Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 25 (2006), 535–554


# The Hilbert Problem for Generalized Q-Holomorphic Functions

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**Abstract.** In this work, we extended classical Hilbert boundary value problem to generalized $Q$-holomorphic functions by replacing the condition that the solution vanishes at infinity by that the solution has a finite order of growth at infinity.

**Keywords.** Generalized Beltrami systems, Q-Holomorphic functions, Hilbert boundary value problem

**Mathematics Subject Classification (2000).** Primary 30G20, secondary 30G35, 35J55

## 1. Introduction

Douglis [5] and Bojarskiĭ [4] developed an analogous of analytic function for elliptic systems in the plane of the form

$$w_{\overline{z}} - q w_z = 0, \tag{1}$$

where $w$ is an $m \times 1$ vector and $q$ is an $m \times m$ quasi-diagonal matrix. Also Bojarskiĭ assumed that all eigenvalues of $Q$ are less than 1. Such systems are considered because they arise from the reduction of general elliptic systems of first order in the plane to a standart canonical form. Subsequently Douglis and Bojarkii's theory has been used to study elliptic systems in the form

$$w_{\overline{z}} - q w_z = aw + b\overline{w}$$

and the solution of such equations were called generalized (or pseudo) hyperanalytic functions. Work in this direction appear in [7, 8, 10, 11]. These results extend the generalized (or "'pseudo"') analytic function theory of Bers [3] and Vekua [15]. Also classical boundary value problems for analytic functions were extended to generalized hyperanalytic functions. A good survey of the methods encountered in hyperanalytic case may be found in [2, 9].

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In [12], Hile noticed that what appears to be the essential property of elliptic systems in the plane for which one can obtain a useful extension of analytic function theory is the self commuting property of the variable matrix $Q$, which means

$$Q(z_1)\,Q(z_2) = Q(z_2)\,Q(z_1)$$

for any two points $z_1, z_2$ in the domain $G_0$ of $Q$. Further such a $Q$ matrix can not be brought into quasi-diagonal form of Bojarskiĭ by a similarity transformation. So Hile [12] attempts to extend the results of Douglis and Bojarskiĭ to a wider class of systems in the same form with equation (1). If $Q(z)$ is self-commuting in $G_0$ and if $Q(z)$ has no eigenvalues of magnitude 1 for each $z$ in $G_0$, then Hile called the system (1) generalized Beltrami system and the solutions of such a system are called *Q-holomorphic functions*. Later in [13, 14] using Vekua and Bers techniques a function theory as given for the equation

$$w_{\overline{z}} - Qw_z + Aw + B\overline{w} = 0, \tag{2}$$

where the unknown $w(z) = \{w_{ij}(z)\}$ is an $m \times s$ complex matrix, $Q(z) = \{q_{ij}(z)\}$ is a self commuting complex matrix with dimension $m \times m$, and $q_{k,k-1} \neq 0$ for $k = 1, \ldots m$. $A = \{a_{ij}(z)\}$ and $B = \{b_{ij}(z)\}$ are commuting with $Q$. Solutions of such equation were called *generalized Q-holomorphic functions*.

In this work, we generalize the classical Hilbert boundary value problem for analytic functions to the solutions of the equation (2) with the jump condition

$$w_+ - Hw_- = h$$

where $H$ is commuting with $Q$ and $h$ is an $m \times s$ complex matrix. In general, there is no similarity principle for generalized $Q$-holomorphic functions; hence the local behavior of these functions is not the same as for $Q$-holomorphic functions. Consequently this forces us to impose some conditions on coefficients of equation (2). For this reason we assume that the coefficients have also compact support. Also we assume that $Q$ is commuting with $\overline{Q}$.

We recall next a few elementary properties associated with the operator $\mathrm{D} := \frac{\partial}{\partial \overline{z}} - Q\frac{\partial}{\partial z}$. First there exits a so called generating solution $\phi(z) := \phi_0(z) + N(z)$ which satisfies the equation $\mathrm{D}\phi = 0$, where $N$ is the nilpotent part of $\phi$ and $\phi_0$ is the main diagonal term of $\phi$ satisfying the Beltrami equation

$$\frac{\partial \phi_0}{\partial \overline{z}} - \lambda \frac{\partial \phi_0}{\partial z} = 0$$

where $|\lambda| \neq 1$. Moreover $\phi(z)$ has the following property:

$$\left\| (\phi(\zeta) - \phi(z))^{-1} \right\| \leq \frac{M}{|\zeta - z|}. \tag{3}$$

## 2. Fundamental kernels

The fundamental kernels permit the formulation of the Hilbert boundary value problem for generalized $Q$-holomorphic functions as a Cauchy type integral relation.

**Lemma 2.1.** *Let $G$ be a regular domain, and $u \in C(\overline{G})$, $Du \in L^p(G)$ where $2 < p < \infty$. Then*

$$\int_\Gamma d\phi\,(z)\,u\,(z) = 2i \iint_G \phi_z\,(z)\,Du\,(z)\,dx\,dy\,,$$

*where $u$ commutes with $Q$ in $G$.*

*Proof.* Let $\{\psi_n\}$ be a sequence in $C_c^1(G)$ such that $\psi_n \to Du$ in $L^p(G)$. By Theorem 3.6 in [13], $J_G\psi_n \to J_G(Du)$ pointwise uniformly in the whole plane $\mathbb{C}$. In Theorem 6 given in [12], we set $v = \phi, u = J\psi_n$, to obtain

$$\int_\Gamma d\phi(z)\,(J_G\psi_n)\,(z) = 2i \iint \phi_z\,(z)\,\psi_n\,(z)\,dx\,dy.$$

Letting $n \to \infty$, we have

$$\int_\Gamma d\phi\,(z)\,(J_G Du)\,(z) = 2i \iint_G \phi_z\,(z)\,Du\,(z)\,dx\,dy\,.$$

By [13, Corollary 3.4], in $G$, $J_G\,(Du) = u - \Phi$, where $\Phi$ is $Q$-holomorphic in $G$. Furthermore $J_G\,(Du)$, $u \in C\left(\overline{G}\right)$, and thus $\Phi \in C\left(\overline{G}\right)$. Now we may use [12, Corollary 7] with $w \equiv \Phi$, to conclude $\int_\Gamma d\phi\,(z)\,\Phi\,(z) = 0$, and the lemma is proved. □

**Definition 2.2.** For fixed $A$ and $B$, we define the operator

$$Cw \equiv Dw + Aw + B\overline{w} \tag{4}$$

and an associated operator

$$\widetilde{C}v \equiv Dv - Av + B^*\overline{v}$$

where $B^*$ is defined by

$$B^* = \phi_z^{-1}\overline{\phi_z B}$$

**Theorem 2.3.** *Let $G$ be a regular domain, and $A$ and $B \in L^p(G)$ with $2 < p < \infty$. If $w$ and $v \in C(\overline{G})$, and satisfy, in $G$, $Cw = 0$, $\widetilde{C}v = 0$, then the integral $-\frac{1}{2i}\int_\Gamma d\phi(z)v(z)w(z)$ is a real matrix, where $v$ is commuting with $Q$.*

*Proof.* Since $w$, $v \in C(\overline{G})$, we have $Dw = -Aw - B\overline{w}$ and $Dv = Av - B^{*}v \in L^{p}(G)$. By Lemma 2.1 and [13, Theorem 3.9],

$$\begin{aligned}\frac{-1}{2i}\int_{\Gamma} d\phi vw &= -\iint_{G}\phi_z D\left(vw\right)dx\,dy\\ &= -\iint_{G}\phi_z\left[vDw + \left(Dv\right)w\right]dx\,dy\\ &= \iint_{G}\left(\overline{\phi_z B}v\overline{w} + \phi_z B\overline{v}w\right)dx\,dy\end{aligned}$$

which is a real matrix. □

**Theorem 2.4.** *Let $A$ and $B$ be commuting with $Q$ in $L^{p,2}(\mathbb{C})$, where $2 < p < \infty$. Then there exist a complex matrix of two variables commuting with $Q$, $X_1(z,\zeta)$ and $X_2(z,\zeta)$, with the properties*

1. *in $\mathbb{C} - \{\zeta\}$, for $j = 1, 2$,*

$$D_z X_j\left(z,\zeta\right) + A\left(z\right)X_j\left(z,\zeta\right) + B\left(z\right)\overline{X_j\left(z,\zeta\right)} = 0$$

*(here $D_z$ denotes our differential operator $D$ where differentiation is with respect to the variable $z$ rather than $\zeta$);*

2. *for $X_j \in B^{\alpha}(\mathbb{C})$, $\alpha = \frac{p-2}{p}$, and $\omega_j(z) = O\left(|z|^{-\alpha}\right)$ as $|z| \to \infty$, $j = 1, 2$,*

$$\begin{aligned}X_1\left(z,\zeta\right) &= \frac{1}{2}\left(\phi\left(\zeta\right) - \phi\left(z\right)\right)^{-1}\exp\left[\omega_1\left(z\right) - \omega_1\left(\zeta\right)\right]\\ X_2\left(z,\zeta\right) &= -\frac{i}{2}\left(\phi\left(\zeta\right) - \phi\left(z\right)\right)^{-1}\exp\left[\omega_2\left(z\right) - \omega_2\left(\zeta\right)\right].\end{aligned}$$

*Proof.* We temporarily fix a point $\zeta$ in $\mathbb{C}$, and define a function $\widehat{B}$ by

$$\widehat{B} = \left(\phi\left(z\right) - \phi\left(\zeta\right)\right)\overline{\left(\phi\left(z\right) - \phi\left(\zeta\right)\right)}^{-1}B\left(z\right).$$

We have $\widehat{B} \in L^{p,2}(\mathbb{C})$, since

$$\left\|\widehat{B}\left(z\right)\right\| \leq \left\|\left(\phi\left(z\right) - \phi\left(\zeta\right)\right)^{-1}\right\|\left\|\left(\phi\left(z\right) - \phi\left(\zeta\right)\right)\right\|\left\|B\left(z\right)\right\| \leq M\left(Q\right)\left\|B\left(z\right)\right\|.$$

Let us consider the integral equation

$$w\left(z\right) + \widetilde{J}w\left(z\right) - \widetilde{J}w\left(\zeta\right) = 1, \quad z \in \mathbb{C}, \tag{5}$$

where $\widetilde{J}w = J\left(Aw + \widehat{B}\overline{w}\right)$. If we define an operator $\widetilde{S}$ by

$$\left(\widetilde{S}w\right)\left(z\right) = \widetilde{J}w\left(z\right) - \widetilde{J}w\left(\zeta\right), \quad z \in \mathbb{C},$$

then equation (5) may be written as

$$w(z) + \left(\widetilde{S}w\right)(z) = 1\,. \tag{6}$$

Since by [13, Theorem 4.3], $\widetilde{J}$ is compact in the space $B(\mathbb{C})$, $\widetilde{S}$ is compact in $B(\mathbb{C})$. Therefore, in order to show that equation (5) has a unique solution in $B(\mathbb{C})$, it is sufficient to show the homogeneous equation has only the zero solution. Suppose that $v \in B(\mathbb{C})$ satisfies

$$v(z) + \widetilde{J}v(z) - \widetilde{J}v(\zeta) = 0, \quad z \in \mathbb{C}.$$

Differetiating this equation, we obtain

$$Dv + Av + \widehat{B}v = 0.$$

Since $v(\zeta) = 0$, by [13, Corollary 4.2], $v = 0$.

Thus we may let $w$ be the unique solution to equation (5). By [13, Theorem 4.4] this solution is commuting with $Q$. Differentiating equation (5), we obtain

$$Dw + Aw + \widehat{B}\overline{w} = 0\,.$$

By [13, Theorem 4.1], $w$ has the form $w(z) = C\exp\omega_1(z)$, where $C$ is a lower diagonal constant matrix, $\omega_1$ is a matrix valued function in $\mathbf{B}^{0,\alpha}$, $\alpha = \frac{p-2}{p}$ and $\omega_1(z) = O(|z|^\alpha)$. But since $w(\zeta) = 1$, we conclude $C = \exp[-\omega_1(\zeta)]$, and $w(z) = \exp[\omega_1(z) - \omega_1(\zeta)] \equiv w(z,\zeta)$. We now set

$$\begin{aligned} X_1(z,\zeta) &\equiv \frac{1}{2}(\phi(\zeta) - \phi(z))^{-1} w(z,\zeta) \\ &= \frac{1}{2}(\phi(\zeta) - \phi(z))^{-1} \exp[\omega_1(z) - \omega_1(\zeta)]\,. \end{aligned}$$

Then, for $z \in \mathbb{C} - \{\zeta\}$,

$$\begin{aligned} D_z X_1(z,\zeta) &= \frac{(\phi(\zeta) - \phi(z))^{-1}}{2} D_z w(z,\zeta) \\ &= \frac{(\phi(\zeta) - \phi(z))^{-1}}{2}\left(-Aw(z,\zeta) - \widehat{B}\overline{w(z,\zeta)}\right) \\ &= -A(z)\,X_1(z,\zeta) - B(z)\,\overline{X_1(z,\zeta)}. \end{aligned}$$

For the proof of $X_2$ we replace the 1 on the right-hand sides of equations (5) and (6) by $-i$. This serves to define the functions $w_2 = -i\exp(\omega_2(z) - \omega_2(\zeta))$ and $X_2(z,\zeta) = \frac{1}{2}(\phi(\zeta) - \phi(z))^{-1} w_2(z,\zeta)$. □

**Definition 2.5.** The fundamental kernels $\Omega_1$ and $\Omega_2$, associated with $A$ and $B$ in $L^{p,2}(\mathbb{C})$, are

$$\Omega_1(z,\zeta) \equiv X_1(z,\zeta) + iX_2(z,\zeta) \tag{7}$$
$$\Omega_2(z,\zeta) \equiv X_1(z,\zeta) - iX_2(z,\zeta), \tag{8}$$

where $X_1$ and $X_2$ are the functions described in Theorem 2.4.

**Theorem 2.6.** *The fundamental kernels $\Omega_1$ and $\Omega_2$ satisfy:*

1. *for each $\zeta$ in $C$, in $C - \{\zeta\}$*

$$D_z\Omega_1(z,\zeta) + A(z)\,\Omega_1(z,\zeta) + B(z)\,\overline{\Omega_2(z,\zeta)} = 0$$
$$D_z\Omega_2(z,\zeta) + A(z)\,\Omega_2(z,\zeta) + B(z)\,\overline{\Omega_1(z,\zeta)} = 0\,;$$

2. *for fixed $\zeta$, and $j = 1, 2$,*

$$\|\Omega_1(z,\zeta)\| = O\left(|z|^{-1}\right) \quad as \;\; |z| \to \infty\,;$$

3. *as $|z-\zeta| \to 0$,*

$$\left\|\Omega_1(z,\zeta) - (\phi(\zeta) - \phi(z))^{-1}\right\| = O\Big(\,|z-\zeta|^{-\frac{2}{p}}\,\Big) \tag{9}$$
$$\|\Omega_2(z,\zeta)\| = O\Big(\,|z-\zeta|^{-\frac{2}{p}}\,\Big). \tag{10}$$

*Proof.* Property 1 is readily verified from 1. of Theorem 2.4. Property 2 follow from the relations

$$\Omega_1(z,\zeta) = \frac{(\phi(z) - \phi(\zeta))^{-1}}{2}\left[\exp(\omega_1(z) - \omega_1(z)) + \exp(\omega_2(z) - \omega_2(z))\right]$$
$$\Omega_2(z,\zeta) = \frac{(\phi(z) - \phi(\zeta))^{-1}}{2}\left[\exp(\omega_1(z) - \omega_1(z)) - \exp(\omega_2(z) - \omega_2(z))\right]$$

because each $\omega_j$ is bounded in $\mathbb{C}$, and by (3).

To show 3., first when $zI + N$ is an $m \times m$ complex matrix where $N$ is nilpotent, it is possible to write

$$\exp(z+N) = (\exp z)\sum_{k=0}^{m-1} \frac{N^k}{k!},$$

and it is easily seen that the matrix valued function $\exp(z+N)$ is uniformly Lipshitz continuous whenever $z+N$ remains bounded. Hence, since $\omega_j$ is in $\mathbf{B}^{\mathbf{0},\alpha}(\mathbb{C})$ we have the result. □

**Theorem 2.7.** *Let $G$ be a regular domain, and let $A$ and $B$ be in $L^{p,2}(\mathbb{C})$ where $2 < p < \infty$. Furthermore let $w$ be in $C(\overline{G})$ and satisfy $Cw = Dw + Aw + B\overline{w} = 0$ in $G$. If $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are the fundamental kernels for the associated equation $\widetilde{C}v = Dv - Av + B^*\overline{v}$, then*

$$-P^{-1}\int_\Gamma \left\{ d\phi\left(\zeta\right)\widetilde{\Omega}_1\left(\zeta,z\right)w\left(\zeta\right) - \overline{d\phi\left(\zeta\right)\widetilde{\Omega}_2\left(\zeta,z\right)w\left(\zeta\right)}\right\} = \begin{cases} w\left(z\right), & \text{if } z \in G, \\ 0, & \text{if } z \notin G, \end{cases}$$

*where the constant matrix $P$ is defined by*

$$P = \int_{|z|=1} \left(zI + \overline{z}Q\right)^{-1}\left(Idz + Qd\overline{z}\right),$$

*is called $P$-value for the generalized Beltrami system (see* [12, p. 107]*).*

*Proof.* Let $\widetilde{X_1}$ and $\widetilde{X_2}$ be the corresponding solutions of $\widetilde{C}v = 0$ as described in Theorem 2.4. Using Theorem 2.3, we obtain the formulas, for $j = 1, 2$,

$$\begin{aligned}&\int_\Gamma \left\{ d\phi\left(\zeta\right)\widetilde{X_j}\left(\zeta,z\right)w\left(\zeta\right) - \overline{d\phi\left(\zeta\right)\widetilde{X_j}\left(\zeta,z\right)w\left(\zeta\right)}\right\} \\ &\quad = \begin{cases} \displaystyle\int_{|\zeta-z|=\epsilon} \left\{ d\phi\left(\zeta\right)\widetilde{X_j}\left(\zeta,z\right)w\left(\zeta\right) - \overline{d\phi\left(\zeta\right)\widetilde{X_j}\left(\zeta,z\right)w\left(\zeta\right)}\right\} & \text{if } z \in G \\ 0, & \text{if } z \notin G, \end{cases}\end{aligned}$$

where $\varepsilon$ is a sufficiently small positive number. We multiply by $i$ the equation for $j = 2$ and add to the equation for $j = 1$ to obtain

$$\begin{aligned}&\int_\Gamma \left\{ d\phi\left(\zeta\right)\widetilde{\Omega_1}\left(\zeta,z\right)w\left(\zeta\right) - \overline{d\phi\left(\zeta\right)\widetilde{\Omega_2}\left(\zeta,z\right)w\left(\zeta\right)}\right\} \\ &\quad = \begin{cases} \displaystyle\int_{|\zeta-z|=\epsilon} \left\{ d\phi\left(\zeta\right)\widetilde{\Omega_1}\left(\zeta,z\right)w\left(\zeta\right) - \overline{d\phi\left(\zeta\right)\widetilde{\Omega_2}\left(\zeta,z\right)w\left(\zeta\right)}\right\} & \text{if } z \in G \\ 0, & \text{if } z \notin G. \end{cases}\end{aligned}$$

Using (9) and (10) we obtain, for $z$ in $G$,

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{|\zeta-z|=\epsilon} \left\{ d\phi\left(\zeta\right)\widetilde{\Omega_1}\left(\zeta,z\right)w\left(\zeta\right) - \overline{d\phi\left(\zeta\right)\widetilde{\Omega_2}\left(\zeta,z\right)w\left(\zeta\right)}\right\} \\ &\qquad = \lim_{\epsilon\to 0}\int_{|\zeta-z|=\epsilon} d\phi\left(\zeta\right)\left(\phi\left(z\right)-\phi\left(\zeta\right)\right)w\left(\zeta\right).\end{aligned}$$

But Hile showed (see [12, p. 114]), using the continuity of $w$, that the latter limit is $Pw(z)$. Thus the theorem is proved. □

**Theorem 2.8.** *Let $A$ and $B$ be in $L^{p,2}(C)$, where $2 < p < \infty$. Let $\Omega_1$, $\Omega_2$ be the fundamental kernels for the equation $Cw = Dw + Aw + B\overline{w} = 0$ and $\widetilde{\Omega_1}$, $\widetilde{\Omega_2}$ be the fundamental kernels for the equation $\widetilde{C}v = Dv - Av + B^*\overline{v} = 0$. Then, for $z \neq \zeta$,*

$$\Omega_1(z,\zeta) = -\widetilde{\Omega_1}(\zeta,z), \qquad \Omega_2(z,\zeta) = -\overline{\widetilde{\Omega_2}(\zeta,z)}.$$

*Proof.* Let $z$, $\zeta$ be fixed, $z \neq \zeta$, and let $\epsilon$ be small enough such that $0 < \epsilon < |z-\zeta| < \frac{1}{\epsilon}$. Then by the previous theorem, for $j = 1, 2$,

$$\begin{aligned} X_j(z,\zeta) = &-P^{-1}\int_{|s-\zeta|=\frac{1}{\epsilon}} d\phi(s)\,\widetilde{\Omega}_1(s,z)\,X_j(s,\zeta) \\ &+ P^{-1}\int_{|s-\zeta|=\frac{1}{\epsilon}} \overline{d\phi(s)\,\widetilde{\Omega}_2(s,z)\,X_j(s,\zeta)} \\ &+ P^{-1}\int_{|s-\zeta|=\epsilon} d\phi(s)\,\widetilde{\Omega}_1(s,z)\,X_j(s,\zeta) \\ &+ P^{-1}\int_{|s-\zeta|=\epsilon} \overline{d\phi(s)\,\widetilde{\Omega}_2(s,z)\,X_j(s,\zeta)}\,. \end{aligned}$$

Using Theorem 2.6 and the relations (9), (10), we obtain the estimates

$$\begin{aligned} \left\|\widetilde{\Omega_j}(s,z)\right\|,\ \|X_j(s,z)\| &= O\left(|s|^{-1}\right) && \text{as } |s| \to \infty \\ \left\|X_1(s,\zeta) - \frac{(\phi(\zeta)-\phi(z))^{-1}}{2}\right\| &= O\left(|s-\zeta|^{-\frac{2}{p}}\right) && \text{as } |s-\zeta| \to \infty \\ \left\|X_2(s,\zeta) - \frac{(\phi(\zeta)-\phi(z))^{-1}}{2i}\right\| &= O\left(|s-\zeta|^{-\frac{2}{p}}\right) && \text{as } |s-\zeta| \to \infty. \end{aligned}$$

Letting $\epsilon \to 0$, we therefore obtain

$$\begin{aligned} X_1(z,\zeta) = \lim_{\epsilon\to 0}\frac{P^{-1}}{2}\Bigg\{&\int_{|s-\zeta|=\epsilon} d\phi(s)\,\widetilde{\Omega_1}(s,z)\,[\phi(\zeta)-\phi(z)]^{-1} \\ &- \int_{|s-\zeta|=\epsilon} \overline{d\phi(s)\,\widetilde{\Omega_2}(s,z)\,[\phi(\zeta)-\phi(z)]^{-1}}\Bigg\} \\ X_2(z,\zeta) = \lim_{\epsilon\to 0}\frac{-iP^{-1}}{2}\Bigg\{&\int_{|s-\zeta|=\epsilon} d\phi(s)\,\widetilde{\Omega_1}(s,z)\,[\phi(\zeta)-\phi(z)]^{-1} \\ &- \int_{|s-\zeta|=\epsilon} \overline{d\phi(s)\,\widetilde{\Omega_2}(s,z)\,[\phi(\zeta)-\phi(z)]^{-1}}\Bigg\}. \end{aligned}$$

As in the proof of the previous theorem, we remark that Hile has shown (see [12, p. 114]) that the above limits are

$$X_1(z,\zeta) = -\frac{1}{2}\left[\widetilde{\Omega_1}(\zeta,z) + \widetilde{\Omega_2}(\zeta,z)\right]$$
$$X_2(z,\zeta) = -\frac{1}{2i}\left[\widetilde{\Omega_1}(\zeta,z) - \widetilde{\Omega_2}(\zeta,z)\right].$$

The relations (7), (8) complete the proof. □

**Theorem 2.9.** *Let $G$ be a regular domain, and $A$ and $B$ in $L^{p,2}(\mathbb{C})$ where $2 < p < \infty$. Furthermore, let $w$ be in $C\left(\overline{G}\right)$ and satisfy $Cw = Dw + Aw + B\overline{w} = 0$ in $G$. Then*

$$P^{-1}\int_\Gamma \left[d\phi(\zeta)\,\Omega_1(z,\zeta)\,w(\zeta) - d\overline{\phi(\zeta)}\Omega_2(z,\zeta)\,\overline{w(\zeta)}\right] = \begin{cases} w(z), & \text{if } z \in G \\ 0, & \text{if } z \notin \overline{G}. \end{cases}$$

## 3. The Plemelj formulas

Let $\Gamma = \Gamma_0 + \Gamma_1 + \cdots + \Gamma_r$ be a collection of $r+1$ disjoint contours in $C^{1,\alpha}(\mathbb{C})$ and let the interior of the contour $\Gamma_0$ contain the other contours. By $G^+$ we denote the $(r+1)$-connected domain interior for $\Gamma_0$ and exterior for $\Gamma_1, \ldots, \Gamma_r$. By $G^-$ we denote the complement of $G^+ + \Gamma$ in the entire complex plane. To be definite, we assume that the origin lies in $G^+$. As usual, we orient $\Gamma_0$ so that it is counterclockwise positive, and thus clockwise positive for the other contours $\Gamma_k$.

Let us first show that we can define the singular Cauchy integral

$$S\varphi(\tau) := 2P^{-1}\int_\Gamma d\phi(\zeta)\left[\phi(\zeta) - \phi(\tau)\right]^{-1}\varphi(\tau)$$

in a principal value sense when $\tau \in \Gamma$ and $\varphi$ is Hölder continuous on $\Gamma$. We assume that each contour composing $\Gamma$ is parameterized with respect to the arc length from some fixed point on the contour. For $\tau \in \Gamma$, $0 < \epsilon$, let $\Gamma_\epsilon = \Gamma\backslash\{\zeta : |\zeta - \tau| < \epsilon\}$ and $\tau_i(\epsilon)$ be the endpoints of $\Gamma_\epsilon$.

For simplification, we introduce $\Delta(\zeta,\tau) = \frac{N(\zeta)-N(\tau)}{\phi_0(\zeta)-\phi_0(\tau)}$ . Then

$$\int_{\Gamma_\epsilon} d\phi(\zeta)\left[\phi(\zeta) - \phi(\tau)\right]^{-1} = \int_{\Gamma_\epsilon}\left(d\phi_0(\zeta) + dN(\zeta)\right)\sum_{k=0}^{m-1}(-1)^k\Delta^k(\zeta,\tau).$$

Since

$$\Delta^k\frac{dN(\zeta)}{\phi_0(\zeta) - \phi_0(\tau)} = \frac{1}{k+1}d\Delta^{k+1} + \Delta^{k+1}\frac{d\phi_0(\zeta)}{\phi_0(\zeta) - \phi_0(\tau)}$$

we have

$$d\phi(\zeta)\left[\phi(\zeta)-\phi(\tau)\right]^{-1}=\frac{d\phi_0(\zeta)}{\phi_0(\zeta)-\phi_0(\tau)}+\sum_{k=1}^{m-1}\frac{(-1)^{k-1}}{k}d\Delta^k(\zeta,\tau).$$

Thus

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Gamma_\epsilon} d\phi(\zeta)\left[\phi(\zeta)-\phi(\tau)\right]^{-1}&=\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\frac{d\phi_0(\zeta)}{\phi_0(\zeta)-\phi_0(\tau)}\\&\quad+\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\sum_{k=1}^{m-1}\frac{(-1)^{k-1}}{k}d\left(\frac{N(\zeta)-N(\tau)}{\phi_0(\zeta)-\phi_0(\tau)}\right)^k.\end{aligned}$$

Define for $\zeta,\ \tau\in\Gamma$

$$\Theta(\zeta,\tau)=\begin{cases}\frac{N(\zeta)-N(\tau)}{\phi_0(\zeta)-\phi_0(\tau)}, & \zeta\neq\tau\\ \left.\frac{\dot N(\zeta)}{\dot\phi_0(\zeta)}\right|_{s=s_\tau}, & \zeta=\tau\,,\end{cases}$$

where $s_\tau$ is the value of the arc-length parameter corresponding to $\tau$, and overdots denote differentation with respect to $s$. Since $N$, $\phi_0$ and $\tau(s)$ all have Hölder continuous first derivatives with respect to their arguments, we conclude that $\Theta(\zeta,\tau)$ is Hölder continuous with respect to each argument, separetely. Moreover

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} d\phi(\zeta)\left[\phi(\zeta)-\phi(\tau)\right]^{-1}&=\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon}\frac{d\phi_0(\zeta)}{\phi_0(\zeta)-\phi_0(\tau)}\\&\quad+\lim_{\varepsilon\to 0}\sum_{k=1}^{m-1}\frac{(-1)^{k-1}}{k}\left[\Theta(\zeta,\tau)\right]^k\Big|_{\tau_1(\varepsilon)}^{\tau_2(\varepsilon)}\\&=\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon}\frac{d\phi_0(\zeta)}{\phi_0(\zeta)-\phi_0(\tau)}.\end{aligned}$$

Since $\phi_0$ is a solution of the Beltrami equation, it can be shown that the Beltrami equation has a solution $\rho(z)\in C^{1,\alpha}(\mathbb{C})$ (see [15, Chapter II]). This solution can be found by $\rho(z)=\phi_0(z)$ in the case of $|\lambda|\le q_0<1$ and by $\rho(z)=\phi_0(\overline{z})$ in the case of $|\lambda|\ge q_0>1$ with nonegative constant $q_0$. Hence we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Gamma_\epsilon}\frac{d\phi_0(\zeta)}{\phi_0(\zeta)-\phi_0(\tau)}&=\lim_{\varepsilon\to 0}\log\frac{\phi_0(\tau_2(\epsilon))-\phi_0(\tau)}{\phi_0(\tau_1(\epsilon))-\phi_0(\tau)}\\&=\begin{cases}\pi i, & |\lambda|\le q_0<1\\ -\pi i, & |\lambda|\ge q_0>1\end{cases}\\&=\frac{P}{2}.\end{aligned}$$

Thus in view of the assumed Hölder continuity of $\varphi$,

$$\begin{aligned}&\lim_{\varepsilon\to 0} 2P^{-1}\int_{\Gamma_\epsilon} d\phi(\zeta)\left[\phi(\zeta)-\phi(\tau)\right]^{-1}\varphi(\zeta)\\&\quad=\lim_{\varepsilon\to 0} 2P^{-1}\int_{\Gamma_\epsilon} d\phi(\zeta)\left[\phi(\zeta)-\phi(\tau)\right]^{-1}(\varphi(\zeta)-\varphi(\tau))+\varphi(\tau).\end{aligned}$$

Note that the integrand of the last integral is weakly singular, and hence this term is defined in the usual sense. Hence for the principal value of $S\varphi$, we have

$$\begin{aligned}(S\varphi)(\tau)&=\lim_{\varepsilon\to 0} 2P^{-1}\int_{\Gamma_\epsilon} d\phi(\zeta)\left[\phi(\zeta)-\phi(\tau)\right]^{-1}\varphi(\zeta)\\&=2P^{-1}\int_{\Gamma} d\phi(\zeta)\left[\phi(\zeta)-\phi(\tau)\right]^{-1}(\varphi(\zeta)-\varphi(\tau))+\varphi(\tau).\end{aligned}\tag{11}$$

If

$$\Phi\varphi(z):=P^{-1}\int_{\Gamma} d\phi(\zeta)\left[\phi(\zeta)-\phi(\tau)\right]^{-1}\varphi(\zeta)\,,\quad z\in G^+\cup G^-\,,\tag{12}$$

then for $\tau\in\Gamma$

$$\begin{aligned}(\Phi\varphi)_+(\tau)&=P^{-1}\int_{\Gamma} d\phi(\zeta)\left[\phi(\zeta)-\phi(\tau)\right]^{-1}(\varphi(\zeta)-\varphi(\tau))\\&\quad+\lim_{\substack{z\in G^+\\ z\to\tau}} P^{-1}\int_{\Gamma} d\phi(\zeta)\left[\phi(\zeta)-\phi(\tau)\right]^{-1}\varphi(\tau)\\&=\frac{1}{2}(S\varphi)(\tau)+\frac{1}{2}\varphi(\tau).\end{aligned}$$

With an analogous argument, that is, for $\tau\in\Gamma$, $z\in G^-$ and $z\to\tau$ we obtain

$$(\Phi\varphi)_-(\tau)=\frac{1}{2}(S\varphi)(\tau)-\frac{1}{2}\varphi(\tau).$$

Also it is easily seen that the Hölder continuity of $\Phi$ follows from the Hölder continuity of the Cauchy integral (see [6, p. 41]) and the Hölder continuity of first derivatives of Beltrami homeomorphisms. Thus we have

**Theorem 3.1.** *Let $\varphi$ be an $m\times s$ complex matrix in $C^\alpha(\Gamma)$ and let $S\varphi$ and $\Phi\varphi$ be as defined in* (11) *and* (12). *Then*

$$\begin{aligned}(\Phi\varphi)_+(\tau)&=\frac{1}{2}(S\varphi)(\tau)+\frac{1}{2}\varphi(\tau)\\(\Phi\varphi)_-(\tau)&=\frac{1}{2}(S\varphi)(\tau)-\frac{1}{2}\varphi(\tau).\end{aligned}$$

*Moreover, $\Phi$ is Hölder continuous in $G^+$ and $G^-$.*

**Lemma 3.2.** *If $\varphi$ is an $m \times s$ complex matrix in $C^{\alpha}(\Gamma)$, then*

$$w(z) = P^{-1} \int_{\Gamma} \left\{ d\phi(\zeta)\, \Omega_1(z,\zeta)\, \varphi(\zeta) - \overline{d\phi(\zeta)} \Omega_2(z,\zeta)\, \overline{\varphi(\zeta)} \right\} \tag{13}$$

*is a generalized Q-holomorphic function in each component of $\mathbb{C} - \Gamma$ and fulfills*

$$w_+(\tau) = w(\tau) + \frac{1}{2}\varphi(\tau)\,, \quad w_-(\tau) = w(\tau) - \frac{1}{2}\varphi(\tau) \tag{14}$$

*where $\tau \in \Gamma$. The first integral in (13) for $\tau \in \Gamma$ has to be understood in the Cauchy principal-value sense.*

*Proof.* Because of the local behavior of the kernels as $|\zeta - z| \to 0$,

$$\Omega_1(z,\zeta) = [\phi(\zeta) - \phi(z)]^{-1} + O\left(|\zeta - z|^{-\frac{2}{p}}\right)$$
$$\Omega_1(z,\zeta) = O\left(|\zeta - z|^{-\frac{2}{p}}\right)$$

and the Plemelj formula for paremeter dependent integrals (see [6, p. 51]), it follows that

$$(w - \Phi)_+(\tau) = (w - \Phi)(\tau) = (w - \Phi)_-(\tau) \quad (\tau \in \Gamma)\,.$$

Therefore $w - \Phi$ is continuous even on $\Gamma$, and (14) follows. □

**Theorem 3.3.** *In order that the Hölder continuous function $\gamma$ given on $\Gamma$ represents the boundary value of $w_+$ of a solution of (4) which is Hölder continuous in the closure $\widehat{G}^+$ of $G^+$ and vanishes at infinity, the conditions*

$$\operatorname{Re} \int_{\Gamma} d\phi(\zeta)\, v(\zeta)\, \gamma(\zeta) = 0 \tag{15}$$

$$\int_{\Gamma} \left[ d\phi(\zeta)\, \Omega_1(z,\zeta)\, \gamma(\zeta) - \overline{d\phi(\zeta)} \Omega_2(z,\zeta)\, \overline{\gamma(\zeta)} \right] = 0 \quad \left(z \in G^-\right) \tag{16}$$

*are necessary. Here equation (15) holds for every solution $v$ of the associated equation $Dv - Av + B^* \overline{v} = 0$ of (4) defined in $\widehat{G}^+$.*

*Proof.* Equation (15) is readily verified in Theorem 2.3. To see the validity of (16) one has to consider

$$u(z) = \begin{cases} u_1(z) - w(z)\,, & z \in G^+ \\ u_1(z)\,, & z \in G^- \end{cases}$$
$$u_1(z) = P^{-1} \int_{\Gamma} \left[ d\phi(\zeta)\, \Omega_1(z,\zeta)\, \gamma(\zeta) - \overline{d\phi(\zeta)} \Omega_2(z,\zeta)\, \overline{\gamma(\zeta)} \right] \tag{17}$$

which fulfils the boundary condition $u_+ - u_- = (u_1)_+ - (u_1)_- - w_+ = \gamma - w_+$ of $\Gamma$ and $u(z) = O\left(|z|^{-1}\right)(z \to \infty)$. If on $\Gamma$ $w_+ = \gamma$, then $u$ is a continuous generalized $Q$-holomorphic function in $\mathbb{C}$. For $u$, we have

$$\begin{aligned}&\sum_{l=1}^{s}\left(\frac{\partial u_{1l}}{\partial \overline{z}} - \lambda\frac{\partial u_{1l}}{\partial z}\right)e^{1l} + \sum_{i=2}^{m}\sum_{l=1}^{s}\left(\frac{\partial u_{il}}{\partial \overline{z}} - \sum_{j=1}^{i} q_{ij}\frac{\partial u_{jl}}{\partial z}\right)e^{il}\\ &+\sum_{i=1}^{m}\sum_{l=1}^{s}\sum_{j=1}^{i}\left(a_{ij}u_{jl} + b_{ij}\overline{u}_{jl}\right)e^{il} = 0\,,\end{aligned} \tag{18}$$

where $\left(e^{il}\right)$ denotes the $m \times s$ matrix in which the entry at the $ith$ row and $jth$ column is 1 and the other terms are 0. For $i = 1$, we obtain

$$\frac{\partial u_{1l}}{\partial \overline{z}} - \lambda\frac{\partial u_{1l}}{\partial z} + a_0 u_{1l} + b_0\overline{u}_{1l} = 0.$$

Since $u_{1l}$ is bounded and vanishes at infinity, it is equal to zero by the similarity principle [15]. Similarly we continue successively taking $i = 2, \ldots, m$ in (18), we obtain $u \equiv 0$. □

**Theorem 3.4.** *The condition* (16) *is sufficient for* $\gamma$ *to be the boundary value of a generalized* $Q$*-holomorphic function in* $G^+$*.*

*Proof.* The function $u_1$ given in (17) is a generalized $Q-$holomorphic function in $G^+$ and $G^-$ and $(u_1)_+ - (u_1)_- = \gamma$ on $\Gamma$. Since (16) holds, $u_1$ vanishes identically in $G^-$ so that $(u_1)_- \equiv 0$ on $\Gamma$. □

## 4. The Hilbert boundary value problem

We consider the problem

$$\begin{aligned} Dw + Aw + B\overline{w} = 0, &\qquad w(\infty) = 0\\ w_+ - Hw_- = h &\qquad \text{on } \Gamma\,,\end{aligned} \tag{19}$$

where $Q$ is commuting with $\overline{Q}$, $A$, $B$ and $H$ are commuting with $Q$ and $h$ is an $m \times s$ complex matrix. We assume that $A$ and $B$ vanish outside of some bounded domain $G^*$ and $A, B \in L^p(G^*)$ for some $p > 2$. The boundary matrix $H$ is assumed to be Hölder continuous on $\Gamma$ and $\det H \neq 0$.

If $f$ is commuting with $Q$, then $f$ can be written as $f = f_0 I + N_f$, where $N_f$ is nilpotent (see [13, p. 438]). From this fact we may define, for a complex matrix which is commuting with $Q$ as in hyperanalytic case,

$$\begin{aligned}\exp f &= e^{f_0}\left(\sum_{k=0}^{m-1}\frac{1}{k!}\left(N_f\right)^k\right)\\ \log f &= \log f_0 + \sum_{k=0}^{m-1}\frac{(-1)^{k-1}}{k}\left(\frac{N_f}{f_0}\right)^k, \quad f_0 \neq 0\,.\end{aligned} \tag{20}$$

The functions $\exp f$ and $\log f$ are also commuting with $Q$.

**Theorem 4.1.** *Let $f$ and $g$ be complex matrices commuting with $Q$. Then*

1. $e^{f+g} = e^f e^g$;
2. $\log e^f = e^{\log f} = f$, $f_0 \neq 0$.

*Proof.* The assertions 1. and 2. can be obtained by collecting powers of nilpotent parts. □

All other elementary properties of exponents and logorithms can be derived from 1. and 2.

**Definition 4.2.** The *index* of problem (19) is

$$\kappa := \frac{1}{2\pi}\Delta_\Gamma \arg H_0 = \frac{1}{2\pi i}\int_\Gamma d\log H_0 = \sum_{k=0}^{r} \lambda_k,$$

where$\lambda_k = \frac{1}{2\pi}\Delta_{\Gamma_k} \arg H_0$, $k = 0, \ldots, r$, and $\Gamma_k$ is traversed positively.

As in the analytic case (see [6, p. 95]) we seek a canonical factorization of the matrix $H = H_0 I + N_H$ commuting with $Q$, where $N_H$ is the nilpotent part of $H$.

Let us start by seeking a $Q$-holomorphic function commuting with $Q$ and satisfying the jump condition

$$\chi_+(\tau) - H(\tau)\chi_-(\tau) = 0, \quad \tau \in \Gamma.$$

For a matrix commuting with $Q$, it is clear that $\det H \neq 0$ is equivalent to $H_0(\tau) \neq 0$. Taking logarithms, we obtain

$$\log \chi_+(\tau) - \log \chi_-(\tau) = \log H(\tau). \tag{21}$$

Observe from (20) that all considerations concerning single valuedness reduce to those of $\log H_0$. Thus $\log H$ is single valued if the change in the argument of $H_0$ is zero after traversing any of the bounding curves $\Gamma_k$.

Let $z_k$ be a fixed point in the interior of $\Gamma_k$, $k = 1, \ldots, r$. Also let

$$P(z) = \prod_{k=1}^{r} [\phi(z) - \phi(z_k)]^{\lambda_k}.$$

The main diagonal term of $\phi(z)^{-\kappa} P(z) H(z)$ is

$$\phi_0(z)^{-\kappa} \prod_{k=1}^{r} [\phi_0(z) - \phi_0(z_k)]^{\lambda_k} H_0(z),$$

and

$$\Delta_{\Gamma_k} \arg \phi_0(z)^{-\kappa} \prod_{l=1}^{r} [\phi_0(z) - \phi_0(z_l)]^{\lambda_l} H_0(z) = 0, \quad k = 0, \ldots, r.$$

Thus if $\hat{H} = \log\left[\phi(z)^{-\kappa} P(z) H(z)\right]$, then $\hat{H}$ is single valued and Hölder continuous on $\Gamma$. Instead of (21) we write

$$\log[P(\tau)\chi_+(\tau)] - \log[\phi(\tau)^{\kappa}\chi_-(\tau)] = \hat{H},$$

and we have as a consequence of the Plemelj formulas

$$\chi(z) = \begin{cases} P^{-1}(z) \exp P^{-1} \int_\Gamma d\phi(\zeta) [\phi(\zeta) - \phi(z)]^{-1} \hat{H}, & z \in G^+ \\ \phi(z)^{-\kappa} \exp P^{-1} \int_\Gamma d\phi(\zeta) [\phi(\zeta) - \phi(z)]^{-1} \hat{H}, & z \in G^-. \end{cases}$$

Thus $H(\tau) = \chi_+(\tau)(\chi_0)_-(\tau)\phi(\tau)^{\kappa}$ for $\tau \in \Gamma$, where

$$[\chi_0(z)]^{-1} = \exp P^{-1} \int_\Gamma d\phi(\zeta) [\phi(\zeta) - \phi(z)]^{-1} \hat{H}, \quad z \in G^-$$

and since $\phi$, $\hat{H}$ are commuting with $Q$, $\chi_0$ is also commuting with $Q$ (see [13, p. 438]).

**Definition 4.3.** $H(\tau) = \chi_+(\tau)(\chi_0)_-(\tau)\phi(\tau)^{\kappa}$, $\tau \in \Gamma$, is a *canonical factorization* of $H$ if

1. $\chi(z)$ is a $Q$-holomorphic function, invertible and commuting with $Q$ in $G^+$;
2. $\chi_0(z)$ is a $Q$-holomorphic function in $G^-$, invertible in $G^- \cup \{\infty\}$ and commuting with $Q$;
3. $\kappa$ is an integer.

Let us return to the Hilbert problem for solutions to $Dw + Aw + B\overline{w} = 0$. Define for a given function $w$

$$\widetilde{w} = \begin{cases} \chi^{-1}(z)\, w(z), & z \in G^+ \\ \chi_0(z)\, w(z), & z \in G^-. \end{cases}$$

The problem (19) is clearly equivalent to

$$\begin{aligned} D\widetilde{w} + A\widetilde{w} + \widetilde{B}\overline{\widetilde{w}} &= 0, & \widetilde{w}(\infty) &= 0 \\ \widetilde{w}_+(\tau) - \phi(\tau)^{\kappa}\widetilde{w}_-(\tau) &= \widetilde{h} & \text{on } \Gamma, \end{aligned} \tag{22}$$

where $\widetilde{h} = \chi^{-1} h$ and

$$\widetilde{B} = \begin{cases} \chi^{-1} B \chi, & z \in G^+ \\ \chi_0 B \overline{\chi_0}^{-1}, & z \in G^-. \end{cases}$$

`Case 1:` $\kappa = 0$. If $\widetilde{\Omega_1}$ and $\widetilde{\Omega_2}$ are fundamental kernels for $A$ and $\widetilde{B}$, then

$$\widetilde{w} = P^{-1} \int_\Gamma \left[ d\phi(\zeta)\, \widetilde{\Omega_1}(z,\zeta)\, \widetilde{h}(\zeta) - \overline{d\phi(\zeta)}\widetilde{\Omega_2}(z,\zeta)\, \overline{\widetilde{h}(\zeta)} \right] \tag{23}$$

is the unique solution of (22). Thereby the following theorem is proved.

**Theorem 4.4.** *In case $\kappa = 0$, the general solution of* (22) *is given by* (23).

`Case 2:` $\kappa > 0$. We seek first a special solution having the property that $\lim_{z\to\infty} \phi(z)^\kappa \widetilde{w}(z) = 0$. For such a solution $\widetilde{w}$, let

$$\widetilde{w_1} := \begin{cases} \widetilde{w}(z), & z \in G^+ \\ \phi(z)^\kappa \widetilde{w}(z), & z \in G^-. \end{cases}$$

Then $\widetilde{w_1}$ is a solution to the Hilbert problem

$$\begin{aligned} D\widetilde{w_1} + A\widetilde{w_1} + \widetilde{B_1}\overline{\widetilde{w_1}} &= 0, \qquad \widetilde{w_1}(\infty) = 0 \\ \widetilde{(w_1)}_+(\tau) - \widetilde{(w_1)}_-(\tau) &= \widetilde{h} \qquad \text{on } \Gamma, \end{aligned}$$

where

$$\widetilde{B_1} = \left\{ \begin{matrix} \widetilde{B}(z), & z \in G^+ \\ \phi(z)^\kappa \overline{\phi(z)}^{-\kappa} \widetilde{B}, & z \in G^- \end{matrix} \right\} \in L^{p,2}(\mathbb{C}).$$

The solution of this problem is uniquely defined by

$$\widetilde{w}_1 = P^{-1} \int_\Gamma \left\{ d\phi(\zeta)\, \widetilde{\Omega_1^{(1)}}(z,\zeta)\, \widetilde{h}(\zeta) - \overline{d\phi(\zeta)}\widetilde{\Omega_2^{(1)}}(z,\zeta)\, \overline{\widetilde{h}(\zeta)} \right\}, \tag{24}$$

where $\widetilde{\Omega_k^{(1)}}$ $(k = 1, 2)$ are fundamental kernels belonging to $A$, $\widetilde{B_1}$. With this solution we find a special solution to problem (22) which is

$$\widetilde{w}(z) = \begin{cases} \widetilde{w_1}(z), & z \in G^+ \\ \phi(z)^{-\kappa} \widetilde{w_1}(z), & z \in G^-. \end{cases}$$

To complete the solution to the problem we must characterize all solutions to the homogeneous problem ($\widetilde{h} = 0$). Let $(\widehat{F}_k, \widehat{G}_k)$ be a generating pair associated with $A$ and $\widetilde{B_1}\overline{\phi}^k \phi^{-k}$, $k = 0, \ldots, \kappa-1$, then (see [14, p. 944]) $\widehat{F}_k$, $\widehat{G}_k$ are bounded and continuous solutions of $Dw + Aw + \widetilde{B_1}\overline{\phi}^k \phi^{-k} w = 0$ in the entire plane such that $\widehat{F_k}(\infty) = I$, $\widehat{G_k}(\infty) = iI$. Then for $k = 0, \ldots, \kappa - 1$, the functions

$$\begin{aligned} \widetilde{F_k} &:= \begin{cases} \phi(z)^k \widehat{F}_k(z), & z \in G^+ \\ \phi(z)^{k-\kappa} \widehat{F}_k(z) & z \in G^- \end{cases} \\ \widetilde{G_k} &:= \begin{cases} \phi(z)^k \widehat{G}_k(z), & z \in G^+ \\ \phi(z)^{k-\kappa} \widehat{G}_k(z) & z \in G^- \end{cases} \end{aligned} \tag{25}$$

form a set of solutions for problem (22) with $\widetilde{h}=0$ such that $\widetilde{F_k}$ and $\widetilde{G_k}$ have poles of order $k-\kappa$ at infinity.

Let $\sum_{k=0}^{\kappa-1}\left(\widetilde{F_k}\lambda_k+\widetilde{G_k}\mu_k\right)=0$, where $\lambda_k$ and $\mu_k$ are $m\times s$ real matrices, then from

$$\sum_{k=0}^{\kappa-1}\left(\phi^{k-\kappa+1}\widehat{F}_k\lambda_k+\phi^{k-\kappa+1}\widehat{G}_k\mu_k\right)=0$$

it follows by letting z tend to infinity that $\lambda_{\kappa-1}\equiv 0$, $\mu_{\kappa-1}\equiv 0$ (see [14, p. 945]). Proceding term by term one obtains $\lambda_k\equiv 0$, $\mu_k\equiv 0$ $(0\le k\le \kappa-1)$. On the other hand $\sum_{k=0}^{\kappa-1}\left(\widetilde{F_k}\lambda_k+\widetilde{G_k}\mu_k\right)$ is a solution of (22) for $\widetilde{h}=0$.

**Theorem 4.5.** *Every solution of the homogeneous problem* (22) *has the form*

$$\sum_{k=0}^{\kappa-1}\left(\widetilde{F_k}\lambda_k+\widetilde{G_k}\mu_k\right),$$

*where* $\widetilde{F_k}$, $\widetilde{G_k}$, $(k=0,\dots,\kappa-1)$ *are defined by* (25) *and* $\lambda_k$ *and* $\mu_k$ *are real* $m\times s$ *matrices.*

*Proof.* Let $\widetilde{w}$ be arbitary solution of homogeneous problem (22) . As $A$ and $\widetilde{B}$ vanish near infinity, $\widetilde{w}$ is $Q-$holomorphic there and satisfies the asymptotic condition $\widetilde{w}\,(z)=O\big(\left|z\right|^{l-\kappa}\big)$ for some $0\le l\le\kappa-1$. Let

$$w_0:=\widetilde{w}-\sum_{k=0}^{l}\left(\widetilde{F_k}\lambda_k+\widetilde{G_k}\mu_k\right),$$

where $\lambda_k$ and $\mu_k$ real matrices are to be determined. If we choose $\lambda_l+i\mu_l=\lim_{z\to\infty}\phi\,(z)^{\kappa-l}\,\widetilde{w}\,(z)$, then $w_0$ must be $O\big(\left|z\right|^{l-\kappa-1}\big)$ at infinity. Proceeding in this manner, we conclude that with the choices

$$\lambda_\nu+i\mu_\nu=\lim_{z\to\infty}\phi\,(z)^{\kappa-\nu}\left\{\widetilde{w}-\sum_{k=\nu+1}^{l}\left(\widetilde{F_k}\lambda_k+\widetilde{G_k}\mu_k\right)\right\}$$

for $\nu=l,l-1,\dots,0$, the function $w_0$ is $O\left(\left|z\right|^{-\kappa-1}\right)$. Consequently,

$$\widetilde{w_0}:=\begin{cases}w_0, & z\in G^+\\ \phi^\kappa w_0, & z\in G^-\end{cases}$$

defines a function in $\mathbb{C}$ and vanishes at infinity, and consequently $\widetilde{w_0}=0$. Therefore $\widetilde{w}=\sum_{k=0}^{l}\left(\widetilde{F_k}\lambda_k+\widetilde{G_k}\mu_k\right)$. □

This proves the following theorem.

**Theorem 4.6.** *The general solution of* (22) *with nonnegative index has the form*

$$\widetilde{w}(z) = \sum_{k=0}^{\kappa-1}\left(\widetilde{F_k}\lambda_k + \widetilde{G_k}\mu_k\right) + \begin{cases} \widetilde{w_1}(z), & z \in G^+ \\ \phi(z)^{-\kappa}\,\widetilde{w_1}(z), & z \in G^-, \end{cases}$$

*where* $\lambda_k$, $\mu_k$ $(0 \le k \le \kappa - 1)$ *are arbitary real* $m\times s$ *matrices. Here* $\widetilde{w_1}$ *is defined by* (24)*, and* $\widetilde{F_k}$ *and* $\widetilde{G_k}$ *are special solutions of the homogeneous problem* (22) *given by the formula* (25)

`Case 3:` $\kappa < 0$. In the problem (22) again we let

$$\widetilde{w}_1 := \begin{cases} \widetilde{w}(z), & z \in G^+ \\ \phi(z)^{\kappa}\,\widetilde{w}(z), & z \in G^-, \end{cases}$$

then $\widetilde{w}_1$ is a solution to

$$D\widetilde{w}_1 + A\widetilde{w}_1 + \widetilde{B}_1\overline{\widetilde{w}}_1 = 0, \quad \widetilde{w}_1(\infty) = O\left(|z|^{\kappa-1}\right)$$
$$(\widetilde{w}_1)_+ - (\widetilde{w}_1)_- = \widetilde{h} \quad \text{on } \Gamma,$$

where

$$\widetilde{B}_1(z) := \begin{cases} \widetilde{B}(z), & z \in G^+ \\ \phi(z)^{\kappa}\,\overline{\phi(z)}^{-\kappa}\,\widetilde{B}(z), & z \in G^-. \end{cases}$$

The solution, if any, is given by

$$\widetilde{w_1} = P^{-1}\int_{\Gamma}\left\{d\phi(\zeta)\,\widetilde{\Omega_1^{(1)}}(z,\zeta)\,\widetilde{h}(\zeta) - d\overline{\phi(\zeta)}\widetilde{\Omega_2^{(1)}}(z,\zeta)\,\overline{\widetilde{h}(\zeta)}\right\}. \tag{26}$$

Since $w = \chi\widetilde{w_1}$ and $\chi$ has a pole of order $-\kappa$ at infinity $\widetilde{w_1}$ vanishes at infinity and, moreover, has a zero of order greater than or equal to one. So $w$ has a pole of order not exceeding $-\kappa-1$. In view of the assumption that $A$ and $\widetilde{B_1}$ vanish in a neigborhood of infinity, $\widetilde{w_1}$ is $Q$-holomorphic at infinity and therefore it is expanded in a series of negative powers of $\phi$, i.e., $\widetilde{w_1} = \sum_{k=1}^{\infty}\phi^{-k}c_k$ for $|z|$ big enough (see [12, p. 116]). Hence the first $\kappa$ coefficients have to vanish, namely $c_k = 0$ $(0 \le k \le -\kappa - 1)$ in order that $w$ be regular at infinity. As the coefficients $A$ and $\widetilde{B_1}$ vanish in a neigborhood of infinity, we have

$$\Omega_1^{(1)}(z,\zeta) = [\phi(\zeta) - \phi(z)]^{-1}$$
$$\Omega_2^{(1)}(z,\zeta) = 0$$

so that

$$\widetilde{w_1}(z) = P^{-1} \int_\Gamma d\phi\,(\zeta)\,[\phi\,(\zeta) - \phi\,(z)]^{-1}\,\widetilde{h}\,(\zeta) \quad \left(z \in G^-\right).$$

By this, the coefficients $c_k$ have the form

$$c_k = P^{-1} \int_\Gamma d\phi\,(\zeta)\,\phi^{k-1}\,(\zeta)\,\widetilde{h}(\zeta) \quad (k \in \mathbb{N})$$

and the additional conditions on $\widetilde{h}$ are seen to be

$$\int_\Gamma d\phi\,(\zeta)\,\phi^{k-1}\,(\zeta)\,\widetilde{h}(\zeta) = 0 \quad (1 \le k \le -\kappa - 1)\,. \tag{27}$$

**Theorem 4.7.** *In the case of negative index, the nonhomegenous problem* (22) *is in general unsolvable. In order that it be solvable it is necassary and sufficient that $\widetilde{h}$ satisfies $-\kappa - 1$ condition* (27)*. In this case the unique solution (as well as in the case $\kappa = -1$) is given by*

$$\widetilde{w} = \begin{cases} \widetilde{w_1}, & z \in G^+ \\ \phi^{-\kappa}\widetilde{w_1}, & z \in G^-\,, \end{cases}$$

*where $\widetilde{w_1}$ is defined by* (26)*.*

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Received March 29, 2005; revised September 26, 2005