Turk J Math
35 (2011) , 493 – 499.

doi:10.3906/mat-0910-59

Rotational embeddings in $\mathbb{E}^4$ with pointwise 1-type gauss map

Kadri Arslan, Bengü Kılıç Bayram, Betül Bulca,
Young Ho Kim, Cengizhan Murathan and Günay Öztürk*

Abstract

In the present article we study the rotational embedded surfaces in $\mathbb{E}^4$. The rotational embedded surface was first studied by G. Ganchev and V. Milousheva as a surface in $\mathbb{E}^4$. The Otsuki (non-round) sphere in $\mathbb{E}^4$ is one of the special examples of this surface. Finally, we give necessary and sufficient conditions for the flat Ganchev-Milousheva rotational surface to have pointwise 1-type Gauss map.

Key word and phrases: Rotation surface, gauss map, finite type, Pointwise 1-type.

1. Introduction

Since the late 1970's, the study of submanifolds of Euclidean space or pseudo-Euclidean space with the notion of finite type immersion has been extensively carried out. An isometric immersion $x : M \rightarrow \mathbb{E}^m$ of a submanifold M in Euclidean m-space $\mathbb{E}^m$ is said to be of finite type if x identified with the position vector field of M in $\mathbb{E}^m$ can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is,

$$x = x_0 + \sum_{i=1}^{k} x_i,$$

where x_0 is a constant map $x_1, x_2, ..., x_k$ non-constant maps such that $\Delta x = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq k$. If $\lambda_1, \lambda_2, ..., \lambda_k$ are different, then M is said to be of k-type. Similarly, a smooth map ϕ of an n-dimensional Riemannian manifold M of $\mathbb{E}^m$ is said to be of finite type if ϕ is a finite sum of $\mathbb{E}^m$-valued eigenfunctions of Δ ([4], [5]). Granted, this notion of finite type immersion is naturally extended in particular to the Gauss map G on M in Euclidean space ([8]). Thus, if a submanifold M of Euclidean space has 1-type Gauss map G, then G satisfies $\Delta G = \lambda(G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C ([1], [2], [3], [11]). However, the Laplacian of the Gauss map of some typical well-known surfaces such as a helicoid, a catenoid and a right cone in Euclidean 3-space $\mathbb{E}^3$ take a somewhat different form; namely, $\Delta G = f(G + C)$ for some non-constant function f and some constant vector C. Therefore, it is worth studying the class of solution surfaces satisfying

2000 *AMS Mathematics Subject Classification:* 53C40, 53C42.
This paper was prepared during the fourth named author's visit to the Uludag University, Bursa, Turkey in July 2009.
*supported by KOSEF R01-2007-000-20014-0 (2007).

such an equation. A submanifold M of a Euclidean space $\mathbb{E}^m$ is said to have pointwise 1-type Gauss map if its Gauss map G satisfies

$$\Delta G = f(G + C) \tag{1}$$

for some non-zero smooth function f on M and a constant vector C. A pointwise 1-type Gauss map is called proper if the function f defined by (1) is non-constant. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector C in (1) is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind ([6], [9], [12], [13]). In [9], one of the present authors characterized the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind. Also, together with B.-Y. Chen, they proved that surfaces of revolution with pointwise 1-type Gauss map of the first kind coincides with surfaces of revolution with constant mean curvature. Moreover, they characterized the rational surfaces of revolution with pointwise 1-type Gauss map ([6]).

In [16], D. W. Yoon studied Vranceanu rotation surfaces in Euclidean 4-space $\mathbb{E}^4$. He obtained the complete classification theorems for the flat Vranceanu rotation surfaces with 1-type Gauss map and an equation in terms of the mean curvature vector. For more details see also [15].

In this article we will investigate rotational embedded surface with pointwise 1-type Gauss map in Euclidean 4-space $\mathbb{E}^4$.

The rotational embedded surface was studied by G. Ganchev and V. Milousheva as a surface in $\mathbb{E}^4$ which is defined by the following surface patch with respect to an orthonormal system of coordinates

$$X(s,t) = (f_1(s), f_2(s), f_3(s)\cos t, f_3(s)\sin t), \tag{2}$$

where $\alpha(s) = (f_1(s), f_2(s), f_3(s))$ is a space curve parametrized by the arc-length, i.e., $({f_1}')^2+({f_2}')^2+({f_3}')^2 = 1$ and $f_3(s) > 0$ ([10]).

We prove the following theorem.

Theorem A. *Let M be a flat rotational embedded surface in Euclidean 4-space $\mathbb{E}^4$. Then M has pointwise 1-type Gauss map if and only if*

$$\begin{aligned} f_1(s) &= \int \mu \cos\left(\frac{\lambda}{a\mu}\ln|as+b|\right)ds, \\ f_2(s) &= \int \mu \sin\left(\frac{\lambda}{a\mu}\ln|as+b|\right)ds, \\ f_3(s) &= as+b. \end{aligned}$$

for some constants $\lambda \neq 0, \mu > 0, a \neq 0$ and b.

2. Preliminaries

Let $x : M \to \mathbb{E}^m$ be an isometric immersion from an n-dimensional connected Riemannian manifold M into an m-dimensional Euclidean space $\mathbb{E}^m$. Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\mathbb{E}^m$ and ∇ the induced connection on M. Then the Gaussian and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{3}$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{4}$$

for vector fields X, Y tangent to M and a vector field ξ normal to M, where h denotes the second fundamental form, D the normal connection and A_ξ the shape operator in the direction of ξ that is related with h by

$$\langle h(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle,$$

where $\langle \cdot, \cdot\rangle$ is the standard inner product in $\mathbb{E}^4$ and that in the submanifold M as well.

If we define a covariant differentiation $\overline{\nabla} h$ of the second fundamental form h on the direct sum of the tangent bundle and the normal bundle $TM \oplus T^\perp M$ of M by

$$(\overline{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

then we have the Codazzi equation

$$(\overline{\nabla}_X h)(Y, Z) = (\overline{\nabla}_Y h)(X, Z). \tag{5}$$

Let us now define the Gauss map G of a submanifold M into $G(n, m)$ in $\wedge^n \mathbb{E}^m$, where $G(n, m)$ is the Grassmannian manifold consisting of all oriented n-planes through the origin of $\mathbb{E}^m$ and $\wedge^n \mathbb{E}^m$ is the vector space obtained by the exterior product of n vectors in $\mathbb{E}^m$. In a natural way, we can identify $\wedge^n \mathbb{E}^m$ with some Euclidean space $\mathbb{E}^N$ where $N = \begin{pmatrix} m \\ n \end{pmatrix}$. Let $\{e_1, ..., \ e_n, \ e_{n+1}, ..., \ e_m\}$ be an adapted local orthonormal frame field in $\mathbb{E}^m$ such that $e_1, e_2, ..., \ e_n$ are tangent to M and $e_{n+1}, e_{n+2} ..., \ e_m$ normal to M. The map $G : M \rightarrow G(n, m)$ defined by $G(p) = (e_1 \wedge e_2 \wedge ... \wedge \ e_n)(p)$ is called the Gauss map of M, that is a smooth map which carries a point p in M into the oriented n-plane in $\mathbb{E}^m$ obtained from the parallel translation of the tangent space of M at p in $\mathbb{E}^m$.

For any real valued function f on M the Laplacian of f is defined by the relation

$$\Delta f = -\sum_i (\widetilde{\nabla}_{e_i} \widetilde{\nabla}_{e_i} f - \widetilde{\nabla}_{\nabla_{e_i} e_i} f). \tag{6}$$

3. Proof of Theorem

Let M be a rotational embedded surface in $\mathbb{E}^4$ defined by the patch (2). We choose a moving frame e_1, e_2, e_3, e_4 such that e_1, e_2 are tangent to M and e_3, e_4 are normal to M in the the following ([10]):

$$\begin{aligned}
e_1 &= \frac{\frac{\partial X}{\partial s}}{\left\| \frac{\partial X}{\partial s} \right\|}, \quad e_2 = \frac{\frac{\partial X}{\partial t}}{\left\| \frac{\partial X}{\partial t} \right\|}, \\
e_3 &= \frac{1}{\kappa}(f_1{}''(s), f_2{}''(s), f_3{}''(s) \cos t, f_3{}''(s) \sin t), \\
e_4 &= \frac{1}{\kappa}(f_2{}'(s) f_3{}''(s) - f_2{}''(s) f_3{}'(s), f_1{}''(s) f_3{}'(s) - f_1{}'(s) f_3{}''(s), \\
&\quad (f_1{}'(s) f_2{}''(s) - f_1{}''(s) f_2{}'(s)) \cos t, (f_1{}'(s) f_2{}''(s) - f_1{}''(s) f_2{}'(s)) \sin t),
\end{aligned}$$

where

$$\kappa = \sqrt{(f_1'')^2 + (f_2'')^2 + (f_3'')^2} \neq 0 \tag{7}$$

is the curvature of the space curve α.

Hence, the coefficients of the first fundamental form of the surface are

$$\begin{aligned} E &= \langle X_s(s,t), X_s(s,t)\rangle = 1, \\ F &= \langle X_s(s,t), X_t(s,t)\rangle = 0, \\ G &= \langle X_t(s,t), X_t(s,t)\langle = {f_3}^2(s). \end{aligned}$$

Since $EG - F^2 = {f_3}^2(s)$ does not vanish, the surface patch $X(s,t)$ is regular.

We denote by $\widetilde{\alpha}$ the projection of α on the 2-dimensional plane Oe_1e_2. So the curvature of $\widetilde{\alpha}$ is defined by $\kappa_1 = f_1'f_2'' - f_2'f_1''$. Then with respect to the frame field $\{e_1, e_2, e_3, e_4\}$, the Gaussian and Weingarten formulas (3)–(4) of M look like [10]:

$$\begin{aligned} \widetilde{\nabla}_{e_1}e_1 &= \kappa e_3, \\ \widetilde{\nabla}_{e_1}e_2 &= 0, \\ \widetilde{\nabla}_{e_2}e_2 &= -\frac{f_3'}{f_3}e_1 - \frac{f_3''}{\kappa f_3}e_3 - \frac{\kappa_1}{\kappa f_3}e_4, \end{aligned} \tag{8}$$

$$\widetilde{\nabla}_{e_2}e_1 = \frac{f_3'}{f_3}e_2 \tag{9}$$

and

$$\begin{aligned} \widetilde{\nabla}_{e_1}e_3 &= -\kappa e_1 + \tau e_4, \\ \widetilde{\nabla}_{e_2}e_3 &= \frac{f_3''}{\kappa f_3}e_2, \\ \widetilde{\nabla}_{e_1}e_4 &= -\tau e_3, \\ \widetilde{\nabla}_{e_2}e_4 &= \frac{\kappa_1}{\kappa f_3}e_2. \end{aligned} \tag{10}$$

Where, τ is the second curvature of space curve α. The Gauss curvature of M is obtained by equating

$$K = -\frac{f_3''}{f_3}. \tag{11}$$

Putting

$$\begin{aligned} A(s) &= -\left(\kappa^2 + \frac{(f_3'')^2 + \kappa_1^2}{\kappa^2 f_3^2}\right), \\ B(s) &= -\left(\kappa' + \frac{f_3''f_3'}{\kappa f_3^2} + \frac{\kappa f_3'}{f_3}\right), \\ D(s) &= -\left(\kappa\tau + \frac{\kappa_1 f_3'}{\kappa f_3^2}\right), \end{aligned} \tag{12}$$

we get, by using (6), (8) and (10),

$$-\Delta G = A(s)e_1 \wedge e_2 + B(s)e_2 \wedge e_3 + D(s)e_2 \wedge e_4. \tag{13}$$

We now suppose that the rotational embedded surface M is of pointwise 1-type Gauss map in $\mathbb{E}^4$. From (1) and (13),

$$\begin{aligned} f + f \quad \langle \quad & C, e_1 \wedge e_2\rangle = -A(s), \\ f \quad \langle \quad & C, e_2 \wedge e_3\rangle = -B(s), \\ f \quad \langle \quad & C, e_2 \wedge e_4\rangle = -D(s). \end{aligned} \tag{14}$$

Since ΔG is a linear combination of $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_1 \wedge e_4$, $e_2 \wedge e_3$, $e_2 \wedge e_4$ and $e_3 \wedge e_4$, we also have

$$\begin{aligned} f \quad \langle \quad & C, e_1 \wedge e_3\rangle = 0, \\ f \quad \langle \quad & C, e_1 \wedge e_4\rangle = 0, \\ f \quad \langle \quad & C, e_3 \wedge e_4\rangle = 0. \end{aligned} \tag{15}$$

By differentiating (15) covariantly with respect to s, we get

$$\begin{aligned} \frac{f_3{}'}{f_3} &< C, e_2 \wedge e_3 > + \frac{f_3{}''}{\kappa f_3} < C, e_1 \wedge e_2 >= 0, \\ \frac{f_3{}'}{f_3} &< C, e_2 \wedge e_4 > + \frac{\kappa_1}{\kappa f_3} < C, e_1 \wedge e_2 >= 0, \\ \frac{f_3{}''}{\kappa f_3} &< C, e_2 \wedge e_4 > - \frac{\kappa_1}{\kappa f_3} < C, e_2 \wedge e_3 >= 0. \end{aligned} \tag{16}$$

Since M is flat, (11) implies $f_3{}'' = 0$. Thus $f_3(s) = as + b$ for some constants $a \neq 0$ and b. Hence, substituting $f_3{}'' = 0$ into (16) and using (14) we obtain,

$$\begin{aligned} f_3{}'B(s) &= 0, \\ f_3{}'D + \frac{\kappa_1}{\kappa}(A(s) + f) &= 0, \\ \kappa_1 B(s) &= 0. \end{aligned} \tag{17}$$

Suppose $Q = \{p \in M : B(s) \neq 0\}$ is a non-empty set. Then, from the third formula of (16) we have $\kappa_1 = f_1{}'f_2{}'' - f_2{}'f_1{}'' = 0$. Consequently, using this equality with $(f_1{}')^2 + (f_2{}')^2 + (f_3{}')^2 = 1$, we get $(f_1{}')^2 + (f_2{}')^2 = 1 - a^2$. Therefore, $f_1{}', f_2{}', f_3{}'$ are constant functions and $\kappa = \sqrt{(f_1{}'')^2 + (f_2{}'')^2 + (f_3{}'')^2} = 0$, which is a contradiction. So, $B(s) = 0$. Furthermore, if we make use of the second equation of (12) with $f_3{}'' = 0$, then we obtain $\kappa = \frac{\lambda}{as+b}$, where λ is a nonzero constant. We may put

$$f_1{}' = \mu \cos\theta(s), \;\; f_2{}' = \mu \sin\theta(s) \tag{18}$$

for some function $\theta(s)$, where $1-a^2=\mu^2$. Furthermore, substituting (18), $\kappa=\frac{\lambda}{as+b}$ and $f_3=as+b$ into (7) with some computation implies $\frac{d\theta}{ds}=\frac{\lambda}{\mu}(\frac{1}{as+b})>0$. Solving this equation, we get $\theta(s)=\frac{\lambda}{a\mu}\ln|as+b|$. So, we obtain

$$\begin{aligned} f_1(s) &= \int \mu\cos(\frac{\lambda}{a\mu}\ln|as+b|)ds, \\ f_2(s) &= \int \mu\sin(\frac{\lambda}{a\mu}\ln|as+b|)ds, \\ f_3(s) &= as+b. \end{aligned}$$

The converse is easily verified. Thus, our theorem is proved.

Corollary 3.1 *Let M be a rotational embedded surface in Euclidean 4-space given by the surface patch (2). Then the Gauss map of M cannot be harmonic.*

Proof. Suppose the Gauss map of the rotational embedded surface is harmonic. Then by (13), $A(s)=B(s)=D(s)=0$. Thus, from the first equation of (12) we get $\kappa=0$, which is a contradiction. □

References

[1] Baikoussis, C. and Blair, D. E.: On the Gauss map of ruled surfaces. Glasgow Math. J. 34, 355-359 (1992).

[2] Baikoussis, C., Chen, B. Y and Verstraelen, L.: Ruled surfaces and tubes with finite type Gauss map. Tokyo J. Math. 16, 341-349 (1993).

[3] Baikoussis, C. and Verstraelen, L.: On the Gauss map of helicoidal surfaces. Rend. Sem. Mat. Messina Ser. II 16, 31-42 (1993)

[4] Chen, B. Y.: Total mean curvature and submanifolds of finite type. Series in Pure Mathematics. 1. World Scientific Publishing Co. Singapore, (1984).

[5] Chen, B. Y.: Finite type submanifolds and generalizations. Universita degli Studi di Roma "La Sapienza", Dipartimento di Matematica. IV, 68 p. (1985).

[6] Chen, B. Y., Choi, M. and Kim, Y. H.: Surfaces of revolution with pointwise 1-type Gauss map. J. Korean Math. Soc. 42, no. 3, 447-455 (2005).

[7] Chen, B. Y. and Ishikawa, S.: On classification of some surfaces of revolution of finite type. Tsukuba J. Math. 17, no. 1, 287-298 (1993).

[8] Chen, B. Y. and Piccinni, P.: Submanifolds with finite type Gauss map. Bull. Austral. Math. Soc. 35, no. 2, 161-186 (1987).

[9] Choi, M. and Kim, Y. H.: Characterization of the helicoid as ruled surfaces with pointwise 1-type Gauss map. Bull. Korean Math. Soc. 38, no. 4, 753-761 (2001).

[10] Ganchev, G. and Milousheva, V.: On the theory of surfaces in the four-dimensional Euclidean space. Kodai Math. J. 31, 183-198 (2008).

[11] Kim, Y. H. and Yoon, D. W.: Ruled surfaces with finite type Gauss map in Minkowski spaces. Soochow J. Math. 26, no. 1, 85-96 (2000).

[12] Kim, Y. H. and Yoon, D. W.: Ruled surfaces with pointwise 1-type Gauss map. J. Geom. Phys. 34, no. 3-4, 191-205 (2000).

[13] Kim, Y. H. and Yoon, D. W.: On the Gauss map of ruled surfaces in Minkowski space. Rocky Mountain J. Math. 35, no. 5, 1555-1581 (2005).

[14] Seaman, W.: Helicoids of constant mean curvature and their Gauss maps. Pacific J. Math. 110, no. 2, 387-396 (1984).

[15] Yoon, D. W.: Rotation surfaces with finite type Gauss map in E^4. Indian J. Pure Appl. Math. 32(12), 1803-1808 (2001).

[16] Yoon, D. W.: Some properties of the Clifford torus as rotation surface. Indian J. Pure Appl. Math. 34(6), 907-915 (2003).

Kadri ARSLAN, Betül BULCA,
Cengizhan MURATHAN
Department of Mathematics
Uludağ University
16059 Bursa, TURKEY
e-mail: arslan@uludag.edu.tr,

Received: 09.10.2009

Bengü KILIÇ BAYRAM
Department of Mathematics
Balıkesir University
Balıkesir, TURKEY
e-mail: benguk@balikesir.edu.tr

Young Ho KIM
Department of Mathematics
Kyunpook National University
Taegu, KOREA
e-mail: yhkim@knu.ac.kr

Günay ÖZTÜRK
Department of Mathematics
Kocaeli University
Kocaeli, TURKEY
e-mail: ogunay@kocaeli.edu.tr