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Invariant solutions and conservation laws to nonconservative FP equation

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ABSTRACT

We generate conservation laws for the one dimensional nonconservative Fokker–Planck (FP) equation, also known as the Kolmogorov forward equation, which describes the time evolution of the probability density function of position and velocity of a particle, and associate these, where possible, with Lie symmetry group generators. We determine the conserved vectors by a composite variational principle and then check if the condition for which symmetries associate with the conservation law is satisfied. As the Fokker–Planck equation is evolution type, no recourse to a Lagrangian formulation is made. Moreover, we obtain invariant solutions for the FP equation via potential symmetries.

1. Introduction

In all areas of physics, conservation laws are essential since they allow us to draw conclusions of a physical system under study in an efficient way. A variety of powerful methods [1–12], have been used to investigate conservation laws of partial differential equations (PDEs).

Recently, Ibragimov proposed a general method [13] to find conservation laws for partial differential equations (PDEs). This new approach does not require the existence of a Lagrangian and is based on the concept of an adjoint equation for non-linear PDEs [14]. In [13], it is proved that the adjoint equation inherits all the symmetries of the given PDEs.

The FP equation models a wide variety of phenomena arising in diverse fields: probability theory (describing the Markov process, an FP equation appears as the master equation), laser physics (the statistics of light may very well be treated by a FP equation), electronics (supersonic conductors, Josephson tunnelling junction, relaxation of dipoles, second-order phase-locked loops, an optimal portfolio problem) [15,16].

In the case of one space variable, to which we restrict ourselves here just for the sake of simplicity, the FP equation is included in the parabolic equation

$$u_t = a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u \tag{1}$$

where u is the probability density function, t and x are the time and space coordinates, respectively, and a, b and c are smooth functions of t and x, assumed to be given.

In this work, we consider the nonconservative form of Eq. (1), namely $a(t,x) = 1$, $b(t,x) = x$ and $c(t,x) = 0$. We give the complete Lie-point symmetries, conservation laws and invariant solutions corresponding to potential symmetries for the nonconservative FP equation of the form

$$u_{xx} + xu_x - u_t = 0. \tag{2}$$

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In [17], Mahomed studies invariant characterization for Eq. (1). He provides the condition for which Eq. (2) is reducible to one of four canonical forms and thus the point symmetries are readily available.

It is well known that if one knows the conservation laws of a given equation and those conservation laws can be associated with symmetries (Lie-point or Lie-Bäcklund) of the given equation then new conservation laws can be obtained.Using this fact, we also yield new conservation laws. In addition, we put Eq. (2) into the conservative form by using one of the conservation laws and then we derive potential (nonlocal) symmetries.Utilizing one of these potential symmetries we obtain an invariant solution by considering the algorithm of Pucci and Saccomandi [18] (see for other alternative methods [19,20]).

The outline of this work is as follows. In the next section we present the necessary preliminaries. In Section 3, we discuss the Fokker–Planck equation and its Lie-point symmetries. Section 4 is devoted to calculation of the conservation laws. Then in Section 5, we obtain the potential symmetries and an invariant solution corresponding to one potential symmetry. Some conclusions are given in Section 6.

2. Necessary preliminaries

We first state some notations and theorems. The summation convention over repeated indices, one upper and one lower, will be used.

Let

$$x = (x^1, \ldots, x^n) \tag{3}$$

be the independent variable with coordinates x^i, and

$$u = (u^1, \ldots, u^m) \tag{4}$$

be the dependent variable with the coordinates u^α. The derivatives of u with respect to x are

$$u_i^\alpha = D_i(u^\alpha), \qquad u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots \tag{5}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, \ldots, n \tag{6}$$

is the operator of total differentation. The collection of all first derivatives u_i^α is denoted by $u_{(1)}$. Similarly, the collections of all higher-order derivatives are denoted by $u_{(2)}, u_{(3)}, \ldots$. The variables u^α are also called differential variables. A function $f(x, u, u_{(1)}, \ldots)$ of a finite number of a variables $x, u, u_{(1)}, u_{(2)}, \ldots$ is called a differential function if it is locally analytic. The set of all differential functions of all finite orders is denoted by A.

We denote a rth order ($r \geq 1$) system of m PDEs of n independent variables $x = (x^1, \ldots, x^n)$ with components x^i and m dependent variables $u = (u^1, \ldots, u^m)$ with components u^α by

$$F_\beta(x, u, u_{(1)} \ldots, u_{(r)}) = 0, \quad \beta = 1, \ldots, m. \tag{7}$$

The system (7) admits a Lie point symmetry with generator

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha} \tag{8}$$

if $pr\, X^r F_\beta = 0$ on the solution space of Eq. (7), where $pr\, X^r$ is the rth prolongation of X vector field.

The vector

$$C = (C^1(x, u, u_{(1)}, \ldots, u_{(r-1)}), \ldots, C^n(x, u, u_{(1)}, \ldots, u_{(r-1)})) \tag{9}$$

is a conserved vector of Eq. (7) if

$$D_i(C^i) = 0 \tag{10}$$

on the solution space of (7). The expression (10) is a conservation law of Eq. (7). Here D_i is the total derivative with respect to x^i and in what follows we will employ in the case of independent variables $(x^0, x^1) = (t, x)$ and $(u^1, u^2) = (u, v)$ as dependent variables.

Let an operator (8) be a symmetry of a system of rth order partial differential equations [13]

$$F_\alpha(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \quad \alpha = 1, \ldots, m. \tag{11}$$

Then the quantities (Noether identity)

$$C^i = L\xi^i + (\eta^\alpha - \xi^j u_j^\alpha)\left[\frac{\partial L}{\partial u_i^\alpha} - D_j\left(\frac{\partial L}{\partial u_{ij}^\alpha}\right) + D_j D_k\left(\frac{\partial L}{\partial u_{ijk}^\alpha}\right) - \cdots\right]$$
$$+ D_j(\eta^\alpha - \xi^j u_j^\alpha)\left[\frac{\partial L}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial L}{\partial u_{ijk}^\alpha}\right) + \cdots\right] + D_j D_k(\eta^\alpha - \xi^j u_j^\alpha)\left[\frac{\partial L}{\partial u_{ijk}^\alpha} - \cdots\right] + \cdots \tag{12}$$

furnish a conserved vector $C = (C^1, \ldots, C^n)$ for the Eq. (11) considered together with the adjoint system

$$F^*_\alpha(x, u, v, u_{(1)}, v_{(1),} \ldots, u_{(r)}, v_{(r)}) \equiv \frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m \tag{13}$$

where

$$L = v^\beta F_\beta(x, u, u_{(1)}, \ldots, u_{(r)}), \tag{14}$$
$$W^\alpha = \eta^\alpha - \xi_j u^\alpha_j,$$

are a formal Lagrangian and a Lie characteristic function, respectively and $v = (v^1, \ldots, v^m)$ are adjoint dependent variables, i.e. $v = v(x)$ and

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum (-1)^r D_{i_1} \ldots D_{i_r} \frac{\partial}{\partial u^\alpha_{i_1 \ldots i_r}}, \quad \alpha = 1, \ldots, m. \tag{15}$$

In the case of a second order Lagrangian, the Noether identity, (12), is replaced by

$$C^i = \xi^i L + \left[\frac{\partial L}{\partial u^\alpha_i} - D_k \left(\frac{\partial L}{\partial u^\alpha_{ik}} \right) \right] W^\alpha + D_k(W^\alpha) \frac{\partial L}{\partial u^\alpha_{ik}}. \tag{16}$$

Suppose that X is a Lie-point symmetry of Eq. (7), such that the conserved form of Eq. (7), given by (10), is invariant under X. Then

$$X(T^i) + T^i D_j(\xi^j) - T^j D_j(\xi^i) = 0. \tag{17}$$

A Lie point symmetry X is said to be associated with a conserved vector T of the system (7) if X and T satisfy (17) [21].

Bluman et al. [22] have introduced the concept of potential symmetry for any differential equation that can be written as a conservation law. In the case of considered here, this means that the definition of the conservation laws (10) can be written as

$$D_x G(x, u, u_{(1)}) - D_t H(x, u, u_{(1)}) = 0 \tag{18}$$

where D_x and D_t are the total derivative operators. Introducing an auxiliary potential variable $w = w(x, t)$, it is possible to form the potential system, $S = 0$,

$$w_t = G, \qquad w_x = H \tag{19}$$

which is obviously equivalent to (18).

To compute the classical point symmetries of (18), we introduce the infinitesimal generator

$$X = \xi(x, t, u, w) \frac{\partial}{\partial x} + \tau(x, t, u, w) \frac{\partial}{\partial t} + \eta(x, t, u, w) \frac{\partial}{\partial u} + \varphi(x, t, u, w) \frac{\partial}{\partial w} \tag{20}$$

and its first-order prolongation

$$pr\, X^1 = X + \eta^x \frac{\partial}{\partial u_x} + \eta^t \frac{\partial}{\partial u_t} + \varphi^x \frac{\partial}{\partial w_x} + \varphi^t \frac{\partial}{\partial w_t} \tag{21}$$

where

$$\eta^x = D_x \eta - u_x D_x \xi - u_t D_x \tau, \qquad \eta^t = D_t \eta - u_x D_t \xi - u_t D_t \tau, \tag{22}$$

$$\varphi^x = D_x \varphi - w_x D_x \xi - w_t D_x \tau, \qquad \varphi^t = D_t \varphi - w_x D_t \xi - w_t D_t \tau. \tag{23}$$

Considering relation

$$pr\, X^1 S = 0 \tag{24}$$

we obtain the defining equations of the classical point symmetries admitted by Eq. (19). Any admitted symmetry with infinitesimal generator X where ξ, τ or η depend on w is called a potential symmetry of Eq. (19); potential symmetries are non-local symmetries.

Now, we give the following theorems [18]:

Theorem 2.1. *The necessary conditions for Eq.* (19), *of order* $m > 2$, *to admit potential symmetries are that*

$$\frac{\partial G}{\partial u_{0,m-1}} = 0 \quad \text{and} \quad H = H(x, t, u, u_x, u_t). \tag{25}$$

Since, for $m = 2$, $F = F(x, t, u, u_x, u_t)$ and $G = G(x, t, u, u_x, u_t)$, we have shown that potential symmetries can exist only if the density or the flow depends at most on the first derivatives of u.

Theorem 2.2. *Eq. (19) admits potential symmetries only if (19) assumes one of the following forms:*

$$w_x = K_1(x,t,u)u_t + K_2(x,t,u)u_x + K_3(x,t,u), \tag{26}$$

where $K_1 \neq 0$; *otherwise*

$$w_x = K(x,t,u,u_x) \tag{27}$$

and in this case it is $\tau = \tau(t)$.

Now, let R be a PDE, which can be written in a conserved form with a choice of H and G, which satisfies the necessary conditions of the last theorems. We suppose to have determined a potential symmetry of R. It is interesting to clarify how it is possible to use these symmetries to find exact solutions by reduction methods.

Given a point symmetry for S, the invariant surface conditions are:

$$\begin{aligned} &\xi(x,t,u,w)u_x + \tau(x,t,u,w)u_t - \eta(x,t,u,w) = 0, \\ &\xi(x,t,u,w)w_x + \tau(x,t,u,w)w_t - \phi(x,t,u,w) = 0. \end{aligned} \tag{28}$$

The solutions of the associated characteristic system are given by three independent integrals:

$$s_1(x,t,u,w) = c_0, \qquad s_2(x,t,u,w) = c_1, \qquad s_3(x,t,u,w) = c_2 \tag{29}$$

with $\frac{\partial(s_1,s_2,s_3)}{\partial(u,w)}$ of rank 2.

The solutions of Eq. (28) are defined as one-parameter families of characteristic curves Eq. (29), we obtain

$$\begin{aligned} &u = U(x,t,z,h_1(z),h_2(z)), \\ &w = V(x,t,z,h_1(z),h_2(z)), \\ &G(x,t,z,h_1(z),h_2(z)) = 0. \end{aligned} \tag{30}$$

Eq. $(30)_3$, defines implicitly the similarity variable z as a function of x; t. We point out that Eq. $(30)_1$ is a family of solutions of the second-order equation:

$$\eta^*(x,t,u,u_1,u_2) = 0 \tag{31}$$

that is obtained by eliminating w between the two equations of (28).

The invariant solutions of S are given by Eq. (30), where $h_i(z)$; $i = 1$; 2 are the solutions of the ordinary system $\overline{\varphi}$, which is obtained by substitution in S. R being a differential consequence of S. The solutions of S give those solutions F_E of R which verify the differential relation:

$$\widetilde{\eta}(x,t,u,u_1,u_2,\ldots,u_{m-1}) = 0 \tag{32}$$

that is obtained by eliminating w between Eq. $(28)_1$, and

$$\xi H + \tau G - \phi = 0. \tag{33}$$

We can determine a family F_E^* of R solutions by direct substitution of Eq. $(30)_1$ and $(30)_3$ into R. We obtain, in this way, a relation involving z; h_1; h_2, the derivative up to order m, and one parameter given by x or t. By imposing that the relation is identically zero for any value of the parameter; this will result in an ordinary system $\overline{\overline{\varphi}}$ on the $h_i(z)$. F_E^* is given by Eq. $(30)_1$, where $h_i(z)$ are solutions of $\overline{\overline{\varphi}}$; then F_E^* is a family of solutions for Eq. (31). On the other hand, F_E, besides Eq. (31), verifies also Eq. (29), then F_E is enclosed in F_E^*.

3. Lie group analysis of FP equation

The one dimensional, nonconservative FP equation is in the form of

$$u_{xx} + xu_x - u_t = 0. \tag{34}$$

We consider a one-parameter Lie group of infinitesimal transformations in (x,t,u) given by

$$x^* = x^*(x,t,u,\varepsilon), \qquad t^* = t^*(x,t,u,\varepsilon), \qquad u^* = u^*(x,t,u,\varepsilon), \tag{35}$$

where ε is the group parameter. The infinitesimal generator of the group (35) can be expressed in the following vector form:

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} \tag{36}$$

in which ξ, τ, η are infinitesimal functions of group variables (independent and dependent variables). Then the corresponding one-parameter Lie group of transformations becomes

$$\begin{aligned} &x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2), \qquad t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2), \\ &u^* = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2). \end{aligned} \tag{37}$$

Applying the classical method to (34) yields a system of determining equations. Thus the Lie algebra of infinitesimal symmetries of the FP equation is spanned by the six vector fields

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = u\frac{\partial}{\partial u}, \qquad X_3 = e^{-t}\frac{\partial}{\partial x}, \qquad X_4 = e^t\frac{\partial}{\partial x} - e^t ux\frac{\partial}{\partial u}, \tag{38}$$
$$X_5 = -\frac{1}{2}e^{2t}x\frac{\partial}{\partial x} - \frac{1}{2}e^{2t}\frac{\partial}{\partial t} + \left(\frac{1}{2}u + \frac{1}{2}ux^2\right)e^{2t}\frac{\partial}{\partial u}, \qquad X_6 = \frac{1}{2}e^{-2t}x\frac{\partial}{\partial x} - \frac{1}{2}e^{-2t}\frac{\partial}{\partial t},$$

and the infinite-dimensional subalgebra

$$X_\infty = f(t,x)\frac{\partial}{\partial u}, \tag{39}$$

where $f(t,x)$ is an arbitrary solution of

$$f_{xx} + xf_x - f_t = 0.$$

4. Calculation of conservation laws

The formal Lagrangian (14) of Eq. (34) is

$$L = (u_{xx} + xu_x - u_t)v. \tag{40}$$

Substituting (40) to (13) we yield the following adjoint equation

$$F^* = v_t + v_{xx} - xv_x - v = 0. \tag{41}$$

Let us investigate, the self-adjointness of Eq. (34). If we substitute $u = v$ in Eq. (41), $u_{xx} - xu_x - u + u_t = 0$ is obtained, i.e. $F \neq F^*$. Therefore, (34) is not self-adjoint.

Eq. (41) admits

$$X = \frac{\partial}{\partial t} \tag{42}$$

time translation symmetry. It is clear that, $v = e^{\frac{x^2}{2}}$ is invariant solution of (41) by using time translation symmetry.

Adjoint equation (13) admits the symmetries of Eq. (7). Indeed, adjoint equation admits the operator X extended to the variables v by the formula

$$Y = X + \eta_*\frac{\partial}{\partial v}, \qquad \eta_* = -[\lambda + D_i(\xi^i)]v \tag{43}$$

where

$$X(F) = \lambda F. \tag{44}$$

Let us illustrate this fact. We consider X_4 from (38) and its prolongation, respectively,

$$X_4 = e^t\frac{\partial}{\partial x} - e^t ux\frac{\partial}{\partial u}, \tag{45}$$

$$pr\,X_4 = e^t\frac{\partial}{\partial x} - e^t ux\frac{\partial}{\partial u} - e^t(xu + xu_t + u_x)\frac{\partial}{\partial u_t} - e^t(u + xu_x)\frac{\partial}{\partial u_x} - e^t(2u_x + xu_{xx})\frac{\partial}{\partial u_{xx}}. \tag{46}$$

We obtained λ as $-xe^t$ by employing the invariance condition. Therefore, η_* is $xe^t v$.

Consequently, the extension of the operator X_4 to v has the form

$$Y = e^t\frac{\partial}{\partial x} - e^t ux\frac{\partial}{\partial u} + xe^t v\frac{\partial}{\partial v}. \tag{47}$$

At this stage, we find the conservation laws. For instance, we consider the symmetry X_2 in (38). Applying the formula (16) to the symmetry $X_2 = u\frac{\partial}{\partial u}$, where $\xi^0 = 0,\ \xi^1 = 0,\ \eta = u$, and to the formal Lagrangian (40), we derive the following conserved vectors of the conservation law (10)

$$C_2^0 = u\frac{\partial L}{\partial u_t} = -vu, \tag{48}$$

$$C_2^1 = [xv - D_x(v)]u + vD_x(u) = xuv - uv_x + vu_x. \tag{49}$$

Remark. It is noticed that each symmetry and formal Lagrangian satisfies a $X(L) + LD_i(\xi^i) = 0$ invariance condition [13].

If one uses $v = e^{\frac{x^2}{2}}$ (solution of the adjoint equation (41)) in (48) the conserved vectors are the following:

$$C_2^0 = -e^{\frac{x^2}{2}}u, \qquad C_2^1 = e^{\frac{x^2}{2}}u_x. \tag{50}$$

After lengthy calculations, we obtain the following conservation laws corresponding to each symmetry of (38)–(39):

$$\begin{aligned}
&C_1^0 = vu_{xx} + xvu_x, \qquad C_1^1 = -xvu_t + v_x u_t - vu_{xt}\\
&C_2^0 = -vu, \qquad C_2^1 = xuv - uv_x + vu_x,\\
&C_3^0 = ve^{-t}u_x, \qquad C_3^1 = e^{-t}(u_x v_x - vu_t),\\
&C_4^0 = ve^t(u_x + xu), \qquad C_4^1 = -e^t(v(u_t + x^2 u + xu_x + u) - v_x(xu + u_x))\\
&C_5^0 = -\frac{e^{2t}}{2}v(u_t + xu_x + u + ux^2),\\
&C_5^1 = \frac{e^{2t}v}{2}\left((x^2+2)u_x + 2xu_t + xu(3+x^2) + u_{tx}\right) - \frac{e^{2t}v_x}{2}\left(u(1+x^2) + xu_x + u_t\right),\\
&C_6^0 = -\frac{e^{-2t}}{2}v(u_t - xu_x),\\
&C_6^1 = \frac{e^{-2t}}{2}v(u_{tx} - u_x) + \frac{e^{-2t}}{2}v_x(xu_x - u_t),\\
&C_\infty^0 = -vf, C_\infty^1 = (xv - v_x)f + vf_x.
\end{aligned} \tag{51}$$

We note that the conservation laws of the FP equation can also easily be treated using the partial Noether approach [10]. In [10], it is shown how one can construct the conservation laws of Euler–Lagrange-type equations via Noether-type symmetry operators associated with partial Lagrangians. This is even in the case when a system does not directly have a usual Lagrangian, e.g. scalar evolution equations.

Let us generate another new conservation law from (51). Eq. (34) has conserved components, (C_2^0 and C_2^1)

$$C^0 = -e^{\frac{x^2}{2}}u, \qquad C^1 = e^{\frac{x^2}{2}}u_x \tag{52}$$

with associated symmetry $X_1 = \frac{\partial}{\partial x}$ in the sense of (17). Clearly Eq. (34) admits $X_2 = u\frac{\partial}{\partial u}$. The action of X_2 on (52) yields

$$C_*^0 = X(C^0) + C^0 D_x(\xi^1) - C^1 D_x(\xi^0) = -e^{\frac{x^2}{2}}u = C^0 \tag{53}$$

and

$$C_*^1 = e^{\frac{x^2}{2}}u = C^1 \tag{54}$$

and therefore X_2 does not produce a new conservation law. In additional, the canonical Lie–Bäcklund operator does not give new conserved quantities, due to the following results:

$$X_2 = u\frac{\partial}{\partial u}, \qquad X_2(C^0) = -e^{\frac{x^2}{2}}u = C_*^0, \tag{55}$$

$$X_2(C^1) = C_*^1. \tag{56}$$

But, for instance from X_3, we can yield new conservation laws using (17). The action of X_3 on (52) yields

$$C_*^0 = -xe^{-t+\frac{x^2}{2}}u, \tag{57}$$

$$C_*^1 = (xu_x - u)e^{-t+\frac{x^2}{2}}. \tag{58}$$

Again, if a canonical approach is used, we obtain (57),

$$X_3 C^0 = C_*^0, \qquad X_3 C^1 = C_*^1. \tag{59}$$

Let us determine the point symmetries associated with the conserved components

$$C^0 = -e^{\frac{x^2}{2}}u, \qquad C^1 = e^{\frac{x^2}{2}}u_x \tag{60}$$

of the FP equation. The symmetry conditions on (60) are

$$X(C^0) + C^0 D_x(\xi^x) - C^1 D_x(\xi^t) = 0, \qquad X(C^1) + C^1 D_t(\xi^t) - C^0 D_t(\xi^x) = 0 \tag{61}$$

where X is the operator

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{xx}\frac{\partial}{\partial u_{xx}}. \tag{62}$$

The expansion of the determining equations (61) and separation by monomials of the first derivatives give X_1, X_4 and X_5. Hence there are three symmetries associated with C^0 and C^1. We note that $\{X_1, X_4, X_5\}$ forms a subalgebra of the Lie-Algebra of point symmetry generators of the Eq. (34).

5. Potential symmetries and invariant solution

In order to find the potential symmetries of Eq. (34), we set it in the following conservative form (for instance, we employ C_2^0 and C_2^1 corresponding to X_2)

$$D_t(-\mathrm{e}^{\frac{x^2}{2}}u) - D_x(-\mathrm{e}^{\frac{x^2}{2}}u_x) = 0. \tag{63}$$

By considering a potential $w(x; t)$ as an auxiliary unknown function, the following system S can be associated with Eq. (63)

$$w_t = -u_x\mathrm{e}^{\frac{x^2}{2}}, \qquad w_x = -\mathrm{e}^{\frac{x^2}{2}}u. \tag{64}$$

Since R is a linear PDE and S is a linear system of PDEs, then, ξ and τ are independent of u and w and are linear in u and w [22]. Also, system S is a particular case of Eq. (27) which is a particular case of $w_x = K(x, t, u, u_x)$.

Using the classical algorithm, we find the following point symmetry generators of Eq. (64):

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \mathrm{e}^t ux\frac{\partial}{\partial u} - \mathrm{e}^t\frac{\partial}{\partial x},$$
$$X_3 = \mathrm{e}^{2t}u(x^2+1)\frac{\partial}{\partial u} - \mathrm{e}^{2t}x\frac{\partial}{\partial x} - \mathrm{e}^{2t}\frac{\partial}{\partial t},$$
$$X_4 = -\mathrm{e}^{-t}\frac{\partial}{\partial x} + w\mathrm{e}^{-t-\frac{x^2}{2}}\frac{\partial}{\partial u} - wx\mathrm{e}^{-t}\frac{\partial}{\partial w}$$
$$X_5 = -x\mathrm{e}^{-2t}\frac{\partial}{\partial x} + \mathrm{e}^{-4t}\frac{\partial}{\partial t} + 2\mathrm{e}^{-2t}(u + wx\mathrm{e}^{-x^2})\frac{\partial}{\partial u} + w\mathrm{e}^{-2t}(-x^2+1)\frac{\partial}{\partial w}.$$

In all the symmetries above, only X_4 and X_5 are potential symmetries for Eq. (34). For the potential symmetry X_4, a potential system with

$$w_x = -\mathrm{e}^{\frac{x^2}{2}}u, \qquad w_t = -\mathrm{e}^{\frac{x^2}{2}}u_x \tag{65}$$

the characteristic system related to invariant surface conditions:

$$-\mathrm{e}^{-t}u_x - w\mathrm{e}^{-t-\frac{x^2}{2}} = 0, \tag{66}$$
$$-\mathrm{e}^{-t}w_x + wx\mathrm{e}^{-t} = 0 \tag{67}$$

or

$$u_x + w\mathrm{e}^{-\frac{x^2}{2}} = 0, \tag{68}$$
$$-w_x + xw = 0, \tag{69}$$

admits the following three integrals

$$c_0 = t, \qquad c_1 = w\mathrm{e}^{-\frac{x^2}{2}}, \qquad c_2 = u + wx\mathrm{e}^{-\frac{x^2}{2}}. \tag{70}$$

If we assume $c_0 = z$ as a parameter, $c_1 = h_1(z)$, and $c_2 = h_2(z)$ in Eq. (70), we obtain

$$u = h_2(z) - xh_1(z), \tag{71}$$
$$w = h_1(z)\mathrm{e}^{\frac{x^2}{2}},$$
$$z = t.$$

We point out that Eq. $(71)_1$ is a family of solutions of the second-order equation:

$$\eta^* \equiv u_{xx} = 0, \tag{72}$$

which is obtained by eliminating w between Eqs. (68) and (69). Now, to find the solutions F_E^*, we introduce the $(71)_1$ in (34) obtaining:

$$h_2' - xh_1' + xh_1 = 0. \tag{73}$$

From Eq. (73), we have the system $\overline{\overline{\varphi}}$ as

$$h_2' = 0, \tag{74}$$
$$h_1 - h_1' = 0$$

which on solving yields

$$h_1(z) = c_3 e^z,\quad h_2(z) = c_4, \tag{75}$$

where c_3, c_4 is a constant. Then, the family F_E^* is therefore

$$u = c_4 - x c_3 e^t. \tag{76}$$

Also, Eq. (71) is a family of solutions of the first-order equation

$$\widetilde{\eta} = x u_x - u = 0 \tag{77}$$

that is obtained by eliminating w between Eqs. (71) and (33).

To find the solutions F_E, we introduce Eq. (71) into Eq. (65) obtaining the system $\overline{\varphi}$:

$$h_2(z) = 0,\quad h_1^{'}(z) - h_1(z) = 0 \tag{78}$$

which on solving yields

$$h_1(z) = c_5 e^t,\quad h_2(z) = 0, \tag{79}$$

where c_5 is a constant. Then, the family F_E is therefore

$$u = -c_5 x e^t. \tag{80}$$

It is clear that F_E is enclosed in F_E^*.

6. Conclusions

In this work, we considered the one-dimensional nonconservative FP equation.This equation belongs to the family of evolution type equations and it does not have the usual Lagrangian. In view of this fact, we applied the composite variational principle, inspired by [13]. First we obtained complete Lie-point symmetries of Eq. (34). Next, we constructed an adjoint equation by applying the formal Lagrangian to the variational derivative. It is seen that adjoint equation (41) admits all the symmetries of Eq. (34). Therefore Eqs. (34) and (41) together are a member of family of Euler–Lagrange type equations. As a result, non-local conservation laws are obtained. Moreover, we derived local conservation laws using one particular solution of Eq. (41). Computing the conservation laws of the studied equation provides many possibilities. For instance, using one conserved quantity of Eq. (34), we put it into a conservative form and yielded potential symmetries. Finally, the invariant solution of Eq. (34) is derived by employing one potential symmetry.

References

[1] E. Noether, Invariante Variationsprobleme, Nachr. Konig. Gesell. Wissen., Gottingen, Math.-Phys. Kl. 2 (1918) 235–257.
[2] T. Wolf, A comparison of four approaches to the calculation of conservation laws, European J. Appl. Math. 13 (2002) 129–152.
[3] R.O. Popovych, N.M. Ivanova, Hierarchy of conservation laws of diffusion–convection equations, J. Math. Phys. 46 (2005) 043502.
[4] P.S. Laplace, Celestial Mechanics (English translation), New York, 1966, (Original work published 1798).
[5] S.C. Anco, G.W. Bluman, Direct construction method for conservation laws of partial differential equations. Part I: examples of conservation law classifications, Eur. J. Appl. Math. 13 (2002) 545–566.
[6] S.C. Anco, G.W. Bluman, Direct construction method for conservation laws of partial differential equations. Part II: general treatment, European J. Appl. Math. 9 (2002) 567–585.
[7] H. Steudel, Uber die zuordnung zwischen invarianzeigenschaften und erhaltungssatzen, Z. Naturforsch. 17 (1962) 129–132.
[8] P.J. Olver, Application of Lie groups to Differential Equations, Springer-Verlag, New York, 1993.
[9] A.H. Kara, F.M. Mahomed, Relationship between symmetries and conservation laws, Int. J. Theor. Phys. 39 (2000) 23–40.
[10] A.H. Kara, F.M. Mahomed, Noether-type symmetries and conservation laws via partial Lagragians, Nonlinear Dynam. (45) (2006) 367–383.
[11] E. Yaşar, New local and nonlocal conservation laws of evolution type Equations, Ph.D. Thesis, Uludag University, 2009 (in Turkish).
[12] E. Yaşar, T. Özer, Conservation laws for one-layer shallow water wave systems, Nonlinear Anal. RWA 11 (2010) 838–848.
[13] N.H. Ibragimov, A new conservation theorem, J. Math. Anal. Appl. 333 (2007) 311–328.
[14] R.W. Atherton, GM Homsy, On the existence and formulation of variational principles for nonlinear differential equations, Stud. Appl. Math. 54 (1975) 31–60.
[15] K.V. Gardiner, Handbook of Stochastic Methods, Springer, Berlin, 1985.
[16] H Risken, The Fokker–Planck Equation, Methods of Solution and Applications, 2nd ed., Springer, Berlin, 1989.
[17] F. Mahomed, Complete invariant characterization of scalar linear (1+1) parabolic equations, J. Nonlinear Math. Phys. 15 (1) (2008) 112–123.
[18] E. Pucci, G. Saccomandi, Potential symmetries and solutions by reduction of partial differential equations, J. Phys. A 26 (1993) 681–690.
[19] T. Özer, Symmetry group analysis and similarity solutions of variant nonlinear long-wave equations, Chaos Solitons Fractals 33 (2008) 722–730.
[20] T. Özer, New exact solutions to the CDF equations, Chaos Solitons Fractals 39 (2009) 1371–1375.
[21] A.H. Kara, F.M. Mahomed, A basis of conservation laws for partial differential equations, J. Nonlinear Math. Phys. 9 (2) (2002) 60–72.
[22] G.W. Bluman, G.J. Reid, S. Kumei, New classes of symmetries for partial differential equations, J. Math. Phys. 29 (1988) 806–811.