k\beta_k\neq 0$. Then, from the definitions of $c\left(\lambda_j\right)$ and $s\left(\lambda_j\right)$ $\left(\lambda_j, j=1,\cdots,n\right)$ it follows that

$$\begin{aligned}I-B_n&=I+\sum_{k=1}^{n}\alpha_k\beta_k I-\sum_{k=1}^{n}(\alpha_k+\beta_k)c(\lambda_k)+\sum_{m=1}^{n}\sum_{\substack{k=1\\k\neq m}}^{n}\alpha_m\beta_k c(\lambda_m-\lambda_k)\\&=\left(1+\sum_{k=1}^{n}\alpha_k\beta_k\right)\left(I-C_n\right),\end{aligned}$$

where

$$C_n=\frac{1}{1+\sum_{k=1}^{n}\alpha_k\beta_k}\left(\sum_{k=1}^{n}(\alpha_k+\beta_k)c(\lambda_k)-\sum_{m=1}^{n}\sum_{\substack{k=1\\k\neq m}}^{n}\alpha_m\beta_k c(\lambda_m-\lambda_k)\right).$$

Using the triangle inequality and estimate (2.1), we obtain

$$\begin{aligned}\|C_n\|_{H\to H}\le\frac{1}{\left|1+\sum_{k=1}^{n}\alpha_k\beta_k\right|}&\left[\sum_{k=1}^{n}|\alpha_k+\beta_k|\,\|c(\lambda_k)\|_{H\to H}\right.\\&\left.+\sum_{m=1}^{n}\sum_{\substack{k=1\\k\neq m}}^{n}|\alpha_m||\beta_k|\,\|c(\lambda_m-\lambda_k)\|_{H\to H}\right]\le q,\end{aligned}$$

where

$$q=\frac{1}{\left|1+\sum\limits_{k=1}^{n}\alpha_k\beta_k\right|}\left[\sum_{k=1}^{n}|\alpha_k+\beta_k|+\sum_{m=1}^{n}\sum_{\substack{k=1\\k\neq m}}^{n}|\alpha_m||\beta_k|\right].$$

Since $q<1,$ the operator $I-C_n$ has a bounded inverse and

$$\left\|(I-C_n)^{-1}\right\|_{H\to H}\leq\frac{1}{1-q}.$$

Therefore, from that it follows $(I-B_n)^{-1}$ exists and

$$\begin{aligned}\left\|(I-B_n)^{-1}\right\|_{H\to H}&\leq\frac{1}{\left|1+\sum\limits_{k=1}^{n}\alpha_k\beta_k\right|}\frac{1}{1-q}\\&=\frac{1}{|1+\sum\limits_{k=1}^{n}\alpha_k\beta_k|-\sum\limits_{k=1}^{n}|\alpha_k+\beta_k|-\sum\limits_{m=1}^{n}|\alpha_m|\sum\limits_{\substack{k=1\\k\neq m}}^{n}|\beta_k|}.\end{aligned}$$

Lemma 2.2 is proved.

Now, we will obtain the formula for solution of the problem (1.1). It is clear that (see [1]) the initial value problem

$$\frac{d^2u}{dt^2}+Au(t)=f(t),0<t<1,u(0)=u_0,u'(0)=u'_0$$

has a unique solution

$$u(t)=c(t)u_0+s(t)u'_0+\int_0^t s(t-s)f(s)ds. \tag{2.3}$$

Using (2.3) and the nonlocal boundary conditions

$$u(0)=\sum_{m=1}^{n}\alpha_m u(\lambda_m)+\varphi,u^{'}(0)=\sum_{k=1}^{n}\beta_k u^{'}(\lambda_k)+\psi,$$

it can be written as follows

$$\begin{cases}u(0)=\sum\limits_{m=1}^{n}\alpha_m\left\{c(\lambda_m)u(0)+s(\lambda_m)u^{'}(0)+\int\limits_0^{\lambda_m}s\left(\lambda_m-s\right)f(s)ds\right\}+\varphi,\\u^{'}(0)=\sum\limits_{k=1}^{n}\beta_k\left\{-As(\lambda_k)u(0)+c(\lambda_k)u^{'}(0)+\int\limits_0^{\lambda_k}c\left(\lambda_k-s\right)f(s)ds\right\}+\psi.\end{cases} \tag{2.4}$$

Denoting

$$\Delta = \begin{vmatrix} I-\sum\limits_{m=1}^{n}\alpha_m c(\lambda_m) & -\sum\limits_{m=1}^{n}\alpha_m s(\lambda_m) \\ \sum\limits_{k=1}^{n}\beta_k As(\lambda_k) & I-\sum\limits_{k=1}^{n}\beta_k c(\lambda_k) \end{vmatrix},$$

and using the definitions of $c\left(\lambda_j\right)$ and $s\left(\lambda_j\right)$ $(\lambda_j, j=1,\cdots,n)$, we can write

$$\Delta=\left(I-\sum_{m=1}^{n}\alpha_m c(\lambda_m)\right)\left(I-\sum_{k=1}^{n}\beta_k c(\lambda_k)\right)+A\sum_{m=1}^{n}\sum_{k=1}^{n}\alpha_m\beta_k s(\lambda_m)s(\lambda_k)=I-B_n$$

Then, using the definition of the operator T, we obtain

$$T=\Delta^{-1}.$$

Solving system (2.4), we get

$$u(0)=T\begin{vmatrix} \sum\limits_{m=1}^{n}\alpha_m\int\limits_0^{\lambda_m}s\left(\lambda_m-s\right)f(s)ds+\varphi & -\sum\limits_{m=1}^{n}\alpha_m s(\lambda_m) \\ \sum\limits_{k=1}^{n}\beta_k\int\limits_0^{\lambda_k}c\left(\lambda_k-s\right)f(s)ds+\psi & I-\sum\limits_{k=1}^{n}\beta_k c(\lambda_k) \end{vmatrix} \tag{2.5}$$

$$=T\left\{\left(I-\sum_{k=1}^{n}\beta_k c(\lambda_k)\right)\left(\sum_{m=1}^{n}\alpha_m\int\limits_0^{\lambda_m}s\left(\lambda_m-s\right)f(s)ds+\varphi\right)\right.$$
$$\left.+\sum_{m=1}^{n}\alpha_m s(\lambda_m)\left(\sum_{k=1}^{n}\beta_k\int\limits_0^{\lambda_k}c\left(\lambda_k-s\right)f(s)ds+\psi\right)\right\},$$

$$u^{'}(0)=T\begin{vmatrix} I-\sum\limits_{m=1}^{n}\alpha_m c(\lambda_m) & \sum\limits_{m=1}^{n}\alpha_m\int\limits_0^{\lambda_m}s\left(\lambda_m-s\right)f(s)ds+\varphi \\ \sum\limits_{k=1}^{n}\beta_k As(\lambda_k) & \sum\limits_{k=1}^{n}\beta_k\int\limits_0^{\lambda_k}c\left(\lambda_k-s\right)f(s)ds+\psi \end{vmatrix} \tag{2.6}$$

$$=T\left\{\left(I-\sum_{m=1}^{n}\alpha_m c(\lambda_m)\right)\left(\sum_{k=1}^{n}\beta_k\int\limits_0^{\lambda_k}c\left(\lambda_k-s\right)f(s)ds+\psi\right)\right.$$

$$-A\sum_{k=1}^{n}\beta_k s(\lambda_k)\left(\sum_{m=1}^{n}\alpha_m\int_0^{\lambda_m}s\left(\lambda_m-s\right)f(s)ds+\varphi\right)\Bigg\}.$$

Consequently, if the function $f(t)$ is not only continuous, but also continuously differentiable on [0,1], $\varphi\in D(A)$, $\psi\in D(A^{\frac{1}{2}})$ and formulas (2.3), (2.5), (2.6) give a solution of the problem (1.1).

Theorem 2.1. *Suppose that $\varphi\in D(A)$, $\psi\in D(A^{\frac{1}{2}})$ and $f(t)$ is continuously differentiable function on $[0,1]$ and the assumption (1.2) holds. Then, there is a unique solution of problem (1.1) and the stability inequalities*

(2.7) $$\max_{0\le t\le 1}\parallel u(t)\parallel_H\le M\left[\parallel\varphi\parallel_H+\parallel A^{-1/2}\psi\parallel_H+\max_{0\le t\le 1}\parallel A^{-1/2}f(t)\parallel_H\right],$$

(2.8) $$\max_{0\le t\le 1}\parallel A^{1/2}u(t)\parallel_H\le M\left[\parallel A^{1/2}\varphi\parallel_H+\parallel\psi\parallel_H+\max_{0\le t\le 1}\parallel f(t)\parallel_H\right],$$

(2.9) $$\max_{0\le t\le 1}\parallel\frac{d^2u(t)}{dt^2}\parallel_H+\max_{0\le t\le 1}\parallel Au(t)\parallel_H\le M[\parallel A\varphi\parallel_H+\parallel A^{1/2}\psi\parallel_H$$
$$+\parallel f(0)\parallel_H+\int_0^1\parallel f'(t)\parallel_H dt]$$

hold, where M does not depend on φ,ψ and $f(t),\ t\in[0,1]$.

Proof. Using formula (1.1) and estimates (2.1), (2.2), we obtain

$$\begin{aligned}\|u(t)\|_H &\le \|c(t)\|_{H\to H}\|T\|_{H\to H}\Bigg\{\left(1+\sum_{k=1}^{n}|\beta_k|\,\|c(\lambda_k)\|_{H\to H}\right)\sum_{m=1}^{n}|\alpha_m|\\
&\times\Bigg(\int_0^{\lambda_m}\left\|A^{\frac{1}{2}}s\left(\lambda_m-s\right)\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f(s)\right\|_H ds\\
&+\|\varphi\|_H)+\sum_{m=1}^{n}|\alpha_m|\left\|A^{\frac{1}{2}}s(\lambda_m)\right\|_{H\to H}\\
&\times\left(\sum_{k=1}^{n}|\beta_k|\int_0^{\lambda_k}\|c\left(\lambda_k-s\right)\|_{H\to H}\left\|A^{-\frac{1}{2}}f(s)\right\|_H ds+\|A^{-\frac{1}{2}}\psi\|_H\right)\Bigg\}\\
&+\left\|A^{\frac{1}{2}}s(t)\right\|_{H\to H}\|T\|_{H\to H}\Bigg\{\left(1+\sum_{m=1}^{n}|\alpha_m|\,\|c(\lambda_m)\|_{H\to H}\right)\end{aligned}$$

$$
\begin{aligned}
&\times\left(\sum_{k=1}^{n}|\beta_k|\int_0^{\lambda_k}\|c(\lambda_k-s)\|_{H\to H}\left\|A^{-\frac{1}{2}}f(s)\right\|_H ds+\|A^{-\frac{1}{2}}\psi\|_H\right)\\
&+\left(\sum_{k=1}^{n}|\beta_k|\left\|A^{\frac{1}{2}}s(\lambda_k)\right\|_{H\to H}\right)\left(\sum_{m=1}^{n}|\alpha_m|\int_0^{\lambda_m}\left\|A^{\frac{1}{2}}s(\lambda_m-s)\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f(s)\right\|_H ds\right.\\
&\left.+\|\varphi\|_H)\right\}+\int_0^t\left\|A^{\frac{1}{2}}s(t-s)\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f(s)\right\|_H ds\\
&\le M\left[\|\varphi\|_H+\|A^{-1/2}\psi\|_H+\max_{0\le t\le 1}\|A^{-1/2}f(t)\|_H\right].
\end{aligned}
$$

for every $t, 0 \le t \le 1$.Therefore, estimate (2.7) is proved.

Applying $A^{\frac{1}{2}}$ to formula (1.1) and using estimates (2.1) and (2.2), in a similar manner one establishes estimate (2.8).

Now, we obtain the estimate for $\| Au(t) \|_H$. Using the integration by parts and applying A to formula (1.1), we can write the formula

$$
\begin{aligned}
Au(t)&=c(t)T\left\{\left(I-\sum_{k=1}^{n}\beta_k c(\lambda_k)\right)\right.\\
&\times\left(\sum_{m=1}^{n}\alpha_m\left(f(\lambda_m)-c(\lambda_m)f(0)-\int_0^{\lambda_m}c(\lambda_m-s)f'(s)ds\right)+A\varphi\right)\\
&\left.+\sum_{m=1}^{n}\alpha_m As(\lambda_m)\left(\sum_{k=1}^{n}\beta_k\left(s(\lambda_k)f(0)+\int_0^{\lambda_k}s(\lambda_k-s)f'(s)ds\right)+\psi\right)\right\}\\
&+As(t)T\left\{\left(I-\sum_{m=1}^{n}\alpha_m c(\lambda_m)\right)\right.\\
&\times\left(\sum_{k=1}^{n}\beta_k\left(s(\lambda_k)f(0)+\int_0^{\lambda_k}s(\lambda_k-s)f'(s)ds\right)+\psi\right)-\left(\sum_{k=1}^{n}\beta_k s(\lambda_k)\right)\\
&\left.\times\left(\sum_{m=1}^{n}\alpha_m\left(f(\lambda_m)-c(\lambda_m)f(0)-\int_0^{\lambda_m}c(\lambda_m-s)f'(s)ds\right)+A\varphi\right)\right\}\\
&+f(t)-c(t)f(0)-\int_0^t c(t-s)f'(s)ds.
\end{aligned}
\tag{2.10}
$$

Using formula (2.10) and estimates (2.1) and (2.2), we get

$$
\begin{aligned}
&\|Au(t)\|_H \le \|c(t)\|_{H\to H}\|T\|_{H\to H}\left\{\left(1+\sum_{k=1}^{n}|\beta_k|\,\|c(\lambda_k)\|_{H\to H}\right)\right.\\
&\times\left(\sum_{m=1}^{n}|\alpha_m|\left(\|f(\lambda_m)\|_H+\|c(\lambda_m)\|_{H\to H}\|f(0)\|_H\right.\right.\\
&\left.\left.+\int_0^{\lambda_m}\|c(\lambda_m-s)\|_{H\to H}\left\|f^{'}(s)\right\|_H ds\right)+\|A\varphi\|_H\right)\\
&+\sum_{m=1}^{n}|\alpha_m|\left\|A^{\frac12}s(\lambda_m)\right\|_{H\to H}\left(\left(\sum_{k=1}^{n}|\beta_k|\left(\left\|A^{\frac12}s(\lambda_k)\right\|_{H\to H}\|f(0)\|_H\right.\right.\right.\\
&\left.\left.+\int_0^{\lambda_k}\left\|A^{\frac12}s(\lambda_k-s)\right\|_{H\to H}\left\|f^{'}(s)\right\|_H ds\right)\right)\\
&\left.\left.+\|A^{\frac12}\psi\|_H\right)\right\}+\left\|A^{\frac12}s(t)\right\|_{H\to H}\|T\|_{H\to H}\left\{\left(1+\sum_{m=1}^{n}|\alpha_m|\,\|c(\lambda_m)\|_{H\to H}\right)\right.\\
&\times\left(\left(\sum_{k=1}^{n}|\beta_k|\left(\left\|A^{\frac12}s(\lambda_k)\right\|_{H\to H}\|f(0)\|_H+\int_0^{\lambda_k}\left\|A^{\frac12}s(\lambda_k-s)\right\|_{H\to H}\left\|f^{'}(s)\right\|_H ds\right)\right.\right.\\
&\left.+\|A^{\frac12}\psi\|_H\right)+\left(\sum_{k=1}^{n}|\beta_k|\left\|A^{\frac12}s(\lambda_k)\right\|_{H\to H}\right)\\
&\times\left(\sum_{m=1}^{n}|\alpha_m|\left(\|f(\lambda_m)\|_H+\|c(\lambda_m)\|_{H\to H}\|f(0)\|_H\right.\right.\\
&\left.\left.\left.+\int_0^{\lambda_m}\|c(\lambda_m-s)\|_{H\to H}\left\|f^{'}(s)\right\|_H ds\right)+\|A\varphi\|_H\right)\right\}\\
&+\|f(t)\|_H+\|c(t)\|_{H\to H}\|f(0)\|_H+\int_0^{t}\|c(t-s)\|_{H\to H}\left\|f^{'}(s)\right\|_H ds\\
&\le M\left[\|A\varphi\|_H+\|A^{\frac12}\psi\|_H+\|f(0)\|_H+\int_0^{t}\|f^{'}(s)\|_H\,ds\right]
\end{aligned}
$$

for every t, $0\le t\le 1$. This shows that

$$
\begin{aligned}
&\max_{0\le t\le 1}\|Au(t)\|_H\\
&\le M\left[\|A\varphi\|_H+\|A^{1/2}\psi\|_H+\|f(0)\|_H+\max_{0\le t\le 1}\|f'(t)\|_H\right].
\end{aligned}
\tag{2.11}
$$

From estimate (2.11) and the triangle inequality it follows estimate (2.9). Theorem 2.1 is proved.

Now, we will consider the applications of Theorem 2.1.

First, the mixed problem for hyperbolic equation

$$\begin{cases} u_{tt}-(a(x)u_x)_x+\delta u=f(t,x),\ 0<t<1,\ 0<x<1, \\ u(0,x)=\sum\limits_{m=1}^{n}\alpha_m u(\lambda_m,x)+\varphi(x), 0\le x\le 1, \\ u_t(0,x)=\sum\limits_{k=1}^{n}\beta_k u_t(\lambda_k,x)+\psi(x), 0\le x\le 1, \\ u(t,0)=u(t,1),\ u_x(t,0)=u_x(t,1),\ 0\le t\le 1 \end{cases} \tag{2.12}$$

under assumption (1.2) is considered. The problem (2.12) has a unique smooth solution $u(t,x)$ for (1.2), $\delta>0$ and the smooth functions $a(x)\geq a>0 \ \ (x\in(0,1))$, $\varphi(x),\psi(x)(x\in[0,1])$ and $f(t,x) \ \ (t,x\in[0,1])$. This allows us to reduce the mixed problem (2.12) to the nonlocal boundary value problem (1.1) in a Hilbert space $H=L_2[0,1]$ with a self-adjoint positive definite operator A^x defined by (2.12).

Theorem 2.2. *For solutions of the mixed problem (2.12), we have the following stability inequalities*

$$\max_{0\le t\le 1}\parallel u_x(t,\cdot)\parallel_{L_2[0,1]}\le M\left[\max_{0\le t\le 1}\parallel f(t,\cdot)\parallel_{L_2[0,1]}+\parallel \varphi_x\parallel_{L_2[0,1]}+\parallel\psi\parallel_{L_2[0,1]}\right],$$

$$\max_{0\le t\le 1}\parallel u_{xx}(t,\cdot)\parallel_{L_2[0,1]}+\max_{0\le t\le 1}\parallel u_{tt}(t,\cdot)\parallel_{L_2[0,1]}$$

$$\le M\left[\max_{0\le t\le 1}\parallel f_t(t,\cdot)\parallel_{L_2[0,1]}+\parallel f(0,\cdot)\parallel_{L_2[0,1]}+\parallel\varphi_{xx}\parallel_{L_2[0,1]}+\parallel\psi_x\parallel_{L_2[0,1]}\right],$$

where M does not depend on $\varphi(x),\psi(x)$ and $f(t,x)$.

The proof of Theorem 2.2 is based on the abstract Theorem 2.1 and the symmetry properties of the operator A^x defined by formula (2.12).

Second, let Ω be the unit open cube in the m-dimensional Euclidean space $\mathbb{R}^m \{x=(x_1,\cdots,x_m):$
$0<x_j<1, 1\le j\le m\}$ with boundary S, $\overline{\Omega}=\Omega\cup S$. In $[0,1]\times\Omega$, the mixed boundary value problem for the multi-dimensional hyperbolic equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2}-\sum\limits_{r=1}^{m}(a_r(x)u_{x_r})_{x_r}=f(t,x), \\ x=(x_1,\ldots,x_m)\in\Omega,\quad 0<t<1, \\ u(0,x)=\sum\limits_{j=1}^{n}\alpha_j u(\lambda_j,x)+\varphi(x), x\in\overline{\Omega}, \\ u_t(0,x)=\sum\limits_{k=1}^{n}\beta_k u_t(\lambda_k,x)+\psi(x), x\in\overline{\Omega}, \\ u(t,x)=0,\quad x\in S \end{cases} \tag{2.13}$$

under assumption (1.2) is considered. Here $a_r(x)$, $(x \in \Omega)$, $\varphi(x), \psi(x)$ $(x \in \overline{\Omega})$ and $f(t,x)$ $(t \in (0,1),\ x \in \Omega)$ are given smooth functions and $a_r(x) \geq a > 0$.

We introduce the Hilbert space $L_2(\overline{\Omega})$ of the all square integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\| f \|_{L_2(\overline{\Omega})} = \{ \int \cdots \int_{x \in \overline{\Omega}} |f(x)|^2 dx_1 \cdots dx_m \}^{\frac{1}{2}}.$$

The problem (2.13) has a unique smooth solution $u(t,x)$ for (1.2) and the smooth functions $\varphi(x), \psi(x), a_r(x)$ and $f(t,x)$. This allows us to reduce the mixed problem (2.13) to the nonlocal boundary value problem (1.1) in a Hilbert space $H = L_2(\overline{\Omega})$ with a self-adjoint positive definite operator A^x defined by (2.13).

Theorem 2.3. *For the solutions of the mixed problem (2.13), the following stability inequalities*

$$\max_{0 \leq t \leq 1} \sum_{r=1}^{m} \| u_{x_r}(t,\cdot) \|_{L_2(\overline{\Omega})}$$
$$\leq M \left[\max_{0 \leq t \leq 1} \| f(t,\cdot) \|_{L_2(\overline{\Omega})} + \sum_{r=1}^{m} \| \varphi_{x_r} \|_{L_2(\overline{\Omega})} + \| \psi \|_{L_2(\overline{\Omega})} \right],$$
$$\max_{0 \leq t \leq 1} \sum_{r=1}^{m} \| u_{x_r x_r}(t,\cdot) \|_{L_2(\overline{\Omega})} + \max_{0 \leq t \leq 1} \| u_{tt}(t,\cdot) \|_{L_2(\overline{\Omega})}$$
$$\leq M \left[\max_{0 \leq t \leq 1} \| f_t(t,\cdot) \|_{L_2(\overline{\Omega})} \right.$$
$$\left. + \| f(0,\cdot) \|_{L_2(\overline{\Omega})} + \sum_{r=1}^{m} \| \varphi_{x_r x_r} \|_{L_2(\overline{\Omega})} + \sum_{r=1}^{m} \| \psi_{x_r} \|_{L_2(\overline{\Omega})} \right]$$

hold, where M does not depend on $\varphi(x)$, $\psi(x)$ and $f(t,x)$.

The proof of Theorem 2.3 is based on the abstract Theorem 2.1, the symmetry properties of the operator A^x defined by formula (2.13) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 2.4. *For the solutions of the elliptic differential problem*

$$A^x u(x) = \omega(x), x \in \Omega, \quad (2.14)$$
$$u(x) = 0, x \in S,$$

the following coercivity inequality holds [3]*:*

$$\sum_{r=1}^{m} \|u_{x_r x_r}\|_{L_2(\overline{\Omega})} \leq M ||\omega||_{L_2(\overline{\Omega})}.$$

3. The First Order of Accuracy Difference Schemes

Throughout this paper for simplicity $\lambda_1 > 2\tau$ and $\lambda_n < 1$ will be considered. Let us associate the boundary value problem (1.1) with the corresponding first order of accuracy difference scheme

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_{k+1} = f_k, \ f_k = f(t_k), \\ t_k = k\tau, \ 1 \leq k \leq N-1, \ N\tau = 1; u_0 = \sum\limits_{r=1}^{n} \alpha_r u_{\left[\frac{\lambda_r}{\tau}\right]} + \varphi, \\ \tau^{-1}(u_1 - u_0) = \sum\limits_{r=1}^{n} \beta_r \left(u_{\left[\frac{\lambda_r}{\tau}\right]+1} - u_{\left[\frac{\lambda_r}{\tau}\right]} \right) \frac{1}{\tau} + \psi. \end{cases} \tag{3.1}$$

A study of discretization, over time only, of the nonlocal boundary value problem also permits one to include general difference schemes in applications, if the differential operator in space variables, A, is replaced by the difference operators A_h that act in the Hilbert spaces and are uniformly self-adjoint positive definite in h for $0 < h \leq h_0$.

In general, we have not been able to obtain the stability estimates for the solution of difference scheme (3.1) under assumption (1.2). Note that the stability of solutions of the difference scheme (3.1) will be obtained under the strong assumption

$$\sum_{k=1}^{n} |\alpha_k| + \sum_{k=1}^{n} |\beta_k| + \sum_{k=1}^{n} |\alpha_k| \sum_{k=1}^{n} |\beta_k| < 1. \tag{3.2}$$

Throughout this section for simplicity we put

$$\begin{aligned} B_n^\tau &= \sum_{k=1}^{n} \beta_k \frac{1}{2} \left(R^{\left[\frac{\lambda_k}{\tau}\right]+1} + \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1} \right) + \sum_{m=1}^{n} \alpha_m \frac{1}{2} \left(R^{\left[\frac{\lambda_m}{\tau}\right]-1} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1} \right) \\ &\quad - \frac{1}{4} \sum_{m=1}^{n} \sum_{k=1}^{n} \alpha_m \beta_k \left(R^{\left[\frac{\lambda_m}{\tau}\right]-1} \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1} R^{\left[\frac{\lambda k}{\tau}\right]+1} \right. \\ &\quad \left. + R^{\left[\frac{\lambda_m}{\tau}\right]} \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]} R^{\left[\frac{\lambda k}{\tau}\right]} \right). \end{aligned}$$

Now, let us give some lemmas that will be needed below.

Lemma 3.1. The following estimates hold:

$$\begin{cases} \|R\|_{H \to H} \leq 1, \ \|\tilde{R}\|_{H \to H} \leq 1, \\ \|\tilde{R}^{-1} R\|_{H \to H} \leq 1, \ \|R^{-1} \tilde{R}\|_{H \to H} \leq 1, \\ \|\tau A^{1/2} R\|_{H \to H} \leq 1, \|\tau A^{1/2} \tilde{R}\|_{H \to H} \leq 1. \end{cases} \tag{3.3}$$

Here and in future $R=\left(I+i\tau A^{1/2}\right)^{-1}$, $\tilde{R}=\left(I-i\tau A^{1/2}\right)^{-1}$.

Lemma 3.2. Suppose that the assumption (3.2) holds. Then, the operator $I-B_n^\tau$ has an inverse $T_\tau=(I-B_n^\tau)^{-1}$and the following estimate is satisfied:

$$\| T_\tau \|_{H\to H}\leq \frac{1}{1-\sum_{k=1}^{n}|\alpha_k|-\sum_{k=1}^{n}|\beta_k|-\sum_{k=1}^{n}|\alpha_k|\sum_{k=1}^{n}|\beta_k|}. \tag{3.4}$$

Proof. Using the definitions of $B_n^\tau, R,$ $\tilde{R}$ and the triangle inequality and estimate (3.3), we obtain

$$\begin{aligned}\|B_n^\tau\|_{H\to H} \leq & \sum_{k=1}^{n}|\beta_k|\frac{1}{2}\left\|R^{\left[\frac{\lambda_k}{\tau}\right]+1}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1}\right\|_{H\to H} \\ &+\sum_{m=1}^{n}|\alpha_m|\frac{1}{2}\left\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\right\|_{H\to H} \\ &+\sum_{m=1}^{n}\sum_{k=1}^{n}\frac{1}{4}|\alpha_m||\beta_k|\left\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}R^{\left[\frac{\lambda_k}{\tau}\right]+1}\right. \\ &\left.+R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]}\right\|_{H\to H}\leq q,\end{aligned}$$

where

$$q=\sum_{k=1}^{n}|\alpha_k|+\sum_{k=1}^{n}|\beta_k|+\sum_{k=1}^{n}|\alpha_k|\sum_{k=1}^{n}|\beta_k|.$$

Since $q<1$, the operator $I-B_n^\tau$ has a bounded inverse and

$$\left\|(I-B_n^\tau)^{-1}\right\|_{H\to H}\leq\frac{1}{1-q}=\frac{1}{1-\sum_{k=1}^{n}|\alpha_k|-\sum_{k=1}^{n}|\beta_k|-\sum_{k=1}^{n}|\alpha_k|\sum_{k=1}^{n}|\beta_k|}.$$

Lemma 3.2 is proved.

Remark 1. Note that the operator function

$$\frac{1}{4}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}R^{\left[\frac{\lambda_k}{\tau}\right]+1}+R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]}\right)$$

is the approximation of $c(\lambda_m-\lambda_k)$. By the definition of $c(t): c(\lambda_m-\lambda_k)=I$ for $m=k.$ It is clear that

$$\frac{1}{4}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}R^{\left[\frac{\lambda_k}{\tau}\right]+1}+R^{\left[\frac{\lambda_m}{\tau}\right]}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}R^{\left[\frac{\lambda_k}{\tau}\right]}\right)$$

$$=R^{\left[\frac{\lambda_k}{\tau}\right]+1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1}$$

for $m = k$. Since $R^{\left[\frac{\lambda_k}{\tau}\right]+1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1} \neq I$,we can not obtain the statements of Lemma 3.2 and later the stability estimates for the solution of difference scheme (3.1) under assumption (1.2).

Now, we will obtain the formula for the solution of problem (3.1). It is clear that the first order of accuracy difference scheme

$$\begin{cases} \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_{k+1} = f_k, \\ \quad f_k = f(t_{k+1}),\ t_{k+1} = (k+1)\tau,\ 1 \le k \le N-1,\ N\tau = 1, \\ u_0 = \mu,\ \tau^{-1}(u_1 - u_0) = \omega \end{cases}$$

has a solution and the following formula holds:

$$\begin{aligned} & u_0 = \mu,\ u_1 = \mu + \tau\omega, \\ & u_k = \frac{1}{2}\left[R^{k-1} + \tilde{R}^{k-1}\right]\mu + (R - \tilde{R})^{-1}\tau(R^k - \tilde{R}^k)\omega \\ & - \sum_{s=1}^{k-1} \frac{\tau}{2i} A^{-1/2}\left[R^{k-s} - \tilde{R}^{k-s}\right] f_s,\ 2 \le k \le N. \end{aligned} \tag{3.5}$$

Applying formula (3.5) and the nonlocal boundary conditions

$$u_0 = \sum_{m=1}^{n} \alpha_m u_{\left[\frac{\lambda_m}{\tau}\right]} + \varphi, \tau^{-1}(u_1 - u_0) = \sum_{k=1}^{n} \tau^{-1}\beta_k \left(u_{\left[\frac{\lambda_k}{\tau}\right]+1} - u_{\left[\frac{\lambda_k}{\tau}\right]}\right) + \psi,$$

we can write

$$\begin{aligned} \mu = & \sum_{m=1}^{n} \alpha_m \left\{ \frac{1}{2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-1} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\right)\mu + (R - \tilde{R})^{-1}\tau\left(R^{\left[\frac{\lambda_m}{\tau}\right]} - \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}\right)\omega \right. \\ & \left. - \sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1} \frac{\tau}{2i} A^{-1/2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-s} - \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\right) f_s \right\} + \varphi, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \omega = & \sum_{k=1}^{n} \tau^{-1}\beta_k \left\{ \frac{1}{2}\left(R^{\left[\frac{\lambda_k}{\tau}\right]} + \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]} - R^{\left[\frac{\lambda_k}{\tau}\right]-1} - \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\right) \right. \\ & \mu + (R - \tilde{R})^{-1}\tau\left(R^{\left[\frac{\lambda_k}{\tau}\right]+1} - \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1} - R^{\left[\frac{\lambda_k}{\tau}\right]} + \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right) \\ & \omega - \frac{\tau}{2i}A^{-1/2}\left[R - \tilde{R}\right] f_{\left[\frac{\lambda_k}{\tau}\right]} - \sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1} \\ & \left. \frac{\tau}{2i}A^{-1/2}\left(R^{\left[\frac{\lambda_k}{\tau}\right]+1-s} - \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s} - R^{\left[\frac{\lambda_k}{\tau}\right]-s} + \tilde{R}^{\left[\frac{\lambda_i}{\tau}\right]-s}\right) f_s \right\} + \psi. \end{aligned} \tag{3.7}$$

Using formulas (3.6) and (3.7), we obtain

$$
\begin{aligned}
\mu =& T_\tau\Bigg\{\Bigg(I-\sum_{k=1}^{n}\tau^{-1}\beta_k\Big((R-\tilde{R})^{-1}\tau(-i\tau A^{1/2})(R^{\left[\frac{\lambda_k}{\tau}\right]+1}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1})\Big)\Bigg)\\
&\times\left(\varphi-\sum_{m=1}^{n}\alpha_m\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{\tau}{2i}A^{-1/2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\right)f_s\right)\\
&+\left(\sum_{m=1}^{n}\alpha_m(R-\tilde{R})^{-1}\tau\left(R^{\left[\frac{\lambda_m}{\tau}\right]}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}\right)\right)\Bigg(\psi-\sum_{k=1}^{n}\tau^{-1}\beta_k\\
&\times\Bigg(\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{\tau}{2i}A^{-1/2}(-i\tau A^{1/2})(R^{\left[\frac{\lambda_k}{\tau}\right]+1-s}\\
&+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s})f_s-\tau^2R\tilde{R}f_{\left[\frac{\lambda_k}{\tau}\right]}\Bigg)\Bigg)\Bigg\},
\end{aligned}\tag{3.8}
$$

$$
\begin{aligned}
\omega =& T_\tau\left\{\left(I-\sum_{m=1}^{n}\alpha_m\frac{1}{2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-1}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\right)\right)\Bigg(\psi-\sum_{k=1}^{n}\tau^{-1}\beta_k\right.\\
&\times\left.\left(\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{\tau}{2i}A^{-\frac{1}{2}}\left(-i\tau A^{\frac{1}{2}}\right)(R^{\left[\frac{\lambda_k}{\tau}\right]+1-s}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s})f_s-\tau^2R\tilde{R}f_{\left[\frac{\lambda_k}{\tau}\right]}\right)\Bigg)\right.\\
&+\left(\sum_{k=1}^{n}\tau^{-1}\beta_k\frac{1}{2}\left(i\tau A^{1/2}\right)\left(-R^{\left[\frac{\lambda_k}{\tau}\right]}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)\right)\\
&\times\left.\left(\varphi-\sum_{m=1}^{n}\alpha_m\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{\tau}{2i}A^{-1/2}(R^{\left[\frac{\lambda_m}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s})f_s\right)\right\}.
\end{aligned}\tag{3.9}
$$

So, formulas (3.5), (3.8) and (3.9) give a solution of problem (3.1).

Theorem 3.1. *Suppose that the assumption (3.2) holds and* $\varphi\in D(A)$, $\psi\in D(A^{\frac{1}{2}})$. *Then, for the solution of the difference scheme (3.1) satisfy the following stability estimates*

$$
\begin{aligned}
&\|u_k\|_H\\
&\le M\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\|A^{-1/2}\psi\|_H+\|\varphi\|_H\}, k=0,2,\cdots,N,\\
&\|u_1\|_H\le M[\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\|\varphi\|_H+\|(I+i\tau A^{1/2})A^{-1/2}\psi\|_H],
\end{aligned}\tag{3.10}
$$

(3.11)
$$\begin{aligned}&\|A^{1/2}u_k\|_H \le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau + \|\ \psi\ \|_H + \|\ A^{1/2}\varphi\ \|_H\right\}, k=0,2,\cdots,N,\\ &\|A^{1/2}u_1\|_H \le M\left[\sum_{s=1}^{N-1}\|f_s\|_H\tau + \|\ A^{1/2}\varphi\ \|_H + \|\ (I+i\tau A^{1/2})\psi\ \|_H\right],\end{aligned}$$

(3.12)
$$\begin{aligned}&\|Au_k\|_H \le M\left\{\sum_{s=2}^{N-1}\|\ f_s - f_{s-1}\ \|_H]\right.\\ &\left.+\|\ f_1\ \|_H + \|\ A^{1/2}\psi\ \|_H + \|\ A\varphi\ \|_H\right\}, k=0,2,\cdots,N,\\ &\|Au_1\|_H \le M\left[\sum_{s=2}^{N-1}\|\ f_s - f_{s-1}\ \|_H\right]\\ &+\|\ f_1\ \|_H + \|\ A\varphi\ \|_H + \|\ (I+i\tau A^{1/2})A^{1/2}\psi\ \|_H]\end{aligned}$$

hold, where M does not depend on τ, φ, ψ and f_s, $1\le s\le N-1$.

Proof. Using formulas(3.8), (3.9) and estimates (3.3), (3.4), we obtain

(3.13)
$$\begin{aligned}&\|\mu\|_H\\ &\le \|T_\tau\|_{H\to H}\left\{\left(1+\sum_{k=1}^{n}|\beta_k|\frac{1}{2}(\|\tilde{R}^{-1}R^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H} + \|R^{-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}R^{-1}\|_{H\to H})\right)\right.\\ &\times\left(\|\varphi\|_H + \sum_{m=1}^{n}|\alpha_m|\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H} + \|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}\right)\|A^{-1/2}f_s\|_H\tau\right)\\ &+\left(\sum_{m=1}^{n}|\alpha_m|\left(\|\tilde{R}^{-1}R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H} + \|R^{-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-1}\|_{H\to H}\right)\right)\\ &\times\sum_{k=1}^{n}|\beta_k|\left(\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]+1-s}\|_{H\to H} + \|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s}\|_{H\to H}\right)\|A^{-1/2}f_s\|_H\tau\right.\\ &\left.\left.+\left\|R\tilde{R}\right\|_{H\to H}\|A^{-1/2}f_{\left[\frac{\lambda_k}{\tau}\right]}\|_H\tau + \|A^{-1/2}\psi\|_H\right)\right\}\\ &\le M\left\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau + \|\ A^{-1/2}\psi\ \|_H + \|\ \varphi\ \|_H\right\}.\end{aligned}$$

$$\|A^{-\frac{1}{2}}\omega\|_H \le \|T_\tau\|_{H\to H}\left\{\left(1+\sum_{m=1}^{n}|\alpha_m|\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H} + \|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}\right)\right)\right.$$
$$\times\left(\|A^{-1/2}\psi\|_H + \sum_{k=1}^{n}|\beta_k|\left(\left\|R\tilde{R}\right\|_{H\to H}\|A^{-1/2}f_{\left[\frac{\lambda_k}{\tau}\right]}\|_H\tau\right.\right.$$

$$
(3.14)\quad
\begin{aligned}
&+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]+1-s}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s}\|_{H\to H}\right)\|A^{-1/2}f_s\|_H\tau\Bigg)\\
&+\left(\sum_{k=1}^{n}|\beta_k|\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}\right)\right)(\|\varphi\|_H\\
&+\sum_{m=1}^{n}|\alpha_m|\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}\right)\|A^{-1/2}f_s\|_H\tau\Bigg)\Bigg\}\\
&\le M\left\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\|\ A^{-1/2}\psi\ \|_H+\|\ \varphi\ \|_H\right\}.
\end{aligned}
$$

Applying $A^{\frac{1}{2}}$ to formulas (3.8), (3.9) and using estimates (3.3) and (3.4), in a similar manner, we obtain

$$
(3.15)\qquad \|A^{1/2}\mu\|_H\le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|\ \psi\ \|_H+\|\ A^{1/2}\varphi\ \|_H\right\},
$$

$$
(3.16)\qquad \|\omega\|_H\le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|\ \psi\ \|_H+\|\ A^{1/2}\varphi\ \|_H\right\}.
$$

Now, we obtain the estimates for $\|\ A\mu\ \|_H$ and $\|A^{1/2}\omega\|_H$. First, applying A to formula (3.8) and using Abel's formula, we can write

$$
(3.17)\quad
\begin{aligned}
A\mu=T_\tau&\left\{\left(\left(I-\sum_{k=1}^{n}\tau^{-1}\beta_k\left((R-\tilde{R})^{-1}\tau(-i\tau A^{1/2})(R^{\left[\frac{\lambda_k}{\tau}\right]+1}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1})\right)\right)\right.\\
&\times\left(A\varphi-\sum_{m=1}^{n}\alpha_m\left[\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{1}{2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\right)(f_{s-1}-f_s)\right.\right.\\
&\left.\left.+\left(R^{\left[\frac{\lambda_m}{\tau}\right]-1}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\right)f_1-\left(R+\tilde{R}\right)f_{\left[\frac{\lambda_m}{\tau}\right]-1}\right]\right)\\
&+\left(\sum_{m=1}^{n}\alpha_m A^{1/2}\tau\left(R-\tilde{R}\right)^{-1}\left(R^{\left[\frac{\lambda_m}{\tau}\right]}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}\right)\right)\\
&\times\left(A^{1/2}\psi+\sum_{k=1}^{n}\beta_k i\left[\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\frac{1}{2}\left(R^{\left[\frac{\lambda_k}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\right)(f_{s+1}-f_s)\right.\right.\\
&\left.\left.\left.+\left(R^{\left[\frac{\lambda_k}{\tau}\right]}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)f_1-\left(R+\tilde{R}\right)f_{\left[\frac{\lambda_k}{\tau}\right]-1}-A^{1/2}\tau^2R\tilde{R}f_{\left[\frac{\lambda_k}{\tau}\right]}\right]\right)\right\}.
\end{aligned}
$$

Second, applying $A^{1/2}$ to formula (3.9) and using Abel's formula, we can write

$$
\begin{aligned}
A^{1/2}\omega = T_\tau \Bigg\{ & \left(\left(I - \sum_{m=1}^{n} \alpha_m \frac{1}{2} \left(R^{\left[\frac{\lambda_m}{\tau}\right]-1} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1} \right) \right. \right. \\
& \times \left(A^{1/2}\psi + \sum_{k=1}^{n} \beta_k i \left[\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2} \frac{1}{2} \left(R^{\left[\frac{\lambda_k}{\tau}\right]-s} - \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s} \right) (f_{s+1} - f_s) \right. \right. \\
& \left. \left. + \left(R^{\left[\frac{\lambda_k}{\tau}\right]} + \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]} \right) f_1 - \left(R + \tilde{R} \right) f_{\left[\frac{\lambda_k}{\tau}\right]-1} + A^{1/2}\tau^2 R\tilde{R} f_{\left[\frac{\lambda_k}{\tau}\right]} \right] \right) \\
& + \left(\sum_{k=1}^{n} \tau^{-1} \beta_k \frac{1}{2} i\tau \left(-R^{\left[\frac{\lambda_k}{\tau}\right]} + \tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]} \right) \right) \Bigg) \\
& \times \left(A\varphi - \sum_{m=1}^{n} \alpha_m \left[\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1} \frac{1}{2} \left(R^{\left[\frac{\lambda_m}{\tau}\right]-s} - \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s} \right) (f_{s-1} - f_s) \right. \right. \\
& \left. \left. + \left(R^{\left[\frac{\lambda_m}{\tau}\right]-1} + \tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1} \right) f_1 - \left(R + \tilde{R} \right) f_{\left[\frac{\lambda_m}{\tau}\right]-1} \right] \right) \Bigg\}.
\end{aligned} \tag{3.18}
$$

Using formulas (3.17), (3.18) and estimates (3.3), (3.4), we obtain

(3.19)

$$
\begin{aligned}
& \|A\mu\|_H \le \|T_\tau\|_{H\to H} \left\{ \left(1 + \sum_{k=1}^{n} |\beta_k| \frac{1}{2} \left(\|\tilde{R}^{-1} R^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H} + \|R^{-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1}\|_{H\to H} \right) \right) \right. \\
& \times \left(\|A\varphi\|_H + \sum_{m=1}^{n} |\alpha_m| \left[\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1} \frac{1}{2} \left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H} + \|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H} \right) \|f_s - f_{s-1}\|_H \right. \right. \\
& \left. \left. + \left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H} + \|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H} \right) \|f_1\|_H + \|R+\tilde{R}\|_{H\to H} \left\| f_{\left[\frac{\lambda_m}{\tau}\right]-1} \right\|_H \right] \right) \\
& + \left(\sum_{m=1}^{n} |\alpha_m| \frac{1}{2} \left(\|\tilde{R}^{-1} R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H} + \|R^{-1}\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}\|_{H\to H} \right) \right) \left(\left\|A^{1/2}\psi\right\|_H + \sum_{k=1}^{n} |\beta_k| \right. \\
& \times \left[\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2} \frac{1}{2} \left(\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\|_{H\to H} + \|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\|_{H\to H} \right) \|f_{s+1} - f_s\|_H + \|R+\tilde{R}\|_{H\to H} \left\| f_{\left[\frac{\lambda_k}{\tau}\right]-1} \right\|_H \right. \\
& \left. \left. \left. + \left(\|R^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H} + \|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H} \right) \|f_1\|_H + \|A^{1/2}\tau^2 R\tilde{R}\|_{H\to H} \left\| f_{\left[\frac{\lambda_k}{\tau}\right]} \right\|_H \right] \right) \right\} \\
& \le M \left\{ \sum_{s=2}^{N-1} \| f_s - f_{s-1} \|_H + \| f_1 \|_H + \| A^{1/2}\psi \|_H + \| A\varphi \|_H \right\},
\end{aligned}
$$

$$
\begin{aligned}
&\left\|A^{1/2}\omega\right\|_H \leq \|T_\tau\|_{H\to H}\Bigg\{\Bigg(1+\sum_{m=1}^{n}|\alpha_m|\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}\right)\Bigg)\\
&\times\left(\left\|A^{1/2}\psi\right\|_H+\sum_{k=1}^{n}|\beta_k|\times\left[\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]-s}\|_{H\to H}\right)\right.\right.\\
&\qquad \|f_{s+1}-f_s\|_H+\|R+\tilde{R}\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\\
&\left.\left.+\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}\right)\|f_1\|_H+\|A^{1/2}\tau^2R\tilde{R}\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\right]\right)\\
&+\left(\sum_{k=1}^{n}|\beta_k|\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}\right)\right)\left(\|A\varphi\|_H+\sum_{m=1}^{n}|\alpha_m|\right.\\
&\times\left[\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}\right)\|f_s-f_{s-1}\|_H\right.\\
&\left.\left.+\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}\right)\|f_1\|_H+\|R+\tilde{R}\|_{H\to H}\left\|f_{\left[\frac{\lambda_m}{\tau}\right]-1}\right\|_H\right]\right)\\
&\leq M\left\{\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H+\|A^{1/2}\psi\|_H+\|A\varphi\|_H\right\}.
\end{aligned}
\tag{3.20}
$$

Now, we will prove estimates (3.10), (3.11) and (3.12). Let $k\geq 2$. Then using formula (3.5) and estimates (3.3), (3.13), (3.14), (3.15) and (3.16), we obtain

$$
\begin{aligned}
&\|u_k\|_H\leq\frac{1}{2}\left[\|R^{k-1}\|_H+\|\tilde{R}^{k-1}\|_H\right]\|\mu\|_H+\frac{1}{2}(\|\tilde{R^{-1}}R^{k-1}\|_{H\to H}\\
&+\|R^{-1}\tilde{R}^{k-1}\|_{H\to H})\|A^{-\frac{1}{2}}\omega\|_H+\sum_{s=1}^{k-1}\frac{\tau}{2}\left[\|R^{k-s}\|_H+\|\tilde{R}^{k-s}\|_H\right]\|A^{-\frac{1}{2}}f_s\|_H\\
&\leq M\left\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\|A^{-1/2}\psi\|_H+\|\varphi\|_H\right\}.\\
&\|A^{\frac{1}{2}}u_k\|_H\leq\frac{1}{2}\left[\|R^{k-1}\|_H+\|\tilde{R}^{k-1}\|_H\right]\|A^{\frac{1}{2}}\mu\|_H+\frac{1}{2}(\|\tilde{R^{-1}}R^{k-1}\|_{H\to H}\\
&+\|R^{-1}\tilde{R}^{k-1}\|_{H\to H})\|\omega\|_H+\sum_{s=1}^{k-1}\frac{\tau}{2}\left[\|R^{k-s}\|_H+\|\tilde{R}^{k-s}\|_H\right]\|f_s\|_H\\
&\leq M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|\psi\|_H+\|A^{\frac{1}{2}}\varphi\|_H\right\}.
\end{aligned}
$$

Now, we obtain the estimates for $\|Au_k\|_H$ for $k\geq 2$. Applying A to formula (3.5) and using Abel's formula, we can write

$$Au_k = \frac{1}{2}\left[R^{k-1}+\tilde{R}^{k-1}\right]A\mu + (R-\tilde{R})^{-1}\tau(R^k-\tilde{R}^k)A\omega$$
$$+\frac{1}{2}\left(\sum_{s=2}^{k-1}\left[R^{k-s}+\tilde{R}^{k-s}\right](f_{s-1}-f_s)+2f_{k-1}-\left[R^{k-1}+\tilde{R}^{k-1}\right]f_1\right). \tag{3.21}$$

Using formula (3.21) and estimates (3.3), (3.19), (3.20), we obtain

$$\|Au_k\|_H \le \frac{1}{2}\left[\|R^{k-1}\|_{H\to H}+\|\tilde{R}^{k-1}\|_{H\to H}\right]\|A\mu\|_H$$
$$+\frac{1}{2}(\|\tilde{R^{-1}}R^{k-1}+R^{-1}\tilde{R}^{k-1})\|A^{\frac{1}{2}}\omega\|_H$$
$$+\frac{1}{2}\left(\sum_{s=2}^{k-1}\left(\left[\|R^{k-s}\|_{H\to H}+\|\tilde{R}^{k-s}\|_{H\to H}\right]\|f_{s-1}-f_s\|_H\right.\right.$$
$$\left.+2\|f_{k-1}\|_H+\left[\|R^{k-1}\|_{H\to H}+\|\tilde{R}^{k-1}\|_{H\to H}\|f_1\|_H\right)$$
$$\le M\left\{\sum_{s=2}^{N-1}\| f_s-f_{s-1}\|_H+\| f_1\|_H+\| A^{1/2}\psi\|_H+\| A\varphi\|_H\right\}.$$

Thus, estimates (3.10), (3.11), (3.12) for any $k\ge 2$ are obtained. From $u_0=\mu$ and (3.13), (3.15), (3.19) it follows estimates (3.10), (3.11) and (3.12) for $k=0$. Note that in a similar manner with estimates (3.14), (3.16), (3.20), (3.3) and (3.4), we obtain

$$\|\tau\omega\|_H \le \|\tau A^{1/2}R\|_{H\to H}\|T_\tau\|_{H\to H}$$
$$\times\left\{\left(1+\sum_{m=1}^{n}|\alpha_m|\frac{1}{2}(\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}+\|\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H})\right)\right.$$
$$\times\left(\|A^{-1/2}(I+i\tau A^{-1/2})\psi\|_H+\sum_{k=1}^{n}|\beta_k|\left(\|\tilde{R}\|_{H\to H}\|A^{-1/2}f_{\left[\frac{\lambda_k}{\tau}\right]}\|_H\tau\right.\right.$$
$$\left.\left.+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\|_{H\to H}+\|R^{-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s}\|_{H\to H}\right)\|A^{-1/2}f_s\|_H\tau\right)$$
$$+\left(\sum_{k=1}^{n}|\beta_k|\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\|_{H\to H}+\|R^{-1}\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}\right)\right)(\|\varphi\|_H \tag{3.22}$$
$$+\sum_{m=1}^{n}|\alpha_m|\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-s-1}\|_{H\to H}+\|R^{-1}\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}\right)$$
$$\left.\|A^{-1/2}f_s\|_H\tau\right)\right\}$$
$$\le M\left\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\| A^{-1/2}(I+i\tau A^{-1/2})\psi\|_H+\|\varphi\|_H\right\},$$

$$
\begin{aligned}
&\|\tau A^{1/2}\omega\|_H \le \|\tau A^{1/2}R\|_{H\to H}\|T_\tau\|_{H\to H}\\
&\times\Bigg\{\Bigg(1+\sum_{m=1}^{n}|\alpha_m|\,\frac{1}{2}(\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}\\
&+\|\tilde R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H})\Bigg)\times\Bigg(\|A(I+i\tau A^{-1/2})\psi\|_H+\sum_{k=1}^{n}|\beta_k|\,\Bigg(\|\tilde R\|_{H\to H}\|f_{\left[\frac{\lambda_k}{\tau}\right]}\|_H\tau\\
&+\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]-s}\|_{H\to H}+\|R^{-1}\tilde R^{\left[\frac{\lambda_k}{\tau}\right]+1-s}\|_{H\to H}\right)\|f_s\|_H\tau\Bigg)\\
&+\left(\sum_{k=1}^{n}|\beta_k|\,\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\|_{H\to H}+\|R^{-1}\tilde R^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}\right)\right)\left(\left\|A^{1/2}\varphi\right\|_H\right.\\
&+\sum_{m=1}^{n}|\alpha_m|\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-s-1}\|_{H\to H}+\|R^{-1}\tilde R^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}\right)\|f_s\|_H\tau\Bigg)\Bigg\}\\
&\le M\left\{\sum_{s=1}^{N-1}\|f_s\|_H\tau+\|\,(I+i\tau A^{-1/2})\psi\,\|_H+\|\,A^{\frac12}\varphi\,\|_H\right\},
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
&\|\tau A\omega\|_H \le \|\tau A^{1/2}R\|_{H\to H}\|T_\tau\|_{H\to H}\\
&\times\Bigg\{\Bigg(1+\sum_{m=1}^{n}|\alpha_m|\frac{1}{2}(\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}+\|\tilde R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H})\Bigg)\times\Big(\|A^{1/2}(I+i\tau A^{-1/2})\psi\|_H\\
&+\sum_{k=1}^{n}|\beta_k|\Bigg[\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-2}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]-s-1}\|_{H\to H}+\|R^{-1}\tilde R^{\left[\frac{\lambda_k}{\tau}\right]-s}\|_{H\to H}\right)\|f_{s+1}-f_s\|_H\\
&+\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\|_{H\to H}+\|R^{-1}\tilde R^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}\right)\|f_1\|_H+\|I+R^{-1}\tilde R\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]-1}\right\|_H\\
&+\|A^{1/2}\tau^2\tilde R\|_{H\to H}\left\|f_{\left[\frac{\lambda_k}{\tau}\right]}\right\|_H\Bigg]\Big)\\
&+\left(\sum_{k=1}^{n}|\beta_k|\,\frac{1}{2}\left(\|R^{\left[\frac{\lambda_k}{\tau}\right]-1}\|_{H\to H}+\|R^{-1}\tilde R^{\left[\frac{\lambda_k}{\tau}\right]}\|_{H\to H}\right)\right)\Big(\|A\varphi\|_H\\
&+\sum_{m=1}^{n}|\alpha_m|\Bigg[\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{1}{2}\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}+\|\tilde R^{\left[\frac{\lambda_m}{\tau}\right]-s}\|_{H\to H}\right)\|f_s-f_{s-1}\|_H\\
&+\left(\|R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}+\|\tilde R^{\left[\frac{\lambda_m}{\tau}\right]-1}\|_{H\to H}\right)\|f_1\|_H+\|R+\tilde R\|_{H\to H}\left\|f_{\left[\frac{\lambda_m}{\tau}\right]-1}\right\|_H\Bigg]\Big)\Bigg\}\\
&\le M\left\{\sum_{s=2}^{N-1}\|\,f_s-f_{s-1}\,\|_H+\|\,f_1\,\|_H+\|\,A^{1/2}(I+i\tau A^{-1/2})\psi\,\|_H+\|\,A\varphi\,\|_H\right\}.
\end{aligned}
\tag{3.24}
$$

Using the formula $u_1=\mu+\tau\omega$ and the triangle inequality and estimates (3.13), (3.15), (3.19), (3.22) and (3.23), we obtain estimates(3.10), (3.11), (3.12) for $k=1$. Theorem 3.1 is proved.

Remark 2. Note that stability estimates (3.10), (3.11) and (3.12) in the case $k=1$ are weaker than respective estimates in the cases $k=0,2,\cdots,N$. However, obtaining this type of estimate is important for applications.We denote by $a^\tau=\{a_k\}_{k=0}^N$ the mesh function of approximation. Then $\|(I+i\tau A^{-1/2})a_1\|_H\sim\|a_1\|_H\ =o(\tau)$ if we assume that $\tau\|Aa_1\|_H$ tends to 0 as $\tau\to 0$ not slower than $\|a_1\|_H$. It takes place in applications by supplementary restriction of the smooth property of the data of space variables. It is clear that the uniformity in τ estimate

$$\|u_1\|_H\leq M\left[\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\|\ A^{-1/2}\psi\ \|_H+\|\ \varphi\ \|_H\right]$$

is absent. However, estimates for the solution of first order of accuracy modified difference scheme for approximately solving the boundary value problem (1.1)

$$\begin{cases}\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_{k+1}=f_k,\ f_k=f(t_k),\\ t_k=k\tau,\ 1\leq k\leq N-1,\ N\tau=1;u_0=\displaystyle\sum_{m=1}^{n}\alpha_m u_{\left[\frac{\lambda_m}{\tau}\right]}+\varphi,\\ (I+\tau^2A)\tau^{-1}(u_1-u_0)=\displaystyle\sum_{k=1}^{n}\tau^{-1}\beta_k\left(u_{\left[\frac{\lambda_k}{\tau}\right]+1}-u_{\left[\frac{\lambda_k}{\tau}\right]}\right)+\psi\end{cases}\tag{3.25}$$

are better than the estimates for the solution of difference scheme (3.1).

Theorem 3.2. *Suppose that the assumption (3.2) holds and $\varphi\in D(A),\ \psi\in D(A^{\frac{1}{2}})$. Then, for the solution of the difference scheme (3.25) the stability inequalities*

$$\max_{0\leq k\leq N}\|u_k\|_H\leq M\{\sum_{s=1}^{N-1}\|A^{-1/2}f_s\|_H\tau+\|\ A^{-1/2}\psi\ \|_H+\|\ \varphi\ \|_H\},$$

$$\max_{0\leq k\leq N}\|A^{1/2}u_k\|_H\leq M\{\sum_{s=1}^{N-1}\|\ f_s\ \|_H\ \tau+\|\ A^{1/2}\varphi\|_H+\|\ \psi\|_H\},$$

$$\max_{1\leq k\leq N-1}\|\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\|_H+\max_{0\leq k\leq N}\|Au_k\ \|_H$$

$$\leq M\{\sum_{s=2}^{N-1}\|\ f_s-f_{s-1}\ \|_H+\|\ f_1\ \|_H+\|\ A^{1/2}\psi\ \|_H+\|\ A\varphi\ \|_H\}$$

hold, where M does not depend on τ,φ,ψ and $f_s, 1\leq s\leq N-1$.

The proof of Theorem 3.2 follows the scheme of the proof of Theorem 3.1 and it is based on the following formulas

$$u_0=\mu, u_1=\mu+\tau R\tilde{R}\omega,$$
$$u_k=\frac{1}{2}\left[R^{k-1}+\tilde{R}^{k-1}\right]\mu+(R-\tilde{R})^{-1}\tau(R^k-\tilde{R}^k)R\tilde{R}\omega$$
$$-\sum_{s=1}^{k-1}\frac{\tau}{2i}A^{-1/2}\left[R^{k-s}-\tilde{R}^{k-s}\right]f_s$$
$$=\frac{1}{2}\left[R^{k-1}+\tilde{R}^{k-1}\right]\mu+(R-\tilde{R})^{-1}\tau(R^k-\tilde{R}^k)R\tilde{R}\omega$$
$$+A^{-1}\frac{1}{2}\left(2f_{k-1}-\left[R^{k-1}+\tilde{R}^{k-1}\right]f_1\right)$$
$$+A^{-1}\frac{1}{2}\sum_{s=2}^{k-1}(\left[R^{k-s}+\tilde{R}^{k-s}\right](f_{s-1}-f_s), 2\le k\le N,$$
$$\mu=T_\tau\left\{\left(I-R\tilde{R}\sum_{k=1}^{n}\tau^{-1}\beta_k\left((R-\tilde{R})^{-1}\tau(-i\tau A^{1/2})(R^{\left[\frac{\lambda_k}{\tau}\right]+1}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1})\right)\right)\right.$$
$$\times\left(\varphi-\sum_{m=1}^{n}\alpha_m\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{\tau}{2i}A^{-1/2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s}\right)f_s\right)$$
$$+R\tilde{R}\left(\sum_{m=1}^{n}\alpha_m(R-\tilde{R})^{-1}\tau\left(R^{\left[\frac{\lambda_m}{\tau}\right]}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]}\right)\right)$$
$$\left.\times\left(\psi-\sum_{k=1}^{n}\tau^{-1}\beta_k\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{\tau}{2i}A^{-1/2}(-i\tau A^{1/2})(R^{\left[\frac{\lambda_k}{\tau}\right]+1-s}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s})f_s\right)\right\},$$
$$\omega=T_\tau R\tilde{R}\left\{\left(I-\sum_{m=1}^{n}\alpha_m\frac{1}{2}\left(R^{\left[\frac{\lambda_m}{\tau}\right]-1}+\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-1}\right)\right)\right.$$
$$\times\left(\psi-\sum_{k=1}^{n}\tau^{-1}\beta_k\sum_{s=1}^{\left[\frac{\lambda_k}{\tau}\right]-1}\frac{\tau}{2i}A^{-1/2}\left(-i\tau A^{1/2}\right)(R^{\left[\frac{\lambda_k}{\tau}\right]+1-s}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]+1-s})f_s\right)$$
$$+\left(\sum_{k=1}^{n}\tau^{-1}\beta_k\frac{1}{2}\left(i\tau A^{1/2}\right)\left(-R^{\left[\frac{\lambda_k}{\tau}\right]}+\tilde{R}^{\left[\frac{\lambda_k}{\tau}\right]}\right)\right)$$
$$\left.\times\left(\varphi-\sum_{m=1}^{n}\alpha_m\sum_{s=1}^{\left[\frac{\lambda_m}{\tau}\right]-1}\frac{\tau}{2i}A^{-1/2}(R^{\left[\frac{\lambda_m}{\tau}\right]-s}-\tilde{R}^{\left[\frac{\lambda_m}{\tau}\right]-s})f_s\right)\right\}$$

and on estimates (3.4) and (3.3).

Remark 3. Note that the estimates for the solution of the modified difference scheme (3.25) better than the estimates for the solution of difference scheme (3.1).

The stability estimates for the solutions of the second order of accuracy implicit difference schemes can be also obtained, unfortunaly, under the strong assumption than (3.2). Of course, stability statements could be also proved for the second order of accuracy explicit difference scheme under the assumption that the condition $\tau||A||_{H\to H}\to 0$ when $\tau\to 0$ is satisfied. In applications, this result permit us to obtain the stability estimates for the solutions of the difference scheme of the nonlocal boundary value problems for hyperbolic equations under the assumption that the magnitudes of the grid steps τ and h with respect to the time and space variables are connected.

Now, we consider the applications of Theorem 3.2.

First, the nonlocal boundary value problem (2.12) for one dimensional hyperbolic equation under assumption (3.2) is considered. The discretization of problem (2.12) is carried out in two steps. In the first step, let us define the grid space

$$[0,1]_h=\{x:x_r=rh,0\le r\le K,Kh=1\}.$$

We introduce the Hilbert space $L_{2h}=L_2\left([0,1]_h\right)$ of the grid functions $\varphi^h(x)=\{\varphi^r\}_1^{K-1}$ defined on $[0,1]_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_{2h}}=\left(\sum_{r=1}^{K-1}\left|\varphi^h(x)\right|^2 h\right)^{1/2}.$$

To the differential operator A generated by the problem (2.12), we assign the difference operator A_h^x by the formula

$$A_h^x\varphi^h(x)=\left\{-(a(x)\varphi_{\bar{x}})_{x,r}+\delta\varphi^r\right\}_1^{K-1}, \tag{3.26}$$

acting in the space of grid functions $\varphi^h(x)=\{\varphi^r\}_0^K$ satisfying the conditions $\varphi^0=\varphi^K$, $\varphi^1-\varphi^0=\varphi^K-\varphi^{K-1}$. With the help of A_h^x we arrive at the nonlocal boundary value problem

$$\begin{cases}\frac{d^2v^h(t,x)}{dt^2}+A_h^xv^h(t,x)=f^h(t,x),\ 0\le t\le 1,\ x\in[0,1]_h,\\ v^h(0,x)=\sum\limits_{j=1}^{n}\alpha_j v^h\left(\lambda_j,x\right)+\varphi^h(x),x\in[0,1]_h,\\ v_t^h(0,x)=\sum\limits_{j=1}^{n}\beta_j v_t^h(\lambda_j,x)+\psi^h(x),x\in[0,1]_h\end{cases} \tag{3.27}$$

for an infinite system of ordinary differential equations.

In the second step, we replace problem (3.27) by the difference scheme (3.28)

$$(3.28)\quad \begin{cases} \frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^x u_{k+1}^h=f_k^h(x),\ f_{k+1}^h(x)=f^h(t_{k+1},x_n),\\ t_{k+1}=(k+1)\tau, 1\le k\le N-1,\ N\tau=1, x\in[0,1]_h,\\ u_0^h(x)=\sum\limits_{j=1}^n \alpha_j u_{[\lambda_j/\tau]}^h(x)+\varphi^h(x), x\in[0,1]_h,\\ (I+\tau^2A_h^x)\frac{u_1^h(x)-u_0^h(x)}{\tau}=\sum\limits_{j=1}^n\beta_j\frac{u_{[\lambda_j/\tau]+1}^h(x)-u_{[\lambda_j/\tau]}^h(x)}{\tau}+\psi^h(x), x\in[0,1]_h. \end{cases}$$

Theorem 3.3. *Let τ and h be sufficiently small numbers. Suppose that the assumption* (3.2) *holds.Then, the solutions of the difference scheme* (3.28) *satisfy the following stability estimates:*

$$\max_{0\le k\le N}\left\|u_k^h\right\|_{L_{2h}}+\max_{0\le k\le N}\left\|\left(u_k^h\right)_x\right\|_{L_{2h}}$$
$$\le M_1\left[\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\varphi_{\bar{x}}^h\right\|_{L_{2h}}\right],$$
$$\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\|_{L_{2h}}+\max_{0\le k\le N}\left\|\left(u_k^h\right)_{\bar{x}x}\right\|_{L_{2h}}$$
$$\le M_1\left[\left\|f_1^h\right\|_{L_{2h}}+\max_{2\le k\le N-1}\left\|\tau^{-1}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}+\left\|\psi_{\bar{x}}^h\right\|_{L_{2h}}+\left\|\left(\varphi_{\bar{x}}^h\right)_x\right\|_{L_{2h}}\right].$$

Here M_1 does not depend on τ, h, $\varphi^h(x)$, $\psi^h(x)$ and $f_k^h, 1\le k<N$.

The proof of Theorem 3.3 is based on the abstract Theorem 3.2 and the symmetry properties of the operator A_h^x defined by (3.26).

Second, the nonlocal boundary value problem (2.13) for the m-dimensional hyperbolic equation under assumption (3.2) is considered. The discretization of problem (3.27) is carried out in two steps.

In the first step, let us define the grid sets

$$\widetilde{\Omega}_h=\{x=x_r=(h_1r_1,\cdots,h_mr_m), r=(r_1,\cdots,r_m),$$
$$0\le r_j\le N_j, h_jN_j=1, j=1,\cdots,m\}, \Omega_h=\widetilde{\Omega}_h\cap\Omega, S_h=\widetilde{\Omega}_h\cap S.$$

We introduce the Banach space $L_{2h}=L_2(\widetilde{\Omega}_h)$ of the grid functions $\varphi^h(x)=\{\varphi(h_1r_1,\cdots,h_mr_m)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_2(\widetilde{\Omega}_h)}=\left(\sum_{x\in\overline{\Omega_h}}\left|\varphi^h(x)\right|^2h_1\cdots h_m\right)^{1/2}.$$

To the differential operator A generated by problem (3.27), we assign the difference operator A_h^x by the formula

$$(3.29)\qquad A_h^xu_x^h=-\sum_{r=1}^m\left(a_r(x)u_{\bar{x}_r}^h\right)_{x_r,j_r}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\widetilde{\Omega}_h)$. With the help of A_h^x we arrive at the nonlocal boundary value problem

$$\begin{cases} \frac{d^2v^h(t,x)}{dt^2} + A_h^x v^h(t,x) = f^h(t,x),\ 0 \le t \le 1,\ x \in \Omega_h, \\ v^h(0,x) = \sum\limits_{l=1}^{n} \alpha_l v^h(\lambda_l, x) + \varphi^h(x), x \in \widetilde{\Omega}_h, \\ \frac{dv^h(0,x)}{dt} = \sum\limits_{l=1}^{n} \beta_l v_t^h(\lambda_l, x) + \psi^h(x), x \in \widetilde{\Omega}_h \end{cases} \tag{3.30}$$

for an infinite system of ordinary differential equations.

In the second step, we replace problem (3.30) by the difference scheme (3.31)

$$\begin{cases} \frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_{k+1}^h = f_k^h(x),\ f_{k+1}^h(x) = f^h(t_{k+1}, x), \\ t_{k+1} = (k+1)\tau, 1 \le k \le N-1,\ N\tau = 1, x \in \Omega_h, \\ u_0^h(x) = \sum\limits_{l=1}^{n} \alpha_l u_{[\lambda_l/\tau]}^h(x) + \varphi^h(x),\ x \in \widetilde{\Omega}_h, \\ (I + \tau^2 A_h^x) \frac{u_1^h(x) - u_0^h(x)}{\tau} = \sum\limits_{l=1}^{n} \beta_l \frac{u_{[\lambda_l/\tau]+1}^h(x) - u_{[\lambda_l/\tau]}^h(x)}{\tau} + \psi^h(x), x \in \widetilde{\Omega}_h. \end{cases} \tag{3.31}$$

Theorem 3.4. *Let τ and $|h|$ be sufficiently small numbers. Suppose that the assumption (3.2) holds. Then, the solutions of the difference scheme (3.31) satisfy the following stability estimates:*

$$\max_{0 \le k \le N} \left\| u_k^h \right\|_{L_{2h}} + \max_{0 \le k \le N} \sum_{r=1}^{m} \left\| \left(u_k^h \right)_{x_r, j_r} \right\|_{L_{2h}}$$
$$\le M_1 \left[\max_{1 \le k \le N-1} \left\| f_k^h \right\|_{L_{2h}} + \left\| \psi^h \right\|_{L_{2h}} + \sum_{r=1}^{m} \left\| \varphi_{\bar{x}_r, j_r}^h \right\|_{L_{2h}} \right],$$
$$\max_{1 \le k \le N-1} \left\| \tau^{-2} \left(u_{k+1}^h - 2u_k^h + u_{k-1}^h \right) \right\|_{L_{2h}} + \max_{0 \le k \le N} \sum_{r=1}^{m} \left\| \left(u_k^h \right)_{\bar{x}_r x_r, j_r} \right\|_{L_{2h}}$$
$$\le M_1 \left[\left\| f_1^h \right\|_{L_{2h}} + \max_{2 \le k \le N-1} \left\| \tau^{-1} \left(f_k^h - f_{k-1}^h \right) \right\|_{L_{2h}} + \sum_{r=1}^{m} \left\| \psi_{\bar{x}_r, j_r}^h \right\|_{L_{2h}} \right.$$
$$\left. + \sum_{r=1}^{m} \left\| \varphi_{\bar{x}_r x_r, j_r}^h \right\|_{L_{2h}} \right].$$

Here M_1 does not depend on $\tau, h,\ \varphi^h(x), \psi^h(x)$ and $f_k^h, 1 \le k < N$.

The proof of Theorem 3.4 is based on the abstract Theorem 3.2, the symmetry properties of the operator A_h^x defined by formula (3.29) and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 3.5. *For the solutions of the elliptic difference problem*

$$A_h^x u^h(x) = \omega^h(x), x \in \Omega_h, \tag{3.32}$$

$$u^h(x) = 0, x \in S_h$$

the following coercivity inequality holds [3]:

$$\sum_{r=1}^{m} \left\| u^h_{x_r \bar{x}_r, j_r} \right\|_{L_{2h}} \leq M ||\omega^h||_{L_{2h}}.$$

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