

NULLITY CONDITIONS IN PARACONTACT GEOMETRY

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ABSTRACT. The paper is a complete study of paracontact metric manifolds for which the Reeb vector field of the underlying contact structure satisfies a nullity condition (the condition (1.2) below, for some real numbers $\tilde{\kappa}$ and $\tilde{\mu}$). This class of pseudo-Riemannian manifolds, which includes para-Sasakian manifolds, was recently defined in [13]. In this paper we show in fact that there is a kind of duality between those manifolds and contact metric (κ, μ) -spaces. In particular, we prove that, under some natural assumption, any such paracontact metric manifold admits a compatible contact metric (κ, μ) -structure (eventually Sasakian). Moreover, we prove that the nullity condition is invariant under \mathcal{D} -homothetic deformations and determines the whole curvature tensor field completely. Finally non-trivial examples in any dimension are presented and the many differences with the contact metric case, due to the non-positive definiteness of the metric, are discussed.

1. Introduction

A contact metric (κ, μ) -space is a contact Riemannian manifold $(M, \varphi, \xi, \eta, g)$ such that the Reeb vector field ξ belongs to the so-called (κ, μ) -nullity distribution, i.e. the curvature tensor field satisfies the condition

$$(1.1) \quad R_{XY}\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

for some real numbers κ and μ , where $2h$ denotes the Lie derivative of φ in the direction of ξ . This new class of Riemannian manifolds was introduced in [5] as a natural generalization both of the Sasakian condition $R_{XY}\xi = \eta(Y)X - \eta(X)Y$ and of those contact metric manifolds satisfying $R_{XY}\xi = 0$ which were studied by D. E. Blair in [3]. Nowadays contact (κ, μ) -manifolds are considered a very important topic in contact Riemannian geometry. In fact in despite of the technical appearance of the definition, there are good reasons for studying (κ, μ) -spaces. The first is that, in the non-Sasakian case (that is for $\kappa \neq 1$), the condition (1.1) determines the curvature tensor field completely; next, (κ, μ) -spaces provide non-trivial examples of some remarkable classes of contact Riemannian manifolds, like CR-integrable contact metric manifolds ([23]), H -contact manifolds ([22]), harmonic contact metric manifolds ([24]), or contact Riemannian manifolds with η -parallel torsion tensor ([17]); moreover, a local classification is known ([6]).

Recently, in [13], an unexpected relationship between contact (κ, μ) -spaces and paracontact geometry was found. It was proved (cf. Theorem 2.6 below) that any (non-Sasakian) contact (κ, μ) -space carries a canonical paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ whose Levi-Civita connection satisfies a condition formally similar to (1.1)

$$(1.2) \quad \tilde{R}_{XY}\xi = \tilde{\kappa}(\eta(Y)X - \eta(X)Y) + \tilde{\mu}(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y)$$

where $2\tilde{h} := \mathcal{L}_\xi \tilde{\varphi}$ and, in this case, $\tilde{\kappa} = (1 - \mu/2)^2 + \kappa - 2$, $\tilde{\mu} = 2$.

We recall that paracontact manifolds are smooth manifolds of dimension $2n+1$ endowed with a 1-form η , a vector field ξ and a field of endomorphisms of tangent spaces $\tilde{\varphi}$ such that $\eta(\xi) = 1$, $\tilde{\varphi}^2 = I - \eta \otimes \xi$

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and $\tilde{\varphi}$ induces an almost paracomplex structure on the codimension 1 distribution defined by the kernel of η (see § 2 for more details). If, in addition, the manifold is endowed with a pseudo-Riemannian metric \tilde{g} of signature $(n, n + 1)$ satisfying

$$\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = -\tilde{g}(X, Y) + \eta(X)\eta(Y), \quad d\eta(X, Y) = \tilde{g}(X, \tilde{\varphi}Y),$$

(M, η) becomes a contact manifold and $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ is said to be a paracontact metric structure on M . The study of paracontact geometry was initiated by Kaneyuki and Williams in [18] and then it was continued by many other authors. Very recently a systematic study of paracontact metric manifolds, and some remarkable subclasses like para-Sasakian manifolds, was carried out by Zamkovoy ([25]). The importance of paracontact geometry, and in particular of para-Sasakian geometry, has been pointed out especially in the last years by several papers highlighting the interplays with the theory of para-Kähler manifolds and its role in pseudo-Riemannian geometry and mathematical physics (cf. e.g. [1], [2], [16], [14], [15]).

These considerations motivate us to study the class of paracontact metric manifolds satisfying the nullity condition (1.2), for some constant real numbers $\tilde{\kappa}$ and $\tilde{\mu}$. We call these pseudo-Riemannian manifolds *paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifolds*. As we will see the class of paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifolds is very large. It contains para-Sasakian manifolds, as well as those paracontact metric manifolds satisfying $\tilde{R}_{XY}\xi = 0$ for all $X, Y \in \Gamma(TM)$ (recently studied in [26]). But, unlike in the contact Riemannian case, a paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} = -1$ in general is not para-Sasakian. There are in fact paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifolds such that $\tilde{h}^2 = 0$ (which is equivalent to take $\tilde{\kappa} = -1$) but with $\tilde{h} \neq 0$. Another important difference with the contact Riemannian case, due to the non-positive definiteness of the metric, is that while for contact metric (κ, μ) -spaces the constant κ can not be greater than 1, here we have no restrictions for the constants $\tilde{\kappa}$ and $\tilde{\mu}$.

In § 3 we study the common properties for the cases $\tilde{\kappa} < -1$, $\tilde{\kappa} = -1$, $\tilde{\kappa} > -1$. We prove for instance that while the values of $\tilde{\kappa}$ and $\tilde{\mu}$ change, the form of (1.2) remains unchanged under \mathcal{D} -homothetic deformations. Moreover we prove that any paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold is *integrable* (in the sense of [25]), i.e. its canonical paracontact connection preserves $\tilde{\varphi}$, and we find some general properties of the curvature.

Since the geometric behavior of the manifold is very different according to the circumstance that $\tilde{\kappa} < -1$ or $\tilde{\kappa} > -1$, we study separately the two cases. In particular, in both cases we prove that the $(\tilde{\kappa}, \tilde{\mu})$ -nullity condition (1.2) determines the whole curvature tensor field completely. In fact we are able to find an explicit formula for the curvature, depending on the tensors $\tilde{\varphi}$, \tilde{h} , $\tilde{\varphi}h$. It is interesting that the same formula holds both for the case $\tilde{\kappa} < -1$ and $\tilde{\kappa} > -1$. Then we find the values of $\tilde{\kappa}$ and $\tilde{\mu}$ for which the pseudo-Riemannian metric in question is η -Einstein, i.e. $\text{Ric} = aI + b\eta \otimes \xi$, for some $a, b \in \mathbb{R}$ and we prove that, unlike the contact metric case, if $\tilde{\kappa} < -1$ there are Einstein paracontact $(\tilde{\kappa}, \tilde{\mu})$ -metrics in dimension greater than 3.

In both cases $\tilde{\kappa} < -1$ and $\tilde{\kappa} > -1$ the geometry of the paracontact metric manifold can be related to the theory of Legendre foliations. Namely, if $\tilde{\kappa} > -1$ then the operator \tilde{h} is diagonalizable and the eigendistributions corresponding to the constant eigenvalues $\pm\tilde{\lambda}$, where $\tilde{\lambda} = \sqrt{1 + \tilde{\kappa}}$, define two mutually orthogonal and totally geodesic Legendre foliations. Whereas, if $\tilde{\kappa} < -1$, the role before played by \tilde{h} is now played by the operator $\tilde{\varphi}\tilde{h}$. Such operator is diagonalizable with the same eigenvalues as \tilde{h} . The main difference with the previous case is that, while the eigendistributions corresponding to $\pm\tilde{\lambda}$ (where now $\tilde{\lambda} = \sqrt{-1 - \tilde{\kappa}}$) still define two mutually orthogonal Legendre foliations, they are not totally geodesic but they are totally umbilical. Then, by using the theory of Legendre foliations, we prove that under some natural assumptions, a paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold carries a contact Riemannian structure compatible with the contact form η , which in turn satisfies a (κ, μ) -nullity condition, for some constant real numbers κ and μ depending on $\tilde{\kappa}$ and $\tilde{\mu}$. Therefore, in view of such a result and [13, Theorem 4.7], it seems that there is a kind of duality between contact and paracontact structures satisfying nullity conditions.

Furthermore, we find non-trivial examples of paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifolds. We construct examples of left-invariant paracontact $(\tilde{\kappa}, \tilde{\mu})$ -structures on Lie groups and, moreover, we show that the tangent sphere bundle of a Riemannian manifold of constant sectional curvature c carries two canonical paracontact $(\tilde{\kappa}_i, \tilde{\mu}_i)$ -structures $(\tilde{\varphi}_i, \xi, \eta, \tilde{g}_i)$, $i \in \{1, 2\}$, (same η and ξ , where ξ is twice the geodesic flow), with

$$\begin{aligned}\tilde{\kappa}_1 &= (1 + c)^2 - 1, & \tilde{\mu}_1 &= 2(1 - |c - 1|), \\ \tilde{\kappa}_2 &= 4c - 1, & \tilde{\mu}_2 &= 2.\end{aligned}$$

Hence, according to the value of c , we obtain examples of paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifolds such that $\tilde{\kappa} < -1$ and $\tilde{\kappa} > -1$. Also we prove that when the base manifold M is flat than the second structure provides an example of paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} = -1$ but which is not para-Sasakian. To the knowledge of the authors these are the first paracontact metric structures defined on the tangent sphere bundle.

Many questions about paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifolds remain open. Apart of the problem of finding other non-trivial examples, the case of strictly non-para-Sasakian paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifolds with $\tilde{\kappa} = -1$ is worthy to be studied. In particular it should be important to find sufficient conditions for such manifolds in order to be para-Sasakian. Other natural questions are to provide a classification of such manifolds, at least in the 3-dimensional case, and to study further the unexpected interplays with contact Riemannian geometry which we have found in this paper.

2. PRELIMINARIES

A differentiable manifold M of dimension $2n + 1$ is said to be a *contact manifold* if it carries a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. It is well known that then there exists a unique vector field ξ (called the *Reeb vector field*) such that $i_\xi \eta = 1$ and $i_\xi d\eta = 0$. The $2n$ -dimensional distribution transversal to the Reeb vector field defined by $\mathcal{D} := \ker(\eta)$ is called the *contact distribution*. Any contact manifold (M, η) admits a Riemannian metric g and a $(1, 1)$ -tensor field φ such that

$$(2.1) \quad \varphi^2 = -I + \eta \otimes \xi, \quad \varphi \xi = 0, \quad \eta(X) = g(X, \xi)$$

$$(2.2) \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \varphi Y) = d\eta(X, Y),$$

for any vector field X and Y on M . The contact manifold (M, η) together with the geometric structure (φ, ξ, η, g) is then called *contact metric manifold* (or *contact Riemannian manifold*). Let h be the operator defined by $h = \frac{1}{2}\mathcal{L}_\xi \varphi$, where \mathcal{L} denotes Lie differentiation. The tensor field h vanishes identically if and only if the vector field ξ is Killing and in this case the contact metric manifold is said to be *K-contact*. It is well known that h and φh are symmetric operators, and

$$\varphi h + h\varphi = 0, \quad h\xi = 0, \quad \eta \circ h = 0, \quad \text{tr}h = \text{tr}\varphi h = 0,$$

where $\text{tr}h$ denotes the trace of h . Since h anti-commutes with φ , if X is an eigenvector of h corresponding to the eigenvalue λ then φX is also an eigenvector of h corresponding to the eigenvalue $-\lambda$. Moreover, for any contact metric manifold M , the following relation holds

$$(2.3) \quad \nabla_X \xi = -\varphi X - \varphi hX$$

where ∇ is the Levi-Civita connection of (M, g) . If a contact metric manifold M is *normal*, in the sense that the tensor field $N_\varphi := [\varphi, \varphi] + 2d\eta \otimes \xi$ vanishes identically, then M is called a *Sasakian manifold*. Equivalently, a contact metric manifold is Sasakian if and only if $R_{XY}\xi = \eta(Y)X - \eta(X)Y$. Any Sasakian manifold is *K-contact* and in dimension 3 the converse also holds (cf. [4]).

As a natural generalization of the above Sasakian condition one can consider contact metric manifolds satisfying

$$(2.4) \quad R_{XY}\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$$

for some real numbers κ and μ . (2.4) is called (κ, μ) -nullity condition. This type of Riemannian manifolds was introduced and deeply studied by Blair, Koufogiorgos and Papantoniou in [5] and a few years earlier by Koufogiorgos for the case $\mu = 0$ ([19]). Among other things, they proved the following result.

Theorem 2.1 ([5]). *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric (κ, μ) -manifold. Then necessarily $\kappa \leq 1$ and $\kappa = 1$ if and only if M is Sasakian. Moreover, if $\kappa < 1$, the contact metric manifold M admits three mutually orthogonal and integrable distributions $\mathcal{D}_h(0)$, $\mathcal{D}_h(\lambda)$ and $\mathcal{D}_h(-\lambda)$ defined by the eigenspaces of h , where $\lambda = \sqrt{1 - \kappa}$.*

The standard contact metric structure on the tangent sphere bundle T_1M satisfies the (κ, μ) -nullity condition if and only if the base manifold M has constant curvature c . In this case $\kappa = c(2 - c)$ and $\mu = -2c$ ([5]). Other examples can be found in [6].

Now we recall the notion of almost paracontact manifold (cf. [18]). An $(2n + 1)$ -dimensional smooth manifold M is said to have an *almost paracontact structure* if it admits a $(1, 1)$ -tensor field $\tilde{\varphi}$, a vector field ξ and a 1-form η satisfying the following conditions:

- (i) $\eta(\xi) = 1$, $\tilde{\varphi}^2 = I - \eta \otimes \xi$,
- (ii) the tensor field $\tilde{\varphi}$ induces an almost paracomplex structure on each fibre of $\mathcal{D} = \ker(\eta)$, i.e. the ± 1 -eigendistributions, $\mathcal{D}^\pm := \mathcal{D}_{\tilde{\varphi}(\pm 1)}$ of $\tilde{\varphi}$ have equal dimension n .

From the definition it follows that $\tilde{\varphi}\xi = 0$, $\eta \circ \tilde{\varphi} = 0$ and the endomorphism $\tilde{\varphi}$ has rank $2n$. When the tensor field $N_{\tilde{\varphi}} := [\tilde{\varphi}, \tilde{\varphi}] - 2d\eta \otimes \xi$ vanishes identically the almost paracontact manifold is said to be *normal*. If an almost paracontact manifold admits a pseudo-Riemannian metric \tilde{g} such that

$$(2.5) \quad \tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = -\tilde{g}(X, Y) + \eta(X)\eta(Y),$$

for all $X, Y \in \Gamma(TM)$, then we say that $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is an *almost paracontact metric manifold*. Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n, n + 1)$. For an almost paracontact metric manifold, there always exists an orthogonal basis $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \xi\}$ such that $\tilde{g}(X_i, X_j) = \delta_{ij}$, $\tilde{g}(Y_i, Y_j) = -\delta_{ij}$ and $Y_i = \tilde{\varphi}X_i$, for any $i, j \in \{1, \dots, n\}$. Such basis is called a $\tilde{\varphi}$ -basis.

If in addition $d\eta(X, Y) = \tilde{g}(X, \tilde{\varphi}Y)$ for all vector fields X, Y on M , $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is said to be a *paracontact metric manifold*. In a paracontact metric manifold one defines a symmetric, trace-free operator $\tilde{h} := \frac{1}{2}\mathcal{L}_\xi\tilde{\varphi}$. It is known ([25]) that \tilde{h} anti-commutes with $\tilde{\varphi}$ and satisfies $\tilde{h}\xi = 0$ and

$$(2.6) \quad \tilde{\nabla}\xi = -\tilde{\varphi} + \tilde{\varphi}\tilde{h},$$

where $\tilde{\nabla}$ is the Levi-Civita connection of the pseudo-Riemannian manifold (M, \tilde{g}) . Moreover $\tilde{h} \equiv 0$ if and only if ξ is a Killing vector field and in this case $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is said to be a *K-paracontact manifold*. A normal paracontact metric manifold is called a *para-Sasakian manifold*. Also in this context the para-Sasakian condition implies the *K-paracontact* condition and the converse holds only in dimension 3. Moreover, in any para-Sasakian manifold

$$(2.7) \quad \tilde{R}_{XY}\xi = -(\eta(Y)X - \eta(X)Y),$$

holds, but unlike contact metric geometry the condition (2.7) not necessarily implies that the manifold is para-Sasakian.

In [25], the author proved the following results, which will be used in next sections:

Theorem 2.2 ([25]). *On a paracontact metric manifold, the following identities hold:*

$$(2.8) \quad (\tilde{\nabla}_{\tilde{\varphi}X}\tilde{\varphi})\tilde{\varphi}Y - (\tilde{\nabla}_X\tilde{\varphi})Y = 2\tilde{g}(X, Y)\xi - \eta(Y)(X - \tilde{h}X + \eta(X)\xi),$$

$$(2.9) \quad \tilde{R}_{\xi X}\xi + \tilde{\varphi}\tilde{R}_{\xi\tilde{\varphi}X}\xi = 2(\tilde{\varphi}^2X - \tilde{h}^2X),$$

$$(2.10) \quad \tilde{R}(\xi, X, Y, Z) = -\tilde{g}(Y, (\tilde{\nabla}_X\tilde{\varphi})Z) + \tilde{g}(X, (\tilde{\nabla}_Y\tilde{\varphi}\tilde{h})Z) - \tilde{g}(X, (\tilde{\nabla}_Z\tilde{\varphi}\tilde{h})Y),$$

$$(2.11) \quad \begin{aligned} \tilde{R}(\xi, X, Y, Z) + \tilde{R}(\xi, X, \tilde{\varphi}Y, \tilde{\varphi}Z) - \tilde{R}(\xi, \tilde{\varphi}X, \tilde{\varphi}Y, Z) - \tilde{R}(\xi, \tilde{\varphi}X, Y, \tilde{\varphi}Z) \\ = -2(\tilde{\nabla}_{\tilde{h}X}\tilde{\Phi})(Y, Z) + 2\eta(Y)\tilde{g}(X - \tilde{h}X, Z) - 2\eta(Z)\tilde{g}(X - \tilde{h}X, Y), \end{aligned}$$

where $\tilde{\Phi} := \tilde{g}(\cdot, \tilde{\varphi}\cdot)$ is the fundamental 2-form of the paracontact metric structure.

Moreover, in any paracontact metric manifold Zamkovoy introduced a canonical connection which plays the same role in paracontact geometry of the generalized Tanaka-Webster connection ([23]) in a contact metric manifold.

Theorem 2.3 ([25]). *On a paracontact metric manifold there exists a unique connection $\tilde{\nabla}^{pc}$, called the canonical paracontact connection, satisfying the following properties:*

- (i) $\tilde{\nabla}^{pc}\eta = 0$, $\tilde{\nabla}^{pc}\xi = 0$, $\tilde{\nabla}^{pc}\tilde{g} = 0$,
- (ii) $(\tilde{\nabla}_X^{pc}\tilde{\varphi})Y = (\tilde{\nabla}_X\tilde{\varphi})Y + \tilde{g}(X - \tilde{h}X, Y)\xi - \eta(Y)(X - \tilde{h}X)$,
- (iii) $\tilde{T}^{pc}(\xi, \tilde{\varphi}Y) = -\tilde{\varphi}\tilde{T}^{pc}(\xi, Y)$,
- (iv) $\tilde{T}^{pc}(X, Y) = 2d\eta(X, Y)\xi$ on $\mathcal{D} = \ker(\eta)$.

Moreover $\tilde{\nabla}^{pc}$ is given by

$$(2.12) \quad \tilde{\nabla}_X^{pc}Y = \tilde{\nabla}_XY + \eta(X)\tilde{\varphi}Y + \eta(Y)(\tilde{\varphi}X - \tilde{\varphi}\tilde{h}X) + \tilde{g}(X - \tilde{h}X, \tilde{\varphi}Y)\xi.$$

An almost paracontact structure $(\tilde{\varphi}, \xi, \eta)$ is said to be *integrable* if $N_{\tilde{\varphi}}(X, Y) \in \Gamma(\mathbb{R}\xi)$ whenever $X, Y \in \Gamma(\mathcal{D})$. For paracontact metric structures, the integrability condition is equivalent to $\tilde{\nabla}^{pc}\tilde{\varphi} = 0$ ([25]).

As pointed out in [8], paracontact geometry is strictly related to the theory of Legendre foliations. Recall that a *Legendrian distribution* on contact manifold (M, η) is an n -dimensional subbundle L of the contact distribution such that $d\eta(X, X') = 0$ for all $X, X' \in \Gamma(L)$. When L is integrable, we speak of *Legendrian foliation*. Legendre foliations have been extensively investigated in recent years from various points of views. In particular Pang ([21]) provided a classification of Legendrian foliations using a bilinear symmetric form $\Pi_{\mathcal{F}}$ on tangent bundle of the foliation \mathcal{F} , defined by

$$(2.13) \quad \Pi_{\mathcal{F}}(X, X') = -(\mathcal{L}_X\mathcal{L}_{X'}\eta)(\xi) = 2d\eta([\xi, X], X').$$

Then he called \mathcal{F} *flat*, *degenerate*, *non-degenerate*, *positive (negative) definite* according to the circumstance that $\Pi_{\mathcal{F}}$ vanishes identically, is degenerate, non-degenerate, positive (negative) definite, respectively. For a non-degenerate Legendre foliation \mathcal{F} , Libermann ([20]) defined a linear map $\Lambda_{\mathcal{F}} : TM \rightarrow T\mathcal{F}$, whose kernel is $T\mathcal{F} \oplus \mathbb{R}\xi$, such that

$$(2.14) \quad \Pi_{\mathcal{F}}(\Lambda_{\mathcal{F}}Z, X) = d\eta(Z, X)$$

for any $Z \in \Gamma(TM)$, $X \in \Gamma(T\mathcal{F})$. The operator $\Lambda_{\mathcal{F}}$ is surjective, satisfies $(\Lambda_{\mathcal{F}})^2 = 0$ and $\Lambda_{\mathcal{F}}[\xi, X] = \frac{1}{2}X$ for all $X \in \Gamma(T\mathcal{F})$. Then we can extend $\Pi_{\mathcal{F}}$ to a symmetric bilinear form on all TM by putting

$$\bar{\Pi}_{\mathcal{F}}(Z, Z') := \begin{cases} \Pi_{\mathcal{F}}(Z, Z') & \text{if } Z, Z' \in \Gamma(T\mathcal{F}) \\ \Pi_{\mathcal{F}}(\Lambda_{\mathcal{F}}Z, \Lambda_{\mathcal{F}}Z'), & \text{otherwise.} \end{cases}$$

An *(almost) bi-Legendrian manifold* (cf. [8]) is by definition a contact manifold (M, η) endowed with two transversal Legendrian distributions (foliations) L_1 and L_2 , so that $TM = L_1 \oplus L_2 \oplus \mathbb{R}\xi$. (L_1, L_2) is then called an *(almost) bi-Legendrian structure* on the contact manifold (M, η) . Any paracontact metric manifold $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ carries a canonical almost bi-Legendrian structure given the eigendistributions

\mathcal{D}^+ and \mathcal{D}^- of $\tilde{\varphi}$ corresponding to the eigenvalues ± 1 . Conversely, every almost bi-Legendrian manifold admits a compatible paracontact metric structure ([8]). We notice also that the integrability in the sense of paracontact geometry, i.e. $\nabla^{pc}\tilde{\varphi} = 0$, is equivalent to the involutiveness of the Legendre distributions \mathcal{D}^\pm (cf. [8, Corollary 3.3]).

Any almost bi-Legendrian manifold admits a canonical connection, called *bi-Legendrian connection*, which plays an important role in the study of almost bi-Legendrian manifolds:

Theorem 2.4 ([7]). *Let (M, η, L_1, L_2) be an almost bi-Legendrian manifold. There exists a unique connection ∇^{bl} such that*

- (i) $\nabla^{bl}L_1 \subset L_1$, $\nabla^{bl}L_2 \subset L_2$, $\nabla^{bl}(\mathbb{R}\xi) \subset \mathbb{R}\xi$,
- (ii) $\nabla^{bl}d\eta = 0$,
- (iii) $T^{bl}(X, Y) = 2d\eta(X, Y)\xi$ for all $X \in \Gamma(L_1)$, $Y \in \Gamma(L_2)$,
 $T^{bl}(X, \xi) = [\xi, X_{L_1}]_{L_2} + [\xi, X_{L_2}]_{L_1}$ for all $X \in \Gamma(TM)$,

where X_{L_1} and X_{L_2} denote, respectively, the projections of X onto the subbundles L_1 and L_2 of TM , according to the decomposition $TM = L_1 \oplus L_2 \oplus \mathbb{R}\xi$.

By using the properties of the bi-Legendrian connection one can point out the relationship between contact metric (κ, μ) -spaces and the theory of Legendre foliations. Namely, we have the following characterization.

Theorem 2.5 ([12]). *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold, which is not K -contact. Then $(M, \varphi, \xi, \eta, g)$ is a contact metric (κ, μ) -manifold if and only if it admits two mutually orthogonal Legendre distributions L and Q and a unique linear connection $\bar{\nabla}$ satisfying the following properties:*

- (i) $\bar{\nabla}L \subset L$, $\bar{\nabla}Q \subset Q$,
- (ii) $\bar{\nabla}\eta = 0$, $\bar{\nabla}d\eta = 0$, $\bar{\nabla}g = 0$, $\bar{\nabla}\varphi = 0$, $\bar{\nabla}h = 0$,
- (iii) $\bar{T}(X, Y) = 2d\eta(X, Y)\xi$ for all $X, Y \in \Gamma(\mathcal{D})$,
 $\bar{T}(X, \xi) = [\xi, X_L]_Q + [\xi, X_Q]_L$ for all $X \in \Gamma(TM)$,

Furthermore $\bar{\nabla}$ is uniquely determined and coincide with the bi-Legendrian connection of (L, Q) , L and Q are integrable and coincides with the eigenspaces $\mathcal{D}_h(\lambda)$ and $\mathcal{D}_h(-\lambda)$ of the operator h .

On the other hand contact (κ, μ) -manifolds are also related to paracontact geometry, as shown by the following result which is one of the motivations for the present paper.

Theorem 2.6 ([13]). *Let $(M, \varphi, \xi, \eta, g)$ be a non-Sasakian contact metric (κ, μ) -space. Then M admits a canonical paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ given by*

$$(2.15) \quad \tilde{\varphi} := \frac{1}{\sqrt{1-\kappa}}h, \quad \tilde{g} := \frac{1}{\sqrt{1-\kappa}}d\eta(\cdot, h\cdot) + \eta \otimes \eta.$$

Furthermore the curvature tensor field of the Levi-Civita connection of (M, \tilde{g}) satisfies a $(\tilde{\kappa}, \tilde{\mu})$ -nullity condition

$$(2.16) \quad \tilde{R}_{XY}\xi = \tilde{\kappa}(\eta(Y)X - \eta(X)Y) + \tilde{\mu}(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y),$$

where

$$\tilde{\kappa} = \kappa - 2 + \left(1 - \frac{\mu}{2}\right)^2, \quad \tilde{\mu} = 2.$$

3. PRELIMINARY RESULTS ON PARACONTACT $(\tilde{\kappa}, \tilde{\mu})$ -MANIFOLDS

Theorem 2.6 motivates the following definition.

Definition 3.1 ([13]). *A paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ -manifold is a paracontact metric manifold for which the curvature tensor field satisfies*

$$(3.1) \quad \tilde{R}_{XY}\xi = \tilde{\kappa}(\eta(Y)X - \eta(X)Y) + \tilde{\mu}(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y)$$

for all vector fields X, Y on M and for some real constants $\tilde{\kappa}$ and $\tilde{\mu}$.

In this section, we discuss some properties of paracontact metric manifolds satisfying the condition (3.1). We start with some preliminary properties.

Lemma 3.2. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ -manifold of dimension $2n + 1$. Then the following identities hold:*

$$(3.2) \quad \tilde{h}^2 = (1 + \tilde{\kappa})\tilde{\varphi}^2,$$

$$(3.3) \quad \tilde{Q}\xi = 2n\tilde{\kappa}\xi,$$

$$(3.4) \quad (\tilde{\nabla}_X \tilde{\varphi})Y = -\tilde{g}(X - \tilde{h}X, Y)\xi + \eta(Y)(X - \tilde{h}X), \quad \text{for } \tilde{\kappa} \neq -1,$$

$$(3.5) \quad (\tilde{\nabla}_X \tilde{h})Y - (\tilde{\nabla}_Y \tilde{h})X = -(1 + \tilde{\kappa})(2\tilde{g}(X, \tilde{\varphi}Y)\xi + \eta(X)\tilde{\varphi}Y - \eta(Y)\tilde{\varphi}X) \\ + (1 - \tilde{\mu})(\eta(X)\tilde{\varphi}\tilde{h}Y - \eta(Y)\tilde{\varphi}\tilde{h}X)$$

$$(3.6) \quad \tilde{\nabla}_\xi \tilde{h} = \tilde{\mu}\tilde{h} \circ \tilde{\varphi}, \quad \tilde{\nabla}_\xi \tilde{\varphi}\tilde{h} = -\tilde{\mu}\tilde{h}.$$

for any vector fields X, Y on M , where \tilde{Q} denotes the Ricci operator of (M, \tilde{g}) .

Proof. (3.2) was proved in [13]. Next, let $\{e_i, \tilde{\varphi}e_i, \xi\}$, $i \in \{1, \dots, n\}$, be a $\tilde{\varphi}$ -basis of M . Then the definition of the Ricci operator directly gives (3.3). For (3.4), notice that using (3.1) one can easily show that

$$(3.7) \quad \tilde{R}_{\xi X}Y = \tilde{\kappa}(\tilde{g}(X, Y)\xi - \eta(Y)X) + \tilde{\mu}(\tilde{g}(\tilde{h}X, Y)\xi - \eta(Y)\tilde{h}X)$$

By virtue of (3.7), the equation (2.11) reduces to

$$(\tilde{\nabla}_{\tilde{h}X} \tilde{\varphi})Y = \tilde{\kappa}(\tilde{g}(X, Y)\xi - \eta(Y)X) - \eta(Y)(X - \tilde{h}X) + \tilde{g}(X - \tilde{h}X, Y)\xi.$$

By replacing X by $\tilde{h}X$ in that equation and using (3.2), we get

$$(1 + \tilde{\kappa})((\tilde{\nabla}_X \tilde{\varphi})Y + \tilde{g}(X - \tilde{h}X, Y)\xi - \eta(Y)(X - \tilde{h}X)) = 0.$$

Hence (3.4) holds. Next, using (3.4) and the symmetry of \tilde{h} , we obtain

$$(3.8) \quad (\tilde{\nabla}_Z \tilde{\varphi}\tilde{h})Y - (\tilde{\nabla}_Y \tilde{\varphi}\tilde{h})Z = \tilde{\varphi}((\tilde{\nabla}_Z \tilde{h})Y - (\tilde{\nabla}_Y \tilde{h})Z)$$

for all $Y, Z \in \Gamma(TM)$. Substituting (3.8) in (2.10), we get

$$\tilde{R}_{YZ}\xi = -\eta(Z)(Y - \tilde{h}Y) + \eta(Y)(Z - \tilde{h}Z) + \tilde{\varphi}((\tilde{\nabla}_Y \tilde{h})Z - (\tilde{\nabla}_Z \tilde{h})Y).$$

Comparing this equation with (3.1), we obtain

$$(3.9) \quad \tilde{\varphi}((\tilde{\nabla}_Y \tilde{h})Z - (\tilde{\nabla}_Z \tilde{h})Y) = (\tilde{\kappa} + 1)(\eta(Z)Y - \eta(Y)Z) + (\tilde{\mu} - 1)(\eta(Z)\tilde{h}Y - \eta(Y)\tilde{h}Z).$$

Using (2.6) and the symmetry of \tilde{h} and $\tilde{\nabla}_Z \tilde{h}$, by a direct computation we have

$$(3.10) \quad \tilde{g}((\tilde{\nabla}_Z \tilde{h})Y - (\tilde{\nabla}_Y \tilde{h})Z, \xi) = 2(1 + \tilde{\kappa})\tilde{g}(\tilde{\varphi}Z, Y).$$

By applying now $\tilde{\varphi}$ to (3.9) and using (3.10), we obtain (3.5). Finally, (3.6) follows from (3.5) by using the properties of \tilde{h} . \square

By (3.4) we get the following corollary

Corollary 3.3. *Any paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifolds such that $\tilde{\kappa} \neq -1$ is integrable.*

In particular from Corollary 3.3 it follows that in any paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} \neq -1$ the canonical Legendre distributions \mathcal{D}^+ and \mathcal{D}^- are integrable and so define two Legendre foliations on M .

Remarkable subclasses of paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifolds are given, in view of (2.7), by para-Sasakian manifolds, and by those paracontact metric manifolds such that $\tilde{R}_{XY}\xi = 0$ for all vector fields X, Y on M . In this last case it was proved ([26]) that in dimension greater than 3 the paracontact metric manifold $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is locally isometric to a product of a flat $(n+1)$ -dimensional manifold and an n -dimensional manifold of negative constant curvature -4 . Notice that, because of (3.2), a paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} = -1$ satisfies $\tilde{h}^2 = 0$. Unlike the contact metric case, since the metric \tilde{g} is pseudo-Riemannian we can not conclude that \tilde{h} vanishes and so the manifold is para-Sasakian. Let us see an explicit counterexample.

The canonical example of paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold is given by the tangent sphere bundle T_1M of a Riemannian manifold (M, g) of constant sectional curvature c . The paracontact metric structure is defined in the following way. Let us consider the standard contact metric structure (φ, ξ, η, g) of T_1M , which is in fact a $(c(2-c), -2c)$ -structure (cf. [4]). Let us define

$$(3.11) \quad \tilde{\varphi}_1 := \frac{1}{|1-c|}\varphi h, \quad \tilde{g}_1 := \frac{1}{|1-c|}d\eta(\cdot, \varphi h \cdot) + \eta \otimes \eta$$

$$(3.12) \quad \tilde{\varphi}_2 := \frac{1}{|1-c|}h, \quad \tilde{g}_2 := \frac{1}{|1-c|}d\eta(\cdot, h \cdot) + \eta \otimes \eta.$$

Then one can easily check that $(\tilde{\varphi}_1, \eta, \xi, \tilde{g}_1)$ and $(\tilde{\varphi}_2, \eta, \xi, \tilde{g}_2)$ define two paracontact metric structures on T_1M . Thus by Theorem 5.9 of [11] we have that $(\tilde{\varphi}_1, \eta, \xi, \tilde{g}_1)$ is a paracontact $(\tilde{\kappa}_1, \tilde{\mu}_1)$ -structure and $(\tilde{\varphi}_2, \eta, \xi, \tilde{g}_2)$ a paracontact $(\tilde{\kappa}_2, \tilde{\mu}_2)$ -structure, where

$$\begin{aligned} \tilde{\kappa}_1 &= (1+c)^2 - 1, & \tilde{\mu}_1 &= 2(1-|c-1|), \\ \tilde{\kappa}_2 &= 4c-1, & \tilde{\mu}_2 &= 2. \end{aligned}$$

Hence we can state the following theorem.

Theorem 3.4. *The tangent sphere bundle of a Riemannian manifold of constant curvature $c \neq 1$ is canonically endowed, via (3.11)–(3.12), with a paracontact $((1+c)^2 - 1, 2(1-|c-1|))$ -structure and with a paracontact $(4c-1, 2)$ -structure.*

Consequently, if the base manifold is flat, $(\tilde{\varphi}_2, \xi, \eta, \tilde{g}_2)$ is a paracontact $(-1, 2)$ -structure on T_1M such that $\tilde{h}_2^2 = 0$, but which is not para-Sasakian because \tilde{h}_2 does not vanish. Indeed, according to (3.12) and [13, Lemma 4.5], one has that $h_2 = \frac{1}{2}\mathcal{L}_\xi h = \varphi h + \varphi$.

Given a paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ and $\alpha > 0$, the change of structure tensors

$$(3.13) \quad \bar{\eta} = \alpha\eta, \quad \bar{\xi} = \frac{1}{\alpha}\xi, \quad \bar{\varphi} = \tilde{\varphi}, \quad \bar{g} = \alpha\tilde{g} + \alpha(\alpha-1)\eta \otimes \eta$$

is called a \mathcal{D}_α -homothetic deformation. One can easily check that the new structure $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is still a paracontact metric structure ([25]). We now show that while \mathcal{D}_α -homothetic deformations destroy conditions like $\tilde{R}_{XY}\xi = 0$, they preserve the class of paracontact $(\tilde{\kappa}, \tilde{\mu})$ -spaces.

Proposition 3.5. *Let $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be a paracontact metric structure obtained from $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ by a \mathcal{D}_α -homothetic deformation. Then we have the following relationship between the Levi-Civita connections $\bar{\nabla}$ and $\tilde{\nabla}$ of \bar{g} and \tilde{g} , respectively,*

$$(3.14) \quad \bar{\nabla}_X Y = \tilde{\nabla}_X Y + \frac{\alpha-1}{\alpha}\tilde{g}(\tilde{\varphi}\tilde{h}X, Y)\xi - (\alpha-1)(\eta(Y)\tilde{\varphi}X + \eta(X)\tilde{\varphi}Y).$$

Furthermore,

$$(3.15) \quad \bar{h} = \frac{1}{\alpha} \tilde{h}.$$

Proof. Using (3.13) and the Koszul formula we obtain, for any $X, Y, Z \in \Gamma(TM)$,

$$(3.16) \quad \begin{aligned} \bar{g}(\bar{\nabla}_X Y, Z) &= \alpha \tilde{g}(\tilde{\nabla}_X Y, Z) + \alpha(\alpha - 1)\eta(\tilde{\nabla}_X Y)\eta(Z) \\ &\quad + \eta(Z)\tilde{g}(\tilde{\varphi}\tilde{h}X, Y) - \eta(Y)\tilde{g}(\tilde{\varphi}X, Z) - \eta(X)\tilde{g}(\tilde{\varphi}Y, Z). \end{aligned}$$

Moreover we have

$$(3.17) \quad \bar{g}(\bar{\nabla}_X Y, Z) = \alpha \tilde{g}(\bar{\nabla}_X Y, Z) + \alpha(\alpha - 1)\eta(\bar{\nabla}_X Y)\eta(Z)$$

and

$$(3.18) \quad \eta(\bar{\nabla}_X Y) = \frac{1}{\alpha^2} \bar{g}(\bar{\nabla}_X Y, \xi).$$

Setting $Z = \xi$ in (3.16) we get

$$(3.19) \quad \bar{g}(\bar{\nabla}_X Y, \xi) = \alpha^2 \eta(\tilde{\nabla}_X Y) + \alpha(\alpha - 1)\tilde{g}(\tilde{\varphi}\tilde{h}X, Y)$$

Then (3.14) easily follows from (3.16), (3.17), (3.18) and (3.19). Finally, by using (3.14) and the definition of \tilde{h} we get (3.15). \square

After a long but straightforward calculation one can prove the following proposition.

Proposition 3.6. *Under the same assumptions of Proposition 3.5, the curvature tensor fields \bar{R} and \tilde{R} are related by*

$$(3.20) \quad \begin{aligned} \alpha \bar{R}_{XY} \bar{\xi} &= \tilde{R}_{XY} \xi - (\alpha - 1)((\tilde{\nabla}_X \tilde{\varphi})Y - (\tilde{\nabla}_Y \tilde{\varphi})X + \eta(Y)(X - \tilde{h}X) - \eta(X)(Y - \tilde{h}Y)) \\ &\quad - (\alpha - 1)^2(\eta(Y)X - \eta(X)Y) \end{aligned}$$

In particular, if $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is a paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold, then $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is a paracontact $(\bar{\kappa}, \bar{\mu})$ -structure, where

$$(3.21) \quad \bar{\kappa} = \frac{\tilde{\kappa} + 1 - \alpha^2}{\alpha^2}, \quad \bar{\mu} = \frac{\tilde{\mu} + 2\alpha - 2}{\alpha}.$$

We pass to discuss some general curvature properties of paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifolds. We start with the following preliminary result.

Theorem 3.7. *Let $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be an integrable paracontact metric manifold. Then the following identity holds*

$$(3.22) \quad \tilde{Q}\tilde{\varphi} - \tilde{\varphi}\tilde{Q} = \tilde{l}\tilde{\varphi} - \tilde{\varphi}\tilde{l} - 4(n-1)\tilde{\varphi}\tilde{h} - \eta \otimes \tilde{\varphi}\tilde{Q} + (\eta \circ \tilde{Q}\tilde{\varphi}) \otimes \xi,$$

where \tilde{l} denotes the Jacobi operator, defined by $\tilde{l}X = \tilde{R}_{X\xi}\xi$.

Proof. Differentiating $\tilde{\nabla}_Y \xi = -\tilde{\varphi}Y + \tilde{\varphi}\tilde{h}Y$, we get

$$(3.23) \quad \tilde{R}_{XY}\xi = -(\tilde{\nabla}_X \tilde{\varphi})Y + (\tilde{\nabla}_Y \tilde{\varphi})X + (\tilde{\nabla}_X \tilde{\varphi}\tilde{h})Y - (\tilde{\nabla}_Y \tilde{\varphi}\tilde{h})X.$$

Using the integrability condition $\tilde{\nabla}^{pc}\tilde{\varphi} = 0$, the properties of \tilde{h} and (3.23) we have

$$(3.24) \quad \tilde{R}_{XY}\xi = -\eta(Y)(X - \tilde{h}X) + \eta(X)(Y - \tilde{h}Y) + \tilde{\varphi}((\tilde{\nabla}_X \tilde{h})Y - (\tilde{\nabla}_Y \tilde{h})X).$$

Since \tilde{h} is a symmetric operator we easily get

$$(3.25) \quad \tilde{g}((\tilde{\nabla}_X \tilde{h})Y - (\tilde{\nabla}_Y \tilde{h})X, \xi) = \tilde{g}(((\tilde{\nabla}_X \tilde{h}) - (\tilde{\nabla}_Y \tilde{h}))\xi, Y - X)$$

Using the formulas (2.6), $\tilde{h}\xi = 0$ and $\tilde{\varphi}\tilde{h} + \tilde{h}\tilde{\varphi} = 0$ in (3.25) we find

$$(3.26) \quad \tilde{g}((\tilde{\nabla}_X \tilde{h})Y - (\tilde{\nabla}_Y \tilde{h})X, \xi) = 2\tilde{g}(\tilde{\varphi}\tilde{h}^2 X, Y).$$

Applying $\tilde{\varphi}$ to (3.24) and using $\tilde{\varphi}^2 = I - \eta \otimes \xi$ and (3.26) we obtain

$$(3.27) \quad (\tilde{\nabla}_X \tilde{h})Y - (\tilde{\nabla}_Y \tilde{h})X = \tilde{\varphi} \tilde{R}_{XY} \xi + 2\tilde{g}(\tilde{\varphi} \tilde{h}^2 X, Y) - \eta(X) \tilde{\varphi}(Y - \tilde{h}Y) + \eta(Y) \tilde{\varphi}(X - \tilde{h}X).$$

Now we suppose that P is a fixed point of M and X, Y, Z are vector fields such that $(\tilde{\nabla}X)_P = (\tilde{\nabla}Y)_P = (\tilde{\nabla}Z)_P = 0$. The Ricci identity for $\tilde{\varphi}$

$$\tilde{R}_{XY} \tilde{\varphi} Z - \tilde{\varphi} \tilde{R}_{XY} Z = (\tilde{\nabla}_X \tilde{\nabla}_Y \tilde{\varphi}) Z - (\tilde{\nabla}_Y \tilde{\nabla}_X \tilde{\varphi}) Z - (\tilde{\nabla}_{[X, Y]} \tilde{\varphi}) Z,$$

at the point P , reduces to the form

$$(3.28) \quad \tilde{R}_{XY} \tilde{\varphi} Z - \tilde{\varphi} \tilde{R}_{XY} Z = \tilde{\nabla}_X (\tilde{\nabla}_Y \tilde{\varphi}) Z - \tilde{\nabla}_Y (\tilde{\nabla}_X \tilde{\varphi}) Z.$$

By virtue of the integrability condition we have, at P ,

$$(3.29) \quad \begin{aligned} \tilde{R}_{XY} \tilde{\varphi} Z - \tilde{\varphi} \tilde{R}_{XY} Z &= \tilde{\nabla}_X (\tilde{\nabla}_Y \tilde{\varphi}) Z - \tilde{\nabla}_Y (\tilde{\nabla}_X \tilde{\varphi}) Z \\ &= \tilde{g}((\tilde{\nabla}_X \tilde{h})Y - (\tilde{\nabla}_Y \tilde{h})X, Z) \xi - \eta(Z) ((\tilde{\nabla}_X \tilde{h})Y - (\tilde{\nabla}_Y \tilde{h})X) \\ &\quad + \tilde{g}(Y - \tilde{h}Y, Z) \tilde{\varphi}(X - \tilde{h}X) - \tilde{g}(X - \tilde{h}X, Z) \tilde{\varphi}(Y - \tilde{h}Y) \\ &\quad - \tilde{g}(\tilde{\varphi}(X - \tilde{h}X), Z) (Y - \tilde{h}Y) + \tilde{g}(\tilde{\varphi}(Y - \tilde{h}Y), Z) (X - \tilde{h}X). \end{aligned}$$

Using (3.27) in (3.29) we find

$$(3.30) \quad \begin{aligned} \tilde{R}_{XY} \tilde{\varphi} Z - \tilde{\varphi} \tilde{R}_{XY} Z &= (\tilde{g}(\tilde{\varphi} \tilde{R}_{XY} \xi - \eta(X) \tilde{\varphi}(Y - \tilde{h}Y) + \eta(Y) \tilde{\varphi}(X - \tilde{h}X), Z)) \xi \\ &\quad - \eta(Z) (\tilde{\varphi} \tilde{R}_{XY} \xi - \eta(X) \tilde{\varphi}(Y - \tilde{h}Y) + \eta(Y) \tilde{\varphi}(X - \tilde{h}X)) \\ &\quad + \tilde{g}(Y - \tilde{h}Y, Z) \tilde{\varphi}(X - \tilde{h}X) - \tilde{g}(X - \tilde{h}X, Z) \tilde{\varphi}(Y - \tilde{h}Y) \\ &\quad - \tilde{g}(\tilde{\varphi}(X - \tilde{h}X), Z) (Y - \tilde{h}Y) + \tilde{g}(\tilde{\varphi}(Y - \tilde{h}Y), Z) (X - \tilde{h}X). \end{aligned}$$

Using (2.5) and the curvature tensor properties we get

$$\tilde{g}(\tilde{\varphi} \tilde{R}_{\tilde{\varphi}X \tilde{\varphi}Y} Z, \tilde{\varphi}W) = -\tilde{g}(\tilde{R}_{ZW} \tilde{\varphi}X, \tilde{\varphi}Y) + \eta(\tilde{R}_{\tilde{\varphi}X \tilde{\varphi}Y} Z) \eta(W).$$

Then by (2.5) and (3.30) we get by a straightforward calculation

$$(3.31) \quad \begin{aligned} \tilde{g}(\tilde{\varphi} \tilde{R}_{\tilde{\varphi}X \tilde{\varphi}Y} Z, \tilde{\varphi}W) &= \tilde{g}(\tilde{R}_{XY} Z, W) + \eta(W) \eta(\tilde{R}_{\tilde{\varphi}X \tilde{\varphi}Y} Z) \\ &\quad + \eta(Y) (-\eta(\tilde{R}_{ZW} X) - \eta(Z) \tilde{g}(W - \tilde{h}W, X) + \eta(W) \tilde{g}(Z - \tilde{h}Z, X)) \\ &\quad - \eta(X) (-\eta(\tilde{R}_{ZW} Y) - \eta(Z) \tilde{g}(W - \tilde{h}W, Y) + \eta(W) \tilde{g}(Z - \tilde{h}Z, Y)) \\ &\quad + \tilde{g}(W - \tilde{h}W, X) \tilde{g}(Z - \tilde{h}Z, Y) - \tilde{g}(Z - \tilde{h}Z, X) \tilde{g}(W - \tilde{h}W, Y) \\ &\quad + \tilde{g}(\tilde{\varphi}(Z - \tilde{h}Z), X) \tilde{g}(W - \tilde{h}W, \tilde{\varphi}Y) - \tilde{g}(W - \tilde{h}W, \tilde{\varphi}X) \tilde{g}(\tilde{\varphi}(Z - \tilde{h}Z), Y). \end{aligned}$$

Replacing in (3.30) X, Y by $\tilde{\varphi}X, \tilde{\varphi}Y$ respectively, and taking the inner product with $\tilde{\varphi}W$, we get

$$(3.32) \quad \begin{aligned} \tilde{g}(\tilde{R}_{\tilde{\varphi}X \tilde{\varphi}Y} \tilde{\varphi}Z - \tilde{\varphi} \tilde{R}_{\tilde{\varphi}X \tilde{\varphi}Y} Z, \tilde{\varphi}W) &= \tilde{g}(\tilde{\varphi}(X + \tilde{h}X), Z) \tilde{g}(\tilde{\varphi}(Y + \tilde{h}Y), W) \\ &\quad - \tilde{g}(\tilde{\varphi}(Y + \tilde{h}Y), Z) \tilde{g}(\tilde{\varphi}(X + \tilde{h}X), W) - \eta(Y) \eta(W) \tilde{g}(X + \tilde{h}X, Z) \\ &\quad + \tilde{g}(X - \eta(X) \xi + \tilde{h}X, Z) \tilde{g}(Y + \tilde{h}Y, W) + \eta(X) \eta(W) \tilde{g}(Y + \tilde{h}Y, Z) \\ &\quad - \tilde{g}(Y - \eta(Y) \xi + \tilde{h}Y, Z) \tilde{g}(X + \tilde{h}X, W) + \eta(Z) \tilde{g}(\tilde{R}_{\tilde{\varphi}X \tilde{\varphi}Y} \xi, W). \end{aligned}$$

Comparing (3.31) to (3.32) we get by direct computation

$$\begin{aligned}
(3.33) \quad \tilde{g}(\tilde{R}_{\tilde{\varphi}X\tilde{\varphi}Y}\tilde{\varphi}Z, \tilde{\varphi}W) &= \tilde{g}(\tilde{R}_{XY}Z, W) + \eta(W)\eta(\tilde{R}_{\tilde{\varphi}X\tilde{\varphi}Y}Z) - \eta(Z)\eta(\tilde{R}_{\tilde{\varphi}X\tilde{\varphi}Y}W) \\
&+ \eta(Y)(-\eta(\tilde{R}_{ZW}X) + 2\eta(Z)\tilde{g}(\tilde{h}X, W) - 2\eta(W)\tilde{g}(\tilde{h}Z, X)) \\
&- \eta(X)(-\eta(\tilde{R}_{ZY}W) + 2\eta(Z)\tilde{g}(\tilde{h}Y, W) - 2\eta(W)\tilde{g}(\tilde{h}Z, Y)) \\
&- 2\tilde{g}(W, X)\tilde{g}(\tilde{h}Z, Y) - 2\tilde{g}(Z, Y)\tilde{g}(\tilde{h}W, X) \\
&+ 2\tilde{g}(W, Y)\tilde{g}(\tilde{h}Z, X) + 2\tilde{g}(Z, X)\tilde{g}(\tilde{h}W, Y)
\end{aligned}$$

Let $\{e_i, \tilde{\varphi}e_i, \xi\}$, $i \in \{1, \dots, n\}$, be a local $\tilde{\varphi}$ -basis. Setting $Y = Z = e_i$ in (3.33), we have

$$\begin{aligned}
(3.34) \quad \sum_{i=1}^n \tilde{g}(\tilde{R}_{\tilde{\varphi}Xe_i}\tilde{\varphi}e_i, \tilde{\varphi}W) &= \sum_{i=1}^n (\tilde{g}(\tilde{R}_{Xe_i}e_i, W) + \eta(W)\eta(\tilde{R}_{\tilde{\varphi}Xe_i}\tilde{\varphi}e_i) + \eta(X)\eta(\tilde{R}_{e_iW}e_i)) \\
&+ 2\eta(X)\eta(W)\tilde{g}(\tilde{h}e_i, e_i) - 2\tilde{g}(W, X)\tilde{g}(\tilde{h}e_i, e_i) - 2\tilde{g}(e_i, e_i)\tilde{g}(\tilde{h}W, X) \\
&+ 2\tilde{g}(W, e_i)\tilde{g}(\tilde{h}e_i, X) + 2\tilde{g}(e_i, X)\tilde{g}(\tilde{h}W, e_i).
\end{aligned}$$

On the other hand, putting $Y = Z = \tilde{\varphi}e_i$ in (3.33), we get

$$\begin{aligned}
(3.35) \quad \sum_{i=1}^n \tilde{g}(\tilde{R}_{\tilde{\varphi}Xe_i}e_i, \tilde{\varphi}W) &= \sum_{i=1}^n (\tilde{g}(\tilde{R}_{X\tilde{\varphi}e_i}\tilde{\varphi}e_i, W) + \eta(W)\eta(\tilde{R}_{\tilde{\varphi}Xe_i}\tilde{\varphi}e_i) + \eta(X)\eta(\tilde{R}_{\tilde{\varphi}e_iW}\tilde{\varphi}e_i)) \\
&+ 2\eta(X)\eta(W)\tilde{g}(\tilde{h}\tilde{\varphi}e_i, \tilde{\varphi}e_i) - 2\tilde{g}(W, X)\tilde{g}(\tilde{h}\tilde{\varphi}e_i, \tilde{\varphi}e_i) - 2\tilde{g}(\tilde{\varphi}e_i, \tilde{\varphi}e_i)\tilde{g}(\tilde{h}W, X) \\
&+ 2\tilde{g}(W, \tilde{\varphi}e_i)\tilde{g}(\tilde{h}\tilde{\varphi}e_i, X) + 2\tilde{g}(\tilde{\varphi}e_i, X)\tilde{g}(\tilde{h}W, \tilde{\varphi}e_i).
\end{aligned}$$

Using the definition of the Ricci operator, (3.34) and (3.35) it is not hard to prove that

$$(3.36) \quad -\tilde{\varphi}\tilde{Q}\tilde{\varphi}X + \tilde{\varphi}\tilde{l}\tilde{\varphi}X + \tilde{Q}X = \tilde{l}X + \sum_{i=1}^n (\eta(\tilde{R}_{\tilde{\varphi}Xe_i}\tilde{\varphi}e_i) - \eta(\tilde{R}_{\tilde{\varphi}X\tilde{\varphi}e_i}e_i))\xi + \eta(X)\tilde{Q}\xi + 4(n-1)\tilde{h}X.$$

Finally, applying $\tilde{\varphi}$ to (3.36) and using $\tilde{\varphi}^2 = I - \eta \otimes \xi$, we obtain the assertion. \square

Corollary 3.8. *Let $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold. Then*

$$(3.37) \quad \tilde{Q}\tilde{\varphi} - \tilde{\varphi}\tilde{Q} = 2(2(n-1) + \tilde{\mu})\tilde{h}\tilde{\varphi}$$

Proof. Using (2.16) and $\tilde{\varphi}\tilde{h} + \tilde{h}\tilde{\varphi} = 0$ we get $\tilde{l}\tilde{\varphi} - \tilde{\varphi}\tilde{l} = 2\tilde{\mu}\tilde{h}\tilde{\varphi}$. On the other hand, by virtue of (3.3) one can easily prove that both $\eta \otimes \tilde{\varphi}\tilde{Q}$ and $(\eta \circ \tilde{Q}\tilde{\varphi}) \otimes \xi$ vanish. Thus (3.37) follows from (3.22). \square

Recall that ([11]) an *almost bi-paracontact structure* on a contact manifold (M, η) is a triplet (ϕ_1, ϕ_2, ϕ_3) where ϕ_3 is an almost contact structure compatible with the contact form η , and ϕ_1, ϕ_2 are two anti-commuting tensors on M such that $\phi_1^2 = \phi_2^2 = I - \eta \otimes \xi$ and $\phi_1\phi_2 = \phi_3$. From the definition it easily follows that $\phi_1\phi_3 = -\phi_3\phi_1 = \phi_2$ and $\phi_3\phi_2 = -\phi_2\phi_3 = \phi_1$. Any almost bi-paracontact manifold is then endowed with four distributions, $\mathcal{D}_1^\pm, \mathcal{D}_2^\pm$, given by the eigendistributions corresponding to the eigenvalues ± 1 of ϕ_1 and ϕ_2 , respectively. One proves that, for each $\alpha \in \{1, 2\}$, \mathcal{D}_α^+ and \mathcal{D}_α^- are transversal n -dimensional subbundles of the contact distribution. In particular it follows that ϕ_1 and ϕ_2 are almost paracontact structures.

Now we prove that any paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold with $\tilde{\kappa} \neq -1$ is canonically endowed with an almost bi-paracontact structure.

Proposition 3.9. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold. If $\tilde{\kappa} \neq -1$ then M admits an almost bi-paracontact structure (ϕ_1, ϕ_2, ϕ_3) given by*

$$(3.38) \quad \phi_1 := \tilde{\varphi}, \quad \phi_2 := \frac{1}{\sqrt{1+\tilde{\kappa}}}\tilde{h}, \quad \phi_3 := \frac{1}{\sqrt{1+\tilde{\kappa}}}\tilde{\varphi}\tilde{h}$$

in the case $\tilde{\kappa} > -1$, and

$$(3.39) \quad \phi_1 := \tilde{\varphi}, \quad \phi_2 := \frac{1}{\sqrt{-1-\tilde{\kappa}}} \tilde{\varphi} \tilde{h}, \quad \phi_3 := \frac{1}{\sqrt{-1-\tilde{\kappa}}} \tilde{h}$$

in the case $\tilde{\kappa} < -1$.

Proof. The proof follows by direct computations, using (3.2) and the property $\tilde{\varphi} \tilde{h} = -\tilde{h} \tilde{\varphi}$. \square

Corollary 3.10. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} \neq -1$. Then the operator \tilde{h} in the case $\tilde{\kappa} > -1$ and the operator $\tilde{\varphi} \tilde{h}$ in the case $\tilde{\kappa} < -1$ are diagonalizable and admit three eigenvalues: 0, associated to the eigenvector ξ , $\tilde{\lambda}$ and $-\tilde{\lambda}$, of multiplicity n , where $\tilde{\lambda} := \sqrt{|1+\tilde{\kappa}|}$. The corresponding eigendistributions $\mathcal{D}_{\tilde{h}}(0) = \mathbb{R}\xi$, $\mathcal{D}_{\tilde{h}}(\tilde{\lambda})$, $\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$ and $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(0) = \mathbb{R}\xi$, $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})$, $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda})$ are mutually orthogonal and one has $\tilde{\varphi}\mathcal{D}_{\tilde{h}}(\pm\tilde{\lambda}) = \mathcal{D}_{\tilde{h}}(\mp\tilde{\lambda})$ and $\tilde{\varphi}\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\pm\tilde{\lambda}) = \mathcal{D}_{\tilde{\varphi}\tilde{h}}(\mp\tilde{\lambda})$. Furthermore,*

$$(3.40) \quad \mathcal{D}_{\tilde{h}}(\pm\tilde{\lambda}) = \left\{ X \pm \frac{1}{\sqrt{1+\tilde{\kappa}}} \tilde{h}X \mid X \in \Gamma(\mathcal{D}^\mp) \right\}$$

in the case $\tilde{\kappa} > -1$, and

$$(3.41) \quad \mathcal{D}_{\tilde{\varphi}\tilde{h}}(\pm\tilde{\lambda}) = \left\{ X \pm \frac{1}{\sqrt{-1-\tilde{\kappa}}} \tilde{\varphi}\tilde{h}X \mid X \in \Gamma(\mathcal{D}^\mp) \right\}$$

in the case $\tilde{\kappa} < -1$, where \mathcal{D}^+ and \mathcal{D}^- denote the eigendistributions of $\tilde{\varphi}$ corresponding to the eigenvalues 1 and -1 , respectively. Finally any two among the four distributions \mathcal{D}^+ , \mathcal{D}^- , $\mathcal{D}_{\tilde{h}}(\tilde{\lambda})$, $\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$ in the case $\tilde{\kappa} > -1$ or \mathcal{D}^+ , \mathcal{D}^- , $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})$, $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda})$ in the case $\tilde{\kappa} < -1$ are mutually transversal.

Proof. The first part of the statement follows from Proposition 3.9, since $\tilde{h} = \tilde{\lambda}\phi_2$ and $\mathcal{D}_{\tilde{h}}(\pm\tilde{\lambda}) = \mathcal{D}_2^\pm$. We prove that $\mathcal{D}_{\tilde{h}}(\tilde{\lambda})$ and $\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$ are mutually orthogonal. Indeed, for any $X \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$ and $Y \in \Gamma(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}))$ we have $\tilde{\lambda}\tilde{g}(X, Y) = \tilde{g}(\tilde{h}X, Y) = \tilde{g}(X, \tilde{h}Y) = -\tilde{\lambda}\tilde{g}(X, Y)$, from which, since $\tilde{\lambda} \neq 0$, we get $\tilde{g}(X, Y) = 0$. Moreover, as $\tilde{\varphi}\tilde{h} = -\tilde{h}\tilde{\varphi}$ one has that if $\tilde{h}X = \tilde{\lambda}X$ then $\tilde{h}\tilde{\varphi}X = -\tilde{\lambda}\tilde{\varphi}X$, so that $\tilde{\varphi}\mathcal{D}_{\tilde{h}}(\pm\tilde{\lambda}) = \mathcal{D}_{\tilde{h}}(\mp\tilde{\lambda})$. The case $\tilde{\kappa} < -1$ can be proved in a similar manner. Finally, (3.40) and (3.41) follow from [11, Proposition 3.3] and the last part is a direct consequence of [11, Proposition 3.2]. \square

Thus paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifolds can be divided into three main classes, according to the circumstance that $\tilde{\kappa}$ is less, equal or greater than -1 . According to (3.21), one can see that these three classes are preserved by \mathcal{D} -homothetic deformations. Notice that the canonical paracontact metric structure $(\tilde{\varphi}_1, \xi, \eta, \tilde{g}_1)$ on the tangent sphere bundle T_1M of a manifold of constant curvature c (cf. Theorem 3.4) always satisfies $\tilde{\kappa}_1 > -1$. Whereas for the other one, $(\tilde{\varphi}_2, \xi, \eta, \tilde{g}_2)$, we have that $\tilde{\kappa}_2$ is less, equal or greater than -1 if and only if, respectively, c is less, equal or greater than 0. Thus T_1M provides examples for all the above three classes of paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifolds.

In the sequel, unless otherwise stated, we will always assume the index of $\mathcal{D}_{\tilde{h}}(\pm\lambda)$ (in the case $\tilde{\kappa} > -1$) and of $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\pm\lambda)$ (in the case $\tilde{\kappa} < -1$) to be constant.

Being \tilde{h} (in the case $\tilde{\kappa} > -1$) or $\tilde{\varphi}\tilde{h}$ (in the case $\tilde{\kappa} < -1$) diagonalizable, one can easily prove the following lemma.

Lemma 3.11. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} \neq -1$. If $\tilde{\kappa} > -1$ (respectively, $\tilde{\kappa} < -1$), then there exists a local orthogonal $\tilde{\varphi}$ -basis $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \xi\}$ of eigenvectors of \tilde{h} (respectively, $\tilde{\varphi}\tilde{h}$) such that $X_1, \dots, X_n \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$ (respectively, $\Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}))$), $Y_1, \dots, Y_n \in \Gamma(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}))$ (respectively, $\Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda}))$), and*

$$(3.42) \quad \tilde{g}(X_i, X_i) = -\tilde{g}(Y_i, Y_i) = \begin{cases} 1, & \text{for } 1 \leq i \leq r \\ -1, & \text{for } r+1 \leq i \leq r+s \end{cases}$$

where $r = \text{index}(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}))$ (respectively, $r = \text{index}(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda}))$) and $s = n - r = \text{index}(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$ (respectively, $s = \text{index}(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}))$).

As pointed out in [8], there is a strict relationship between paracontact metric geometry and the theory of Legendre foliations. Thus it is interesting to investigate on the properties of the bi-Legendrian structure $(\mathcal{D}^+, \mathcal{D}^-)$ canonically associated to a paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold. In the next proposition we prove that it is non-degenerate and we find necessary and sufficient conditions for being positive or negative definite.

Proposition 3.12. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} \neq -1$. Then the canonical Legendre foliations \mathcal{D}^+ and \mathcal{D}^- given by the ± 1 -eigendistributions of $\tilde{\varphi}$ are both non-degenerate. They are positive definite if and only if, respectively, $\text{index}(\mathcal{D}_{\tilde{h}}(\tilde{\lambda})) = 0$ in the case $\tilde{\kappa} > -1$ and $\text{index}(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})) = n$ in the case $\tilde{\kappa} < -1$, and negative definite if and only if $\text{index}(\mathcal{D}_{\tilde{h}}(\tilde{\lambda})) = n$ in the case $\tilde{\kappa} > -1$ and $\text{index}(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})) = 0$ in the case $\tilde{\kappa} < -1$.*

Proof. We consider the case $\tilde{\kappa} > -1$, the proof for the case $\tilde{\kappa} < -1$ being analogous. First notice that the Pang invariants associated to the Legendre foliations \mathcal{D}^+ and \mathcal{D}^- are given by

$$(3.43) \quad \Pi_{\mathcal{D}^+}(X, X') = 2\tilde{g}(\tilde{h}X, X'), \quad \Pi_{\mathcal{D}^-}(Y, Y') = 2\tilde{g}(\tilde{h}Y, Y').$$

Indeed, for any $X \in \Gamma(\mathcal{D}^+)$ and for any $Y \in \Gamma(\mathcal{D}^-)$ one has $\tilde{h}X = [\xi, X]_{\mathcal{D}^-}$ and $\tilde{h}Y = -[\xi, Y]_{\mathcal{D}^+}$ (see [8, Corollary 3.2]). Then we have that, for any $X, X' \in \Gamma(\mathcal{D}^+)$, $\Pi_{\mathcal{D}^+}(X, X') = 2d\eta([\xi, X], X') = 2\tilde{g}([\xi, X], \tilde{\varphi}X') = 2\tilde{g}([\xi, X], X') = 2\tilde{g}([\xi, X]_-, X') = 2\tilde{g}(\tilde{h}X, X')$. Analogously one proves the other equality. Now let $\{X_i, Y_i = \tilde{\varphi}X_i, \xi\}$, $i \in \{1, \dots, n\}$, be a $\tilde{\varphi}$ -basis of eigenvectors of \tilde{h} as in Lemma 3.11. Notice that

$$(3.44) \quad \mathcal{D}^\pm = \{X \pm \tilde{\varphi}X \mid X \in \Gamma(\mathcal{D}_{\tilde{h}}(\pm\tilde{\lambda}))\}.$$

This follows from [11, Proposition 3.3] applied to the canonical almost bi-paracontact structure attached to $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$. Then by (3.43), since $\tilde{h}\mathcal{D}^+ \subset \mathcal{D}^-$, we have that if $\Pi_{\mathcal{D}^+}(X, X') = 0$ for all $X, X' \in \Gamma(\mathcal{D}^+)$ necessarily $\tilde{h}X = 0$. Hence $\tilde{h}^2X = 0$ and, by (3.2), this implies that $X = 0$. Therefore \mathcal{D}^+ is non-degenerate. A similar proof works also for \mathcal{D}^- . Next, by (3.44), in order to check whether \mathcal{D}^+ is positive or negative definite it suffices to evaluate $\Pi_{\mathcal{D}^+}$ on the vector fields of the form $X_i + \tilde{\varphi}X_i = X_i + Y_i$. Using (3.43) we have, for each $i \in \{1, \dots, n\}$,

$$\Pi_{\mathcal{D}^+}(X_i + Y_i, X_i + Y_i) = 2\tilde{g}(\tilde{h}X_i + \tilde{h}Y_i, X_i + Y_i) = 2\tilde{g}(\tilde{\lambda}X_i, X_i) - 2\tilde{g}(\tilde{\lambda}Y_i, Y_i) = \pm 4\tilde{\lambda},$$

where the sign \pm depends on the fact that $\text{index}(\mathcal{D}_{\tilde{h}}(\tilde{\lambda})) = 0$ or $\text{index}(\mathcal{D}_{\tilde{h}}(\tilde{\lambda})) = n$, respectively. On the other hand as before, by (3.44) it is sufficient to evaluate $\Pi_{\mathcal{D}^-}$ on the vector fields of the form $Y_i - \tilde{\varphi}Y_i = Y_i - X_i$. Then

$$\Pi_{\mathcal{D}^-}(Y_i - X_i, Y_i - X_i) = 2\tilde{g}(\tilde{h}Y_i - \tilde{h}X_i, Y_i - X_i) = -2\tilde{g}(\tilde{\lambda}Y_i, Y_i) + 2\tilde{g}(\tilde{\lambda}X_i, X_i) = \pm 4\tilde{\lambda},$$

according to the circumstance that $\text{index}(\mathcal{D}_{\tilde{h}}(\tilde{\lambda})) = 0$ or $\text{index}(\mathcal{D}_{\tilde{h}}(\tilde{\lambda})) = n$, respectively. Conversely, if $\Pi_{\mathcal{D}^-}$ is positive definite then there exists a local basis of \mathcal{D}^- , say $\{Z_1, \dots, Z_n\}$, such that $\Pi_{\mathcal{D}^-}(Z_i, Z_j) = \delta_{ij}$. Then, taking (3.40) into account, we put $X_i := \sqrt{\tilde{\lambda}} \left(Z_i + \frac{1}{\tilde{\lambda}} \tilde{h}Z_i \right)$, for each $i \in \{1, \dots, n\}$. Then $\{X_1, \dots, X_n\}$ is a local basis of $\mathcal{D}_{\tilde{h}}(\tilde{\lambda})$ such that

$$\tilde{g}(X_i, X_j) = \tilde{\lambda} \left(\tilde{g}(Z_i, Z_j) + \frac{1}{\tilde{\lambda}^2} \tilde{g}(\tilde{h}Z_i, \tilde{h}Z_j) + \frac{2}{\tilde{\lambda}} \tilde{g}(\tilde{h}Z_i, Z_j) \right) = \Pi_{\mathcal{D}^-}(Z_i, Z_j) = \delta_{ij},$$

since $\tilde{g}(X, X') = 0$ for all $X, X' \in \Gamma(\mathcal{D}^+)$. Thus $\text{index}(\mathcal{D}_{\tilde{h}}(\tilde{\lambda})) = 0$. The other case is analogous. \square

Remark 3.13. Notice that in the course of the proof of Proposition 3.12 we have proved in fact more than what we have stated. Namely, we have proved that $\Pi_{\mathcal{D}^+}$ and $\Pi_{\mathcal{D}^-}$ have the same signature, which

is given by $(\text{index}(\mathcal{D}_{\tilde{h}}(\tilde{\lambda})), \text{index}(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})))$ in the case $\tilde{\kappa} > -1$ and by $(\text{index}(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})), \text{index}(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda})))$ if $\tilde{\kappa} < -1$

Proposition 3.12 motivates the following definition.

Definition 3.14. *A paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ such that $\tilde{\kappa} \neq -1$ will be called positive definite or negative definite according to the circumstance that the bi-Legendrian structure $(\mathcal{D}^+, \mathcal{D}^-)$ canonically associated to M is positive or negative definite, respectively.*

Positive and negative definite paracontact $(\tilde{\kappa}, \tilde{\mu})$ -structures will play an important role § 4 and § 5. We conclude the section with an example of negative definite paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold.

Example 3.15. *Let \mathfrak{g} be the Lie algebra with basis $\{e_1, e_2, e_3, e_4, e_5\}$ and Lie brackets*

$$(3.45) \quad [e_1, e_5] = \frac{\alpha\beta}{2}e_2 + \frac{\alpha^2}{2}e_3, \quad [e_2, e_5] = -\frac{\alpha\beta}{2}e_1 + \frac{\alpha^2}{2}e_4,$$

$$(3.46) \quad [e_3, e_5] = -\frac{\beta^2}{2}e_1 + \frac{\alpha\beta}{2}e_4, \quad [e_4, e_5] = -\frac{\beta^2}{2}e_2 - \frac{\alpha\beta}{2}e_3,$$

$$(3.47) \quad [e_1, e_2] = \alpha e_2, \quad [e_1, e_3] = -\beta e_2 + 2e_5, \quad [e_1, e_4] = 0,$$

$$(3.48) \quad [e_2, e_3] = \beta e_1 - \alpha e_4, \quad [e_2, e_4] = \alpha e_3 + 2e_5, \quad [e_3, e_4] = -\beta e_3$$

where α, β are real numbers such that $\alpha^2 - \beta^2 \neq 0$. Let G be a Lie group whose Lie algebra is \mathfrak{g} . Define on G a left invariant paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ by imposing that, at the identity,

$$\tilde{g}(e_1, e_1) = -1, \quad \tilde{g}(e_2, e_2) = -1, \quad \tilde{g}(e_3, e_3) = 1, \quad \tilde{g}(e_4, e_4) = 1, \quad \tilde{g}(e_5, e_5) = 1, \quad \tilde{g}(e_i, e_j) = 0 \quad (i \neq j)$$

and $\tilde{\varphi}e_1 = e_3, \tilde{\varphi}e_2 = e_4, \tilde{\varphi}e_3 = e_1, \tilde{\varphi}e_4 = e_2, \tilde{\varphi}e_5 = 0, \xi = e_5$ and $\eta = \tilde{g}(\cdot, e_5)$. Notice that $\tilde{h}e_1 = \tilde{\lambda}e_1, \tilde{h}e_2 = \tilde{\lambda}e_2, \tilde{h}\varphi e_1 = -\tilde{\lambda}\tilde{\varphi}e_1, \tilde{h}\tilde{\varphi}e_2 = -\tilde{\lambda}\tilde{\varphi}e_2, \tilde{h}\xi = 0$. Now let $\tilde{\nabla}$ be the Levi-Civita connection of the pseudo-Riemannian metric \tilde{g} and \tilde{R} be the curvature tensor of \tilde{g} . Using the Koszul formula, we get

$$\begin{aligned} \tilde{\nabla}_{e_1}\xi &= (\tilde{\lambda} - 1)\tilde{\varphi}e_1, & \tilde{\nabla}_{e_2}\xi &= (\tilde{\lambda} - 1)\tilde{\varphi}e_2, & \tilde{\nabla}_{\tilde{\varphi}e_1}\xi &= -(1 + \tilde{\lambda})e_1, & \tilde{\nabla}_{\tilde{\varphi}e_2}\xi &= -(1 + \tilde{\lambda})e_2, \\ \tilde{\nabla}_{\xi}e_1 &= -\frac{\alpha\beta}{2}e_2 - \frac{\tilde{\mu}}{2}\tilde{\varphi}e_1, & \tilde{\nabla}_{\xi}e_2 &= \frac{\alpha\beta}{2}e_1 - \frac{\tilde{\mu}}{2}\tilde{\varphi}e_2, & \tilde{\nabla}_{\xi}\tilde{\varphi}e_1 &= -\frac{\tilde{\mu}}{2}e_1 - \frac{\alpha\beta}{2}\tilde{\varphi}e_2, & \tilde{\nabla}_{\xi}\tilde{\varphi}e_2 &= -\frac{\tilde{\mu}}{2}e_2 + \frac{\alpha\beta}{2}\tilde{\varphi}e_1, \\ \tilde{\nabla}_{e_1}e_1 &= 0, & \tilde{\nabla}_{e_1}e_2 &= 0, & \tilde{\nabla}_{e_1}\tilde{\varphi}e_1 &= -(\tilde{\lambda} - 1)\xi, & \tilde{\nabla}_{e_1}\tilde{\varphi}e_2 &= 0, \\ \tilde{\nabla}_{e_2}e_1 &= -\alpha e_2, & \tilde{\nabla}_{e_2}e_2 &= \alpha e_1, & \tilde{\nabla}_{e_2}\tilde{\varphi}e_1 &= -\alpha\tilde{\varphi}e_2, & \tilde{\nabla}_{e_2}\tilde{\varphi}e_2 &= \alpha\tilde{\varphi}e_1 - (\tilde{\lambda} - 1)\xi, \\ \tilde{\nabla}_{\tilde{\varphi}e_1}e_1 &= \beta e_2 - (1 + \tilde{\lambda})\xi, & \tilde{\nabla}_{\tilde{\varphi}e_1}e_2 &= -\beta e_1, & \tilde{\nabla}_{\tilde{\varphi}e_1}\tilde{\varphi}e_1 &= \beta\tilde{\varphi}e_2, & \tilde{\nabla}_{\tilde{\varphi}e_1}\tilde{\varphi}e_2 &= -\beta\tilde{\varphi}e_1, \\ \tilde{\nabla}_{\tilde{\varphi}e_2}e_1 &= 0, & \tilde{\nabla}_{\tilde{\varphi}e_2}e_2 &= -(1 + \tilde{\lambda})\xi, & \tilde{\nabla}_{\tilde{\varphi}e_2}\tilde{\varphi}e_1 &= 0, & \tilde{\nabla}_{\tilde{\varphi}e_2}\tilde{\varphi}e_2 &= 0, \end{aligned}$$

where $\tilde{\lambda} = \frac{\alpha^2 + \beta^2}{4}$, $\tilde{\kappa} = \frac{(\alpha^2 + \beta^2)^2 - 16}{16}$ and $\tilde{\mu} = \frac{\alpha^2 - \beta^2}{2} + 2$. From the above relations it can be easily proved checked that G is a paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold.

4. PARACONTACT $(\tilde{\kappa}, \tilde{\mu})$ -MANIFOLDS WITH $\tilde{\kappa} > -1$

In this section we deal with paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifolds such that $\tilde{\kappa} > -1$. In this case, according to Corollary 3.10, \tilde{h} is diagonalizable with eigenvectors $0, \pm\tilde{\lambda}$, where $\tilde{\lambda} := \sqrt{1 + \tilde{\kappa}}$. Our first result concerns some remarkable properties of the distributions defined by the eigenspaces of \tilde{h} .

Theorem 4.1. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ -manifold with $\tilde{\kappa} > -1$. Then the eigendistributions $\mathcal{D}_{\tilde{h}}(\tilde{\lambda})$ and $\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$ of \tilde{h} are integrable and define two totally geodesic Legendre foliations of M . Moreover, for any $X \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$, $Y \in \Gamma(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}))$ $\tilde{\nabla}_X Y$ (respectively, $\tilde{\nabla}_Y X$) has no components along $\mathcal{D}_{\tilde{h}}(\tilde{\lambda})$ (respectively, $\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$).*

Proof. Replacing Y with $\tilde{\varphi}Y$ in (3.5), we get

$$(\tilde{\nabla}_X \tilde{h})\tilde{\varphi}Y - (\tilde{\nabla}_{\tilde{\varphi}Y} \tilde{h})X = -(1 + \tilde{\kappa})(2\tilde{g}(X, Y)\xi - 3\eta(X)\eta(Y)\xi + \eta(X)Y) - (1 - \tilde{\mu})\eta(X)\tilde{h}Y$$

for any $X, Y \in \Gamma(TM)$. Then, for any $X, Y, Z \in \Gamma(\mathcal{D})$ we have $\tilde{g}((\tilde{\nabla}_X \tilde{h})\tilde{\varphi}Y - (\tilde{\nabla}_{\tilde{\varphi}Y} \tilde{h})X, Z) = 0$, which is equivalent to

$$(4.1) \quad \tilde{g}(\tilde{\nabla}_X \tilde{h}\tilde{\varphi}Y - \tilde{h}\tilde{\nabla}_X \tilde{\varphi}Y - \tilde{\nabla}_{\tilde{\varphi}Y} \tilde{h}X + \tilde{h}\tilde{\nabla}_{\tilde{\varphi}Y} X, Z) = 0.$$

Now taking $X, Y, Z \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$ in (4.1) it follows that $-2\tilde{\lambda}\tilde{g}(\tilde{\nabla}_X \tilde{\varphi}Y, Z) = 0$. Since we are assuming $\tilde{\lambda} \neq 0$, we get $0 = \tilde{g}(\tilde{\nabla}_X \tilde{\varphi}Y, Z) = X(\tilde{g}(\tilde{\varphi}Y, Z)) - \tilde{g}(\tilde{\varphi}Y, \tilde{\nabla}_X Z) = -\tilde{g}(\tilde{\nabla}_X Z, \tilde{\varphi}Y)$. Thus $\tilde{\nabla}_X Z$ is orthogonal to $\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$. On the other hand, $\tilde{g}(\tilde{\nabla}_X Z, \xi) = X(\tilde{g}(Z, \xi)) - \tilde{g}(Z, \tilde{\nabla}_X \xi) = -\tilde{g}(Z, \tilde{\varphi}X) - \tilde{\lambda}\tilde{g}(Z, \tilde{\varphi}X) = 0$, so we conclude that $\tilde{\nabla}_X Z \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$. Analogously, if $X, Z \in \Gamma(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}))$ then $\tilde{\nabla}_X Z \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$. Hence $\mathcal{D}_{\tilde{h}}(\tilde{\lambda})$ and $\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$ are totally geodesic. Next, if $X \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$ and $Y \in \Gamma(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}))$ then for all $Z \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$ one has $\tilde{g}(\tilde{\nabla}_X Y, Z) = X(\tilde{g}(Y, Z)) - \tilde{g}(Y, \tilde{\nabla}_X Z) = 0$, since $\tilde{\nabla}_X Z \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$. Thus $\tilde{\nabla}_X Y \in \Gamma(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}) \oplus \mathbb{R}\xi)$. In a similar manner one can prove that $\tilde{\nabla}_Y X$ has no components along $\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$. In particular, the total geodesicity of $\mathcal{D}_{\tilde{h}}(\tilde{\lambda})$ and $\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$ implies that they are involutive distributions. Moreover they are also n -dimensional because of [11, Proposition 3.2]. Hence they define two Legendre foliations on M . \square

The geometry of a Legendre foliations is mainly described by its Pang invariant (2.13). Thus we find the explicit expression of the Pang invariants of the Legendre foliations $\mathcal{D}_{\tilde{h}}(\tilde{\lambda})$ and $\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$.

Theorem 4.2. *The Pang invariants of the Legendre foliations $\mathcal{D}_{\tilde{h}}(\tilde{\lambda})$ and $\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$ are given by*

$$(4.2) \quad \Pi_{\mathcal{D}_{\tilde{h}}(\tilde{\lambda})} = -2 \left(1 - \frac{\tilde{\mu}}{2} - \sqrt{1 + \tilde{\kappa}} \right) \tilde{g}|_{\mathcal{D}_{\tilde{h}}(\tilde{\lambda}) \times \mathcal{D}_{\tilde{h}}(\tilde{\lambda})}$$

$$(4.3) \quad \Pi_{\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})} = -2 \left(1 - \frac{\tilde{\mu}}{2} + \sqrt{1 + \tilde{\kappa}} \right) \tilde{g}|_{\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}) \times \mathcal{D}_{\tilde{h}}(-\tilde{\lambda})}.$$

Proof. Let X be a section of $\mathcal{D}_{\tilde{h}}(\tilde{\lambda})$. Then by (2.16) we have

$$(4.4) \quad \tilde{R}_{X\xi}\xi = \tilde{\kappa}X + \tilde{\mu}\tilde{h}X.$$

On the other hand,

$$(4.5) \quad \begin{aligned} \tilde{R}_{X\xi}\xi &= \tilde{\nabla}_X \tilde{\nabla}_\xi \xi - \tilde{\nabla}_\xi \tilde{\nabla}_X \xi - \tilde{\nabla}_{[X, \xi]}\xi \\ &= \tilde{\nabla}_\xi \tilde{\varphi}X - \tilde{\nabla}_\xi \tilde{\varphi}\tilde{h}X - \tilde{\varphi}[\xi, X] + \tilde{\varphi}\tilde{h}[\xi, X] \\ &= (1 - \tilde{\lambda})\tilde{\nabla}_{\tilde{\varphi}X}\xi + (1 - \tilde{\lambda})[\xi, \tilde{\varphi}X] - \tilde{\varphi}[\xi, X] + \tilde{\varphi}\tilde{h}[\xi, X] \\ &= (1 - \tilde{\lambda})(-\tilde{\varphi}^2 X + \tilde{\varphi}\tilde{h}\tilde{\varphi}X) + (1 - \tilde{\lambda})[\xi, \tilde{\varphi}X] - \tilde{\varphi}[\xi, X] + \tilde{\varphi}\tilde{h}[\xi, X] + \tilde{\varphi}\tilde{h}[\xi, X] \\ &= -X + \tilde{\lambda}^2 X + 2\tilde{h}X - \tilde{\lambda}[\xi, \tilde{\varphi}X] + \tilde{\varphi}\tilde{h}[\xi, X] + \tilde{\lambda}\tilde{\varphi}[\xi, X] - \tilde{\lambda}\tilde{\varphi}[\xi, X] \\ &= -X + \tilde{\lambda}^2 X + 2\tilde{\lambda}X - 2\tilde{\lambda}\tilde{h}X + \tilde{\varphi}\tilde{h}[\xi, X] - \tilde{\lambda}\tilde{\varphi}[\xi, X] \\ &= -(1 - \tilde{\lambda})^2 X + \tilde{\varphi}\tilde{h}[\xi, X] - \tilde{\lambda}\tilde{\varphi}[\xi, X]. \end{aligned}$$

By (4.4) and (4.5) it follows that $\tilde{\varphi}\tilde{h}[\xi, X] = \tilde{\lambda}[\xi, X] + (\tilde{\kappa} + \tilde{\mu}\tilde{\lambda} + (1 - \tilde{\lambda})^2)X$. By applying $\tilde{\varphi}$ we obtain

$$\tilde{h}[\xi, X] = \tilde{\lambda}[\xi, X] + (\tilde{\kappa} + \tilde{\mu}\tilde{\lambda} + (1 - \tilde{\lambda})^2)\tilde{\varphi}X.$$

Notice that, since $i_\xi d\eta = 0$, the Reeb vector field is an infinitesimal automorphism with respect to the contact distribution, so that $[\xi, X]$ is still a section of \mathcal{D} . Then by decomposing $[\xi, X]$ in its components

along $\mathcal{D}_{\tilde{h}}(\tilde{\lambda})$ and $\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$, from the last equation it follows that

$$[\xi, X]_{\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})} = -\frac{\tilde{\lambda}^2 - 2\tilde{\lambda} + 1 + \tilde{\kappa} + \tilde{\mu}\tilde{\lambda}}{2\tilde{\lambda}}\tilde{\varphi}X = \left(1 - \frac{\tilde{\mu}}{2} - \sqrt{1 + \tilde{\kappa}}\right)\tilde{\varphi}X.$$

Therefore for any $X, X' \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$

$$\Pi_{\mathcal{D}_{\tilde{h}}(\tilde{\lambda})}(X, X') = 2\tilde{g}([\xi, X]_{\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})}, \tilde{\varphi}X') = -2\left(1 - \frac{\tilde{\mu}}{2} - \sqrt{1 + \tilde{\kappa}}\right)\tilde{g}(X, X')$$

The proof of (4.3) is similar. \square

Proposition 4.3. *In any paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ with $\tilde{\kappa} > -1$ one has*

$$(4.6) \quad (\tilde{\nabla}_X \tilde{h})Y = -\tilde{g}(X, \tilde{\varphi}\tilde{h}^2Y + \tilde{\varphi}\tilde{h}Y)\xi + \eta(Y)((1 + \tilde{\kappa})\tilde{\varphi}X - \tilde{\varphi}\tilde{h}X) - \tilde{\mu}\eta(X)\tilde{\varphi}\tilde{h}Y$$

for any $X, Y \in \Gamma(TM)$.

Proof. Because of Theorem 4.1 we have that

$$(4.7) \quad (\tilde{\nabla}_X \tilde{h})Y = 0$$

for any $X, Y \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$ or $X, Y \in \Gamma(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}))$. Now suppose that $X \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$ and $Y \in \Gamma(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}))$. Let $\{X_1, \dots, X_n, \tilde{\varphi}X_1, \dots, \tilde{\varphi}X_n, \xi\}$ be a $\tilde{\varphi}$ -basis as in Lemma 3.11. Then according to (3.42) we have

$$\begin{aligned} \tilde{h}\tilde{\nabla}_X Y &= \tilde{h}\left(-\sum_{i=1}^r \tilde{g}(\tilde{\nabla}_X Y, \tilde{\varphi}X_i)\tilde{\varphi}X_i + \sum_{i=r+1}^n \tilde{g}(\tilde{\nabla}_X Y, \tilde{\varphi}X_i)\tilde{\varphi}X_i + \tilde{g}(\tilde{\nabla}_X Y, \xi)\xi\right) \\ &= -\tilde{\lambda}\tilde{\varphi}\sum_{i=1}^r \tilde{g}(\tilde{\varphi}\tilde{\nabla}_X Y, X_i)X_i + \tilde{\lambda}\tilde{\varphi}\sum_{i=r+1}^n \tilde{g}(\tilde{\varphi}\tilde{\nabla}_X Y, X_i)X_i \\ &= -\tilde{\lambda}\tilde{\varphi}^2\tilde{\nabla}_X Y \\ &= -\tilde{\lambda}(\tilde{\nabla}_X Y - \tilde{g}(\tilde{\nabla}_X Y, \xi)\xi) \\ &= -\tilde{\lambda}(\tilde{\nabla}_X Y + \tilde{g}(Y, \tilde{\nabla}_X \xi)\xi) \\ &= -\tilde{\lambda}(\tilde{\nabla}_X Y - \tilde{g}(Y, \tilde{\varphi}X)\xi + \tilde{g}(Y, \tilde{\varphi}\tilde{h}X)\xi) \\ &= \tilde{\nabla}_X \tilde{h}Y - \tilde{\lambda}(1 - \tilde{\lambda})\tilde{g}(X, \tilde{\varphi}Y)\xi. \end{aligned}$$

Thus for any $X \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$ and $Y \in \Gamma(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}))$ one has

$$(4.8) \quad (\tilde{\nabla}_X \tilde{h})Y = \tilde{\lambda}(1 - \tilde{\lambda})\tilde{g}(X, \tilde{\varphi}Y)\xi$$

and in a similar way one can prove that

$$(4.9) \quad (\tilde{\nabla}_Y \tilde{h})X = \tilde{\lambda}(1 + \tilde{\lambda})\tilde{g}(X, \tilde{\varphi}Y)\xi.$$

Now let X and Y any two vector fields on M . We decompose X and Y as $X = X_+ + X_- + \eta(X)\xi$, $Y = Y_+ + Y_- + \eta(Y)\xi$, according to the decomposition $TM = \mathcal{D}_{\tilde{h}}(\tilde{\lambda}) \oplus \mathcal{D}_{\tilde{h}}(-\tilde{\lambda}) \oplus \mathbb{R}\xi$. Using (3.6), (4.7), (4.8), (4.9), after a straightforward computation we get

$$\begin{aligned} (\tilde{\nabla}_X \tilde{h})Y &= \tilde{\lambda}(1 - \tilde{\lambda})\tilde{g}(X_+, \tilde{\varphi}Y_-)\xi - \tilde{\lambda}(1 + \tilde{\lambda})\tilde{g}(X_-, \tilde{\varphi}Y_+)\xi + \eta(Y)\tilde{h}\tilde{\varphi}X \\ &\quad + (1 + \tilde{\kappa})\eta(Y)\tilde{\varphi}X + \tilde{\mu}\eta(X)\tilde{h}\tilde{\varphi}Y. \end{aligned}$$

But $\tilde{\lambda}(1 - \tilde{\lambda})\tilde{g}(X_+, \tilde{\varphi}Y_-) - \tilde{\lambda}(1 + \tilde{\lambda})\tilde{g}(X_-, \tilde{\varphi}Y_+) = \tilde{\lambda}\tilde{g}(X_+, \tilde{\varphi}Y_-) - \tilde{\lambda}\tilde{g}(X_-, \tilde{\varphi}Y_+) - \tilde{\lambda}^2(\tilde{g}(X_+, \tilde{\varphi}Y_-) + \tilde{g}(X_-, \tilde{\varphi}Y_+)) = \tilde{g}(\tilde{h}X, \tilde{\varphi}Y) - \tilde{\lambda}^2\tilde{g}(X, \tilde{\varphi}Y)$. Therefore (4.6) follows. \square

We recall the following general result.

Theorem 4.4 ([10]). *Let (M, η) be a contact manifold endowed with a bi-Legendrian structure $(\mathcal{F}_1, \mathcal{F}_2)$ such that $\nabla^{bl}\Pi_{\mathcal{F}_1} = \nabla^{bl}\Pi_{\mathcal{F}_2} = 0$, where ∇^{bl} denotes the bi-Legendrian connection associated to $(\mathcal{F}_1, \mathcal{F}_2)$. Assume that one of the following conditions holds*

- (I) \mathcal{F}_1 and \mathcal{F}_2 are positive definite and there exist two positive numbers a and b such that $\overline{\Pi}_{\mathcal{F}_1} = ab\overline{\Pi}_{\mathcal{F}_2}$ on $T\mathcal{F}_1$ and $\overline{\Pi}_{\mathcal{F}_2} = ab\overline{\Pi}_{\mathcal{F}_1}$ on $T\mathcal{F}_2$,
- (II) \mathcal{F}_1 is positive definite, \mathcal{F}_2 is negative definite and there exist $a > 0$ and $b < 0$ such that $\overline{\Pi}_{\mathcal{F}_1} = ab\overline{\Pi}_{\mathcal{F}_2}$ on $T\mathcal{F}_1$ and $\overline{\Pi}_{\mathcal{F}_2} = ab\overline{\Pi}_{\mathcal{F}_1}$ on $T\mathcal{F}_2$,
- (III) \mathcal{F}_1 and \mathcal{F}_2 are negative definite and there exist two negative numbers a and b such that $\overline{\Pi}_{\mathcal{F}_1} = ab\overline{\Pi}_{\mathcal{F}_2}$ on $T\mathcal{F}_1$ and $\overline{\Pi}_{\mathcal{F}_2} = ab\overline{\Pi}_{\mathcal{F}_1}$ on $T\mathcal{F}_2$.

Then (M, η) admits a compatible contact metric structure $(\varphi_{a,b}, \xi, \eta, g_{a,b})$ such that

- (i) if $a = b$, $(M, \varphi_{a,b}, \xi, \eta, g_{a,b})$ is a Sasakian manifold;
- (ii) if $a \neq b$, $(M, \varphi_{a,b}, \xi, \eta, g_{a,b})$ is a contact metric $(\kappa_{a,b}, \mu_{a,b})$ -manifold, whose associated bi-Legendrian structure is $(\mathcal{F}_1, \mathcal{F}_2)$, where

$$(4.10) \quad \kappa_{a,b} = 1 - \frac{(a-b)^2}{16}, \quad \mu_{a,b} = 2 - \frac{a+b}{2}.$$

Notice that in the proof of Theorem 4.4 the assumptions (I), (II) or (III) are used for constructing the compatible metric structure, whereas the hypothesis $\nabla^{bl}\Pi_{\mathcal{F}_1} = \nabla^{bl}\Pi_{\mathcal{F}_2} = 0$ is necessary only for proving that such contact metric structure satisfies a nullity condition.

Now we prove one of the main results of the section, which puts in relation the theory of paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifolds with contact Riemannian geometry.

Theorem 4.5. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a positive or negative definite paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} > -1$. Then M admits a family of contact Riemannian structures $(\varphi_{a,b}, \xi, \eta, g_{a,b})$ parameterized by real numbers a and b satisfying the relation $ab = 4(1 + \tilde{\kappa})$. Each contact metric structure $(\varphi_{a,b}, \xi, \eta, g_{a,b})$ is explicitly given by*

$$(4.11) \quad \varphi_{a,b} = \begin{cases} \frac{b}{2(1+\tilde{\kappa})}\tilde{h}, & \text{on } \mathcal{D}^+ \\ -\frac{a}{2(1+\tilde{\kappa})}\tilde{h}, & \text{on } \mathcal{D}^- \\ 0, & \text{on } \mathbb{R}\xi \end{cases}$$

$$(4.12) \quad g_{a,b} = \begin{cases} \frac{2}{a}\tilde{g}(\tilde{h}\cdot, \cdot), & \text{on } \mathcal{D}^+ \times \mathcal{D}^+ \\ \frac{2}{b}\tilde{g}(\tilde{h}\cdot, \cdot), & \text{on } \mathcal{D}^- \times \mathcal{D}^- \\ \eta \otimes \eta, & \text{otherwise} \end{cases}$$

Furthermore, if $\tilde{\mu} = 2$ then each structure $(\varphi_{a,b}, \xi, \eta, g_{a,b})$ is a contact metric $(\kappa_{a,b}, \mu_{a,b})$ -structure (eventually Sasakian if $a = b$) with $\kappa_{a,b} = 1 - \frac{(a-b)^2}{16}$ and $\mu_{a,b} = 2 - \frac{a-b}{2}$.

Proof. The result will follow once we will have proved that the canonical bi-Legendrian structure $(\mathcal{D}^+, \mathcal{D}^-)$ of $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ satisfies one among the assumptions (I), (II), (III) of Theorem 4.4. The positive/negative definiteness of \mathcal{D}^+ and \mathcal{D}^- is ensured by the assumption that the paracontact $(\tilde{\kappa}, \tilde{\mu})$ -structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ is positive or negative definite. Hence it remains to prove the existence of real numbers a and b such that the Pang-Libermann invariants $\overline{\Pi}_{\mathcal{D}^+}$ and $\overline{\Pi}_{\mathcal{D}^-}$ are related each other as in the assumptions (I) or (III) of Theorem 4.4. First we find the explicit expression of the Libermann map $\Lambda_{\mathcal{D}^-} : TM \rightarrow \mathcal{D}^+$. By definition $\Lambda_{\mathcal{D}^+}\xi = 0$ and $\Lambda_{\mathcal{D}^-}Y = 0$ for all $Y \in \Gamma(\mathcal{D}^-)$. Let X be a section of \mathcal{D}^+ . Then, using (3.43), we have, for any $Y \in \Gamma(\mathcal{D}^+)$, $2\tilde{g}(\tilde{h}\Lambda_{\mathcal{D}^-}X, Y) = \Pi_{\mathcal{D}^-}(\Lambda_{\mathcal{D}^-}X, Y) = d\eta(X, Y) = \tilde{g}(X, \tilde{\varphi}Y) = -\tilde{g}(X, Y)$. Consequently $2\tilde{h}\Lambda_{\mathcal{D}^-}X = -X$. Applying \tilde{h} and using (3.2) we easily get

$$(4.13) \quad \Lambda_{\mathcal{D}^-}X = -\frac{1}{2(1+\tilde{\kappa})}\tilde{h}X.$$

Thus

$$\begin{aligned}\bar{\Pi}_{\mathcal{D}^-}(X, X') &= \Pi_{\mathcal{D}^-}(\Lambda_{\mathcal{D}^-}X, \Lambda_{\mathcal{D}^-}X') = \frac{1}{4(1+\tilde{\kappa})^2}\Pi_{\mathcal{D}^-}(\tilde{h}X, \tilde{h}X') \\ &= \frac{1}{2(1+\tilde{\kappa})^2}\tilde{g}(\tilde{h}X, X') = \frac{1}{4(1+\tilde{\kappa})}\Pi_{\mathcal{D}^+}(X, X'),\end{aligned}$$

so that

$$(4.14) \quad \Pi_{\mathcal{D}^+}(X, X') = 4(1+\tilde{\kappa})\bar{\Pi}_{\mathcal{D}^-}(X, X')$$

for any $X, X' \in \Gamma(\mathcal{D}^+)$. Arguing in a similar manner one finds that

$$(4.15) \quad \Lambda_{\mathcal{D}^+}Y = \frac{1}{2(1+\tilde{\kappa})}\tilde{h}Y$$

for all $Y \in \Gamma(\mathcal{D}^-)$ and

$$(4.16) \quad \Pi_{\mathcal{D}^-}(Y, Y') = 4(1+\tilde{\kappa})\bar{\Pi}_{\mathcal{D}^+}(Y, Y')$$

for any $Y, Y' \in \Gamma(\mathcal{D}^-)$. Comparing (4.14) with (4.16) we conclude that the bi-Legendrian structure $(\mathcal{D}^+, \mathcal{D}^-)$ satisfies the assumption (I) or (III) of Theorem 4.4, where a and b are any two real numbers such that $ab = 4(1+\tilde{\kappa})$, both positive or negative according to the fact that \mathcal{D}^+ and \mathcal{D}^- are positive or negative definite, respectively. By Theorem 4.4 this proves the existence of a family of contact Riemannian structures $(\varphi_{a,b}, \xi, \eta, g_{a,b})$ on M . The expressions (4.11) and (4.12) follow from [10, (3.4)–(3.5)] and from (4.13), (4.15) and (3.43). Concerning the last part of the theorem we have to prove that the bi-Legendrian structure $(\mathcal{D}^+, \mathcal{D}^-)$ satisfies the further assumption of Theorem 4.4, i.e. that the corresponding bi-Legendrian connection ∇^{bl} preserves the Pang invariant of both the foliations. Notice that by [8, Theorem 3.6], since $\mathcal{D}^+, \mathcal{D}^-$ are integrable, ∇^{bl} coincides in fact with the canonical paracontact connection ∇^{pc} . Now, by (2.12) and (4.6), for any $X, Y \in \Gamma(TM)$,

$$\begin{aligned}(\nabla_X^{pc}\tilde{h})Y &= \tilde{\nabla}_X\tilde{h}Y + \eta(X)\tilde{\varphi}\tilde{h}Y + \eta(\tilde{h}Y)(\tilde{\varphi}X - \tilde{\varphi}\tilde{h}X) + \tilde{g}(X, \tilde{\varphi}\tilde{h}Y)\xi \\ &\quad - \tilde{g}(\tilde{h}X, \tilde{\varphi}\tilde{h}Y)\xi - \tilde{h}\tilde{\nabla}_X Y - \eta(X)\tilde{h}\tilde{\varphi}Y - \eta(Y)(\tilde{h}\tilde{\varphi}X - \tilde{h}\tilde{\varphi}\tilde{h}X) \\ &= (\tilde{\nabla}_X\tilde{h})Y + 2\eta(X)\tilde{\varphi}\tilde{h}Y + \eta(Y)\tilde{\varphi}\tilde{h}X - (1+\tilde{\kappa})\eta(Y)\tilde{\varphi}X + \tilde{g}(X, \tilde{\varphi}\tilde{h}Y)\xi - \tilde{g}(\tilde{h}X, \tilde{\varphi}\tilde{h}Y)\xi \\ &= \tilde{g}(X, \tilde{h}\tilde{\varphi}Y)\xi - (1+\tilde{\kappa})\tilde{g}(X, \tilde{\varphi}Y)\xi + \eta(Y)\tilde{h}\tilde{\varphi}X + (1+\tilde{\kappa})\eta(Y)\tilde{\varphi}X + \tilde{\mu}\eta(X)\tilde{h}\tilde{\varphi}Y \\ &\quad + 2\eta(X)\tilde{\varphi}\tilde{h}Y + \eta(Y)\tilde{\varphi}\tilde{h}X - (1+\tilde{\kappa})\eta(Y)\tilde{\varphi}X + \tilde{g}(X, \tilde{\varphi}\tilde{h}Y)\xi + (1+\tilde{\kappa})\tilde{g}(X, \tilde{\varphi}Y)\xi \\ &= (\tilde{\mu}-2)\eta(X)\tilde{h}\tilde{\varphi}Y.\end{aligned}$$

Consequently, if $\tilde{\mu} = 2$ then $\nabla^{bl}\tilde{h} = \nabla^{pc}\tilde{h} = 0$. On the other hand, by (i) of Theorem 2.3 we have $\nabla^{bl}\tilde{g} = \nabla^{pc}\tilde{g} = 0$. Thus, using the expression (3.43) of $\Pi_{\mathcal{D}^+}$ in terms of the paracontact structure, we get, for all $X, X' \in \Gamma(\mathcal{D}^+)$ and for all $Z \in \Gamma(TM)$

$$\begin{aligned}(\nabla_Z^{bl}\Pi_{\mathcal{D}^+})(X, X') &= 2Z(\tilde{g}(\tilde{h}X, X')) - 2\tilde{g}(\tilde{h}\nabla_Z^{pc}X, X') - 2\tilde{g}(\tilde{h}X, \nabla_Z^{pc}X') \\ &= 2Z(\tilde{g}(\tilde{h}X, X')) - 2\tilde{g}(\nabla_Z^{pc}\tilde{h}X, X') - 2\tilde{g}(\tilde{h}X, \nabla_Z^{pc}X') \\ &= 2(\nabla_Z^{pc}\tilde{g})(\tilde{h}X, X') = 0.\end{aligned}$$

□

Corollary 4.6. *Every positive or negative definite paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} > -1$ admits a K -contact structure.*

Proof. It is sufficient to take $a = b = \pm 2(1+\tilde{\kappa})$ in Theorem 4.5, since in the proof of Theorem 4.4 (cf. [10]) it is shown that if $a = b$ then $h_{a,b} = 0$ and so the contact metric structure is K -contact. □

Actually, we now prove that one can define a distinguished contact metric structure on any positive or negative definite paracontact $(\tilde{\kappa}, \tilde{\mu})$ -space such that $\tilde{\kappa} > -1$.

Theorem 4.7. *Any positive or negative definite paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} > -1$ carries a canonical contact Riemannian structure (ϕ, ξ, η, g) given by*

$$(4.17) \quad \phi := \mp \frac{1}{\sqrt{1+\tilde{\kappa}}} \tilde{\varphi} \tilde{h}, \quad g := -d\eta(\cdot, \phi \cdot) + \eta \otimes \eta,$$

where the sign \mp depends on the positive or negative definiteness of the paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold. Moreover, if $\tilde{\mu} = 2$ then (ϕ, ξ, η, g) is Sasakian, whereas if $\tilde{\mu} \neq 2$ then (ϕ, ξ, η, g) is a non-Sasakian contact metric (κ, μ) -structure, where

$$(4.18) \quad \kappa = 1 - \left(1 - \frac{\tilde{\mu}}{2}\right)^2, \quad \mu = 2 \left(1 \mp \sqrt{1+\tilde{\kappa}}\right),$$

the sign \mp depending, respectively, on the positive or negative definiteness of the paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$.

Proof. Let us define a $(1, 1)$ -tensor field ϕ and a tensor g of type $(0, 2)$ by (4.17). First of all, using (3.2), one can easily prove that $\phi^2 = -I + \eta \otimes \xi$. Next we prove that g is a Riemannian metric. By using the symmetry of the operator \tilde{h} with respect to \tilde{g} , one has, for any $X, Y \in \Gamma(TM)$,

$$(4.19) \quad \begin{aligned} g(X, Y) &= \pm \frac{1}{\sqrt{1+\tilde{\kappa}}} d\eta(X, \tilde{\varphi} \tilde{h}Y) + \eta(X)\eta(Y) \\ &= \pm \frac{1}{\sqrt{1+\tilde{\kappa}}} \tilde{g}(X, \tilde{\varphi}^2 \tilde{h}Y) + \eta(X)\eta(Y) \\ &= \pm \frac{1}{\sqrt{1+\tilde{\kappa}}} \tilde{g}(X, \tilde{h}Y) + \eta(X)\eta(Y) \\ &= \pm \frac{1}{\sqrt{1+\tilde{\kappa}}} \tilde{g}(Y, \tilde{h}X) + \eta(X)\eta(Y) \\ &= g(Y, X), \end{aligned}$$

so that g is symmetric. In order to prove that it is also positive definite, let us consider a $\tilde{\varphi}$ -basis $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \xi\}$ as in Lemma 3.11. Then we have that $g(\xi, \xi) = 1$, $g(X_i, X_i) = \pm \frac{1}{\sqrt{1+\tilde{\kappa}}} \tilde{g}(X_i, \tilde{h}X_i) = \pm \frac{1}{\sqrt{1+\tilde{\kappa}}} \tilde{\lambda} \tilde{g}(X_i, X_i) = \pm \tilde{g}(X_i, X_i) = (\pm 1)(\pm 1) = 1$ and $\tilde{g}(Y_i, Y_i) = \pm \frac{1}{\sqrt{1+\tilde{\kappa}}} \tilde{g}(Y_i, \tilde{h}Y_i) = \mp \tilde{g}(Y_i, Y_i) = (\mp 1)(\mp 1) = 1$. Finally one can straightforwardly check that $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ and $g(X, \phi Y) = d\eta(X, Y)$. Thus (ϕ, ξ, η, g) is a contact Riemannian structure. We prove the second part of the theorem. Let us compute the operator h associated to the contact metric structure (ϕ, ξ, η, g) . We have

$$(4.20) \quad h = \frac{1}{2} \mathcal{L}_\xi \phi = \mp \frac{1}{2\sqrt{1+\tilde{\kappa}}} \mathcal{L}_\xi(\tilde{\varphi} \tilde{h}) = \mp \frac{1}{2\sqrt{1+\tilde{\kappa}}} \left((\mathcal{L}_\xi \tilde{\varphi}) \tilde{h} + \tilde{\varphi}(\mathcal{L}_\xi \tilde{h}) \right).$$

On the other hand, by using (3.6), we have, for any $X \in \Gamma(TM)$,

$$\begin{aligned} (\mathcal{L}_\xi \tilde{h})X &= [\xi, \tilde{h}X] - \tilde{h}[\xi, X] \\ &= \tilde{\nabla}_\xi \tilde{h}X - \tilde{\nabla}_{\tilde{h}X} \xi - \tilde{h} \tilde{\nabla}_\xi X + \tilde{h} \tilde{\nabla}_X \xi \\ &= (\tilde{\nabla}_\xi \tilde{h})X + \tilde{\varphi} \tilde{h}X - \tilde{\varphi} \tilde{h}^2 X - \tilde{h} \tilde{\varphi} X + \tilde{h} \tilde{\varphi} \tilde{h} X \\ &= (2 - \tilde{\mu}) \tilde{\varphi} \tilde{h} - 2(1 + \tilde{\kappa}) \tilde{\varphi} X. \end{aligned}$$

Thus (4.20) becomes

$$(4.21) \quad h = \mp \frac{1}{2\sqrt{1+\tilde{\kappa}}} (2 - \tilde{\mu}) \tilde{h}.$$

We distinguish the cases $\tilde{\mu} \neq 2$ and $\tilde{\mu} = 2$. In the first case by (4.21) we see that h is diagonalizable, it admits the eigenvalues $0, \pm\lambda$, where

$$(4.22) \quad \lambda := 1 - \frac{\tilde{\mu}}{2},$$

and the same eigendistributions as \tilde{h} . We prove that the Legendre foliations $\mathcal{D}_h(\lambda)$, $\mathcal{D}_h(-\lambda)$ and the corresponding bi-Legendrian connection $\bar{\nabla}^{bl}$ satisfy the conditions stated in Theorem 2.5, so concluding that (ϕ, ξ, η, g) is a contact metric (κ, μ) -structure. First of all, notice that $\mathcal{D}_h(\lambda)$ and $\mathcal{D}_h(-\lambda)$ are mutually g -orthogonal. Indeed by using (4.19) one has, for any $X \in \Gamma(\mathcal{D}_h(\lambda))$ and $Y \in \Gamma(\mathcal{D}_h(-\lambda))$, $g(X, Y) = \pm \frac{1}{\sqrt{1+\tilde{\kappa}}} \tilde{g}(X, \tilde{h}Y) = \mp \tilde{g}(X, Y) = 0$, since the eigendistributions of \tilde{h} are \tilde{g} -orthogonal (Corollary 3.10). Next, by definition of bi-Legendrian connection, also the conditions (i), (iii) and $\bar{\nabla}^{bl}\eta = \bar{\nabla}^{bl}d\eta = 0$ of Theorem 2.5 are satisfied. Moreover, $\bar{\nabla}^{bl}h = 0$ because $\bar{\nabla}^{bl}$ preserves $\mathcal{D}_h(\lambda)$ and $\mathcal{D}_h(-\lambda)$. Thus it remains to prove that $\bar{\nabla}^{bl}g = 0$ and $\bar{\nabla}^{bl}\phi = 0$. Let us recall ([7]) that, by definition, $\bar{\nabla}_X^{bl}Y = [X, Y]_{\mathcal{D}_h(-\lambda)}$ and $\bar{\nabla}_Y^{bl}X = [Y, X]_{\mathcal{D}_h(\lambda)}$ for any $X \in \Gamma(\mathcal{D}_h(\lambda))$ and $Y \in \Gamma(\mathcal{D}_h(-\lambda))$. Then for any $X, X' \in \Gamma(\mathcal{D}_h(\lambda))$ and $Y, Y' \in \Gamma(\mathcal{D}_h(-\lambda))$ one has

$$\begin{aligned} (\bar{\nabla}_Y^{bl}\tilde{g})(X, X') &= Y(\tilde{g}(X, X')) - \tilde{g}([Y, X]_{\mathcal{D}_h(\lambda)}, X') - \tilde{g}([Y, X']_{\mathcal{D}_h(-\lambda)}, X) \\ &= Y(\tilde{g}(X, X')) - \tilde{g}([Y, X], X') - \tilde{g}([Y, X'], X) \\ &= -2\tilde{g}(\tilde{\nabla}_X X', Y) = 0, \end{aligned}$$

and, analogously, using the \tilde{g} -orthogonality and totally geodesicity of $\mathcal{D}_{\tilde{h}}(\pm\tilde{\lambda}) = \mathcal{D}_h(\pm\lambda)$, one has that $(\bar{\nabla}_X^{bl}\tilde{g})(Y, Y') = -2\tilde{g}(\tilde{\nabla}_Y Y', X) = 0$, $(\bar{\nabla}_\xi^{bl}\tilde{g})(X, X') = -2\tilde{g}(\tilde{\nabla}_X X', \xi) = 0$ and $(\bar{\nabla}_\xi^{bl}\tilde{g})(Y, Y') = -2\tilde{g}(\tilde{\nabla}_Y Y', \xi) = 0$. Moreover, for any $X, X', X'' \in \Gamma(\mathcal{D}_h(\lambda))$, by using $\bar{\nabla}^{bl}d\eta = 0$,

$$\begin{aligned} (\bar{\nabla}_X^{bl}\tilde{g})(X', X'') &= X(\tilde{g}(X', X'')) - d\eta(\bar{\nabla}_X^{bl}X', \tilde{\varphi}X'') - d\eta(X', \tilde{\varphi}\bar{\nabla}_X^{bl}X'') \\ &= X(\tilde{g}(X', X'')) - X(d\eta(X', \tilde{\varphi}X'')) + d\eta(X', \bar{\nabla}_X^{bl}\tilde{\varphi}X'') \\ &\quad - X(d\eta(\tilde{\varphi}X', X'')) + d\eta(\bar{\nabla}_X^{bl}\tilde{\varphi}X', X'') \\ &= X(\tilde{g}(X', X'')) - X(\tilde{g}(X', X'')) + \tilde{g}(X', \tilde{g}\bar{\nabla}_X^{bl}\tilde{\varphi}X'') \\ &\quad - X(\tilde{g}(\tilde{\varphi}X', \tilde{\varphi}X'')) + \tilde{g}(\bar{\nabla}_X^{bl}\tilde{\varphi}X', \tilde{\varphi}X'') \\ &= -X(\tilde{g}(\tilde{\varphi}X', \tilde{\varphi}X'')) - \tilde{g}(\tilde{\varphi}X', [X, \tilde{\varphi}X'']_{\mathcal{D}_h(\lambda)}) + \tilde{g}([X, \tilde{\varphi}X']_{\mathcal{D}_h(-\lambda)}, \tilde{\varphi}X'') \\ &= -X(\tilde{g}(\tilde{\varphi}X', \tilde{\varphi}X'')) - \tilde{g}(\tilde{\varphi}X', [X, \tilde{\varphi}X'']) + \tilde{g}([X, \tilde{\varphi}X'], \tilde{\varphi}X'') \\ &= 2\tilde{g}(\tilde{\nabla}_{\tilde{\varphi}X'}\tilde{\varphi}X'', X) = 0 \end{aligned}$$

and, by similar computations, for any $Y, Y', Y'' \in \Gamma(\mathcal{D}_h(-\lambda))$, $(\bar{\nabla}_Y^{bl}\tilde{g})(Y', Y'') = 2\tilde{g}(\tilde{\nabla}_{\tilde{\varphi}Y'}\tilde{\varphi}Y'', Y) = 0$, where we used again the total geodesicity of $\mathcal{D}_{\tilde{h}}(\pm\tilde{\lambda})$. Since, by definition, $\bar{\nabla}^{bl}\xi = 0$, we conclude that $\bar{\nabla}^{bl}\tilde{g} = 0$. Thus by (4.19) and (4.21) we have, for all $X, Y, Z \in \Gamma(TM)$,

$$(\bar{\nabla}_X^{bl}g)(Y, Z) = \pm \frac{1}{\sqrt{1+\tilde{\kappa}}} (\bar{\nabla}_X^{bl}\tilde{g})(Y, \tilde{h}Z) \mp \frac{2}{2-\tilde{\mu}} \tilde{g}(Y, (\bar{\nabla}_X^{bl}h)Z) \pm \eta(Z)(\bar{\nabla}_X^{bl}\eta)(Y) \pm \eta(Y)(\bar{\nabla}_X^{bl}\eta)(Z) = 0$$

since $\bar{\nabla}^{bl}\tilde{g} = 0$, $\bar{\nabla}^{bl}h = 0$ and $\bar{\nabla}^{bl}\eta = 0$. On the other hand, from $\bar{\nabla}^{bl}g = 0$, $\bar{\nabla}^{bl}d\eta = 0$ and the relation $d\eta = g(\cdot, \phi\cdot)$ it easily follows that the bi-Legendrian connection $\bar{\nabla}^{bl}$ preserves also the tensor field ϕ . Therefore, according to Theorem 2.5, (ϕ, ξ, η, g) is a contact metric (κ, μ) -structure. In order to find the explicit expression of the constants κ and μ , notice that, by (4.22), $\sqrt{1-\kappa} = \left|1 - \frac{\tilde{\mu}}{2}\right|$, from which it follows that

$$(4.23) \quad \kappa = 1 - \left(1 - \frac{\tilde{\mu}}{2}\right)^2.$$

In order to find $\tilde{\mu}$, notice that since the bi-Legendrian structures $(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}), \mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$ and $(\mathcal{D}_h(-\lambda), \mathcal{D}_h(\lambda))$ coincide, also the corresponding Pang invariants must be equal. More precisely, by (4.21) one can find that

$$(4.24) \quad \mathcal{D}_{\tilde{h}}(\tilde{\lambda}) = \begin{cases} \mathcal{D}_h(\pm|\lambda|), & \text{if } \tilde{\mu} > 2 \\ \mathcal{D}_h(\mp|\lambda|), & \text{if } \tilde{\mu} < 2 \end{cases}$$

$$(4.25) \quad \mathcal{D}_{\tilde{h}}(-\tilde{\lambda}) = \begin{cases} \mathcal{D}_h(\mp|\lambda|), & \text{if } \tilde{\mu} > 2 \\ \mathcal{D}_h(\pm|\lambda|), & \text{if } \tilde{\mu} < 2 \end{cases}$$

where the sign \pm depends on the positive or negative definiteness of the paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$. Let us assume that $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is positive definite and that $\tilde{\mu} > 2$. Then, by using (4.24)–(4.25) and by comparing [10, (11)] with (4.2) we get

$$(4.26) \quad 2 \left(1 - \frac{\mu}{2} + |\lambda|\right) g(X, X') = -2 \left(1 - \frac{\tilde{\mu}}{2} - \sqrt{1 + \tilde{\kappa}}\right) \tilde{g}(X, X')$$

for any $X, X' \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$. By (4.19) and (4.23), (4.26) becomes

$$2 \left(1 - \frac{\mu}{2} - \left(1 - \frac{\tilde{\mu}}{2}\right)\right) \tilde{g}(X, X') = -2 \left(1 - \frac{\tilde{\mu}}{2} - \sqrt{1 + \tilde{\kappa}}\right) \tilde{g}(X, X'),$$

It follows that

$$(4.27) \quad \mu = 2 \left(1 - \sqrt{1 + \tilde{\kappa}}\right).$$

If we assume $\tilde{\mu} < 2$ we use [10, (12)] and we get

$$2 \left(1 - \frac{\mu}{2} - |\lambda|\right) g(X, X') = -2 \left(1 - \frac{\tilde{\mu}}{2} - \sqrt{1 + \tilde{\kappa}}\right) \tilde{g}(X, X')$$

and, as $|\lambda| = 1 - \frac{\tilde{\mu}}{2}$, so we obtain again (4.27). The case when $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is negative definite is similar and one can prove that

$$(4.28) \quad \mu = 2 \left(1 + \sqrt{1 + \tilde{\kappa}}\right).$$

Now let us assume that $\tilde{\mu} = 2$. Then (4.21) implies that the operator h vanishes, so that the contact metric structure (ϕ, ξ, η, g) is K -contact. In particular one has $N_\phi(\xi, X) = \phi^2[\xi, X] - \phi[\xi, \phi X] = -2\phi hX = 0$ for all $X \in \Gamma(TM)$. Moreover, since \mathcal{D}^+ , \mathcal{D}^- , $\mathcal{D}_{\tilde{h}}(\tilde{\lambda})$, $\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$ are Legendre foliations, the canonical almost bi-paracontact structure (3.38) is integrable. Thus by [11, Corollary 3.9] we deduce that $N_\phi(X, Y) = 0$ for all $X, Y \in \Gamma(\mathcal{D})$. Consequently the tensor field N_ϕ vanishes identically and (M, ϕ, ξ, η, g) is a Sasakian manifold. \square

Example 4.8. *Let us consider the paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold $(G, \tilde{\varphi}, \xi, \eta, \tilde{g})$ described in Example 3.15 and let us apply the procedure of Theorem 4.7. Then a canonical contact (κ, μ) -structure (ϕ, ξ, η, g) is defined on G , where, according to (4.18), $\kappa = 1 - \frac{(\alpha^2 - \beta^2)^2}{16}$ and $\mu = 2 \left(1 + \frac{\alpha^2 + \beta^2}{4}\right)$. Explicitly, the contact Riemannian structure is defined as follows*

$$\begin{aligned} \phi e_1 &= e_3, & \phi e_2 &= e_4, & \phi e_3 &= -e_1, & \phi e_4 &= -e_2, & \phi e_5 &= 0, \\ g(e_i, e_j) &= \delta_{ij} & \text{for any } i, j &\in \{1, \dots, 5\}. \end{aligned}$$

In order to understand where such a contact metric (κ, μ) -structure on the Lie group G stays in the Boeckx's classification, let us compute the value of the Boeckx invariant I_G ([6]). An easy computation shows that $I_G = \frac{1 - \frac{\mu}{2}}{\sqrt{1 - \kappa}} = -\frac{\alpha^2 + \beta^2}{|\alpha^2 - \beta^2|}$. Then one can straightforwardly check that $I_G < -1$ provided that $\alpha, \beta \neq 0$, and $I_G = -1$ if $\alpha = 0$ ($\beta \neq 0$) or $\alpha = 0$ ($\beta = 0$). Hence the contact Riemannian manifold

(G, ϕ, ξ, η, g) is locally isometric to one among the contact Riemannian Lie groups described in §4 of [6], namely that one whose Lie algebra has the same constant structures as (3.45)–(3.48).

Remark 4.9. Let (M, ϕ, ξ, η, g) be a (non-Sasakian) contact metric (κ', μ') -space. Then by applying the procedure described in [13] one obtains a paracontact $(\tilde{\kappa}, \tilde{\mu})$ -structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ on M , being

$$\tilde{\kappa} = \kappa - 2 + \left(1 - \frac{\mu}{2}\right)^2, \quad \tilde{\mu} = 2.$$

Since the bi-Legendrian structure $(\mathcal{D}^+, \mathcal{D}^-)$ coincides with that one $(\mathcal{D}_h(\lambda), \mathcal{D}_h(-\lambda))$ defined by the eigendistribution of h , due to [10, Theorem 4] one has that the paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ is positive or negative definite if and only if $I_M^2 > 1$, where I_M denotes the of the contact metric (κ, μ) -structure (ϕ, ξ, η, g) . But $I_M^2 > 1$ if and only if $\frac{(1-\frac{\mu}{2})^2}{1-\kappa} > 1$, that is $\kappa - 1 + (1 - \frac{\mu}{2})^2 > 0$, which is equivalent to require that $\tilde{\kappa} > -1$. Therefore the only positive or negative definite paracontact $(\tilde{\kappa}, \tilde{\mu})$ -structures determined via the above procedure are those ones with $\tilde{\kappa} > -1$. Then we are under the assumption of Theorem 4.7 and so we obtain a new contact Riemannian structure (ϕ', η, ξ, g') . Since $\tilde{\mu} = 2$, (ϕ', η, ξ, g') is in fact a Sasakian structure. Hence we have proved that any non-Sasakian contact metric (κ, μ) -manifold such that $|I_M| > 1$ admits a Sasakian metric compatible with the same underlying contact form η . The same result was proved using different techniques in [10].

Now we pass to study some important curvature properties.

Lemma 4.10. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} > -1$. Then for any vector fields X, Y, Z on M we have*

$$\begin{aligned} \tilde{R}_{XY}\tilde{h}Z - \tilde{h}\tilde{R}_{XY}Z &= (\tilde{\kappa}(\eta(X)\tilde{g}(\tilde{h}Y, Z) - \eta(Y)\tilde{g}(\tilde{h}X, Z)) + \tilde{\mu}(1 + \tilde{\kappa})(\eta(X)\tilde{g}(Y, Z) - \eta(Y)\tilde{g}(X, Z)))\xi \\ (4.29) \quad &+ \tilde{\kappa}(\tilde{g}(X, \tilde{\varphi}Z)\tilde{\varphi}\tilde{h}Y - \tilde{g}(Y, \tilde{\varphi}Z)\tilde{\varphi}\tilde{h}X + \tilde{g}(Z, \tilde{\varphi}\tilde{h}X)\tilde{\varphi}Y - \tilde{g}(Z, \tilde{\varphi}\tilde{h}Y)\tilde{\varphi}X \\ &+ \eta(Z)(\eta(X)\tilde{h}Y - \eta(Y)\tilde{h}X) - \tilde{\mu}((1 + \tilde{\kappa})\eta(Z)(\eta(Y)X - \eta(X)Y) + 2\tilde{g}(X, \tilde{\varphi}Y)\tilde{\varphi}\tilde{h}Z). \end{aligned}$$

Proof. The Ricci identity for \tilde{h} is

$$(4.30) \quad \tilde{R}_{XY}\tilde{h}Z - \tilde{h}\tilde{R}_{XY}Z = (\tilde{\nabla}_X\tilde{\nabla}_Y\tilde{h})Z - (\tilde{\nabla}_Y\tilde{\nabla}_X\tilde{h})Z - (\tilde{\nabla}_{[X,Y]}\tilde{h})Z$$

Using (3.2), (4.6) and the facts that \tilde{h} anti-commutes with $\tilde{\varphi}$ and $\tilde{\nabla}_X\tilde{\varphi}$ is antisymmetric, we get by direct calculation

$$\begin{aligned} (\tilde{\nabla}_X\tilde{\nabla}_Y\tilde{h})Z &= - \left((1 + \tilde{\kappa})\tilde{g}(\tilde{\nabla}_X Y, \tilde{\varphi}Z) + (1 + \tilde{\kappa})\tilde{g}(Y, \tilde{\nabla}_X\tilde{\varphi}Z) + \tilde{g}(\tilde{\nabla}_X Y, \tilde{\varphi}\tilde{h}Z) + \tilde{g}(Y, \tilde{\nabla}_X\tilde{\varphi}\tilde{h}Z) \right) \xi \\ &- \left((1 + \tilde{\kappa})\tilde{g}(Y, \tilde{\varphi}Z) + \tilde{g}(Y, \tilde{\varphi}\tilde{h}Z) \right) \tilde{\nabla}_X\xi + (\eta(\tilde{\nabla}_X Z) + \tilde{g}(Z, \tilde{\nabla}_X\xi))(1 + \tilde{\kappa})\tilde{\varphi}Y - \tilde{\varphi}\tilde{h}Y \\ &+ \eta(Z)((\tilde{\kappa} + 1)\tilde{\nabla}_X\tilde{\varphi}Y - \tilde{\nabla}_X\tilde{\varphi}\tilde{h}Y) \\ &- \tilde{\mu}(\eta(\tilde{\nabla}_X Y) + \tilde{g}(Y, \tilde{\nabla}_X\xi))\tilde{\varphi}\tilde{h}Z - \tilde{\mu}\eta(Y)\tilde{\nabla}_X\tilde{\varphi}\tilde{h}Z. \end{aligned}$$

So, using also (4.6) and (3.5), equation (4.30) yields

$$\begin{aligned} \tilde{R}_{XY}\tilde{h}Z - \tilde{h}\tilde{R}_{XY}Z &= ((1 + \tilde{\kappa})\tilde{g}((\tilde{\nabla}_X\tilde{\varphi})Y - (\tilde{\nabla}_Y\tilde{\varphi})X, Z) + \tilde{g}((\tilde{\nabla}_X\tilde{h}\tilde{\varphi})Y - (\tilde{\nabla}_Y\tilde{h}\tilde{\varphi})X, Z))\xi \\ (4.31) \quad &- ((1 + \tilde{\kappa})\tilde{g}(Y, \tilde{\varphi}Z) - \tilde{g}(Y, \tilde{h}\tilde{\varphi}Z))\tilde{\nabla}_X\xi + ((1 + \tilde{\kappa})\tilde{g}(X, \tilde{\varphi}Z) - \tilde{g}(X, \tilde{h}\tilde{\varphi}Z))\tilde{\nabla}_Y\xi \\ &+ \tilde{g}(Z, \tilde{\nabla}_X\xi)(\tilde{h}\tilde{\varphi}Y + (1 + \tilde{\kappa})\tilde{\varphi}Y) - \tilde{g}(Z, \tilde{\nabla}_Y\xi)(\tilde{h}\tilde{\varphi}X + (\tilde{\kappa} + 1)\tilde{\varphi}X) \\ &+ \eta(Z)((\tilde{\nabla}_X\tilde{h}\tilde{\varphi})Y - (\tilde{\nabla}_Y\tilde{h}\tilde{\varphi})X + (1 + \tilde{\kappa})((\tilde{\nabla}_X\tilde{\varphi})Y - (\tilde{\nabla}_Y\tilde{\varphi})X)) \\ &- \tilde{\mu}(\eta(Y)(\tilde{\nabla}_X\tilde{\varphi}\tilde{h})Z - \eta(X)(\tilde{\nabla}_Y\tilde{\varphi}\tilde{h})Z + 2\tilde{g}(X, \tilde{\varphi}Y)\tilde{\varphi}\tilde{h}Z). \end{aligned}$$

Using now (3.4), (4.6) and $\tilde{h}\xi = 0$, we get

$$(\tilde{\nabla}_X\tilde{\varphi}\tilde{h})Y = \tilde{g}(\tilde{h}^2X - \tilde{h}X, Y)\xi + \eta(Y)(\tilde{h}^2X - \tilde{h}X) - \tilde{\mu}\eta(X)\tilde{h}Y.$$

Therefore, by using (3.4) again, (4.31) reduces to (4.29). \square

Theorem 4.11. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} > -1$. Then we have, for any $X, X', X'' \in \mathcal{D}_{\tilde{h}}(\tilde{\lambda})$ and $Y, Y', Y'' \in \mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$,*

$$(4.32) \quad \tilde{R}_{XX'}X'' = (2(\tilde{\lambda} - 1) + \tilde{\mu})(\tilde{g}(X', X'')X - \tilde{g}(X, X'')X')$$

$$(4.33) \quad \tilde{R}_{XX'}Y = (\tilde{\kappa} + \tilde{\mu})(-\tilde{g}(\tilde{\varphi}X', Y)\tilde{\varphi}X + \tilde{g}(\tilde{\varphi}X, Y)\tilde{\varphi}X')$$

$$(4.34) \quad \tilde{R}_{XY}X' = \tilde{\kappa}\tilde{g}(\tilde{\varphi}Y, X')\tilde{\varphi}X - \tilde{\mu}\tilde{g}(\tilde{\varphi}Y, X)\tilde{\varphi}X'$$

$$(4.35) \quad \tilde{R}_{XY}Y' = -\tilde{\kappa}\tilde{g}(\tilde{\varphi}X, Y')\tilde{\varphi}Y + \tilde{\mu}\tilde{g}(\tilde{\varphi}X, Y)\tilde{\varphi}Y'$$

$$(4.36) \quad \tilde{R}_{Y'Y}X = (\tilde{\kappa} + \tilde{\mu})(-\tilde{g}(\tilde{\varphi}Y', X)\tilde{\varphi}Y + \tilde{g}(\tilde{\varphi}Y, X)\tilde{\varphi}Y')$$

$$(4.37) \quad \tilde{R}_{Y'Y}Y'' = (-2(\tilde{\lambda} + 1) + \tilde{\mu})(\tilde{g}(Y', Y'')Y - \tilde{g}(Y, Y'')Y').$$

Proof. We start by proving (4.33). We can choose a local orthogonal $\tilde{\varphi}$ -basis $\{e_i, \tilde{\varphi}e_i, \xi\}$, $i \in \{1, \dots, n\}$, as in Lemma 3.11. Then we have

$$(4.38) \quad \begin{aligned} \tilde{R}_{XX'}Y &= \tilde{g}(\tilde{R}_{XX'}Y, \xi)\xi - \sum_{i=1}^r \tilde{g}(\tilde{R}_{XX'}Y, e_i)e_i + \sum_{i=r+1}^n \tilde{g}(\tilde{R}_{XX'}Y, e_i)e_i \\ &+ \sum_{i=1}^r \tilde{g}(\tilde{R}_{XX'}Y, \tilde{\varphi}e_i)\tilde{\varphi}e_i - \sum_{i=r+1}^n \tilde{g}(\tilde{R}_{XX'}Y, \tilde{\varphi}e_i)\tilde{\varphi}e_i. \end{aligned}$$

Notice that, because of (3.1), $\tilde{g}(\tilde{R}_{XX'}Y, \xi) = -\tilde{g}(\tilde{R}_{XX'}\xi, Y) = 0$. Moreover, due to Theorem 4.1, also the terms $\tilde{g}(\tilde{R}_{XX'}Y, e_i)$ in (4.38) vanish. On the other hand, if $X \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$ and $Y, Z \in \Gamma(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}))$, then applying (4.29) we get

$$\tilde{R}_{XY}\tilde{h}Z - \tilde{h}\tilde{R}_{XY}Z = -(\tilde{\lambda}\tilde{R}_{XY}Z + \tilde{h}\tilde{R}_{XY}Z) = -2\tilde{\lambda}(\tilde{\kappa}\tilde{g}(X, \tilde{\varphi}Z)\tilde{\varphi}Y - \tilde{\mu}\tilde{g}(X, \tilde{\varphi}Y)\tilde{\varphi}Z)$$

and, taking the inner product with $W \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$, we get

$$(4.39) \quad \tilde{g}(\tilde{R}_{XY}Z, W) = \tilde{\kappa}\tilde{g}(X, \tilde{\varphi}Z)\tilde{g}(\tilde{\varphi}Y, W) - \tilde{\mu}\tilde{g}(X, \tilde{\varphi}Y)\tilde{g}(\tilde{\varphi}Z, W)$$

for any $X, W \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$ and $Y, Z \in \Gamma(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}))$. Therefore, using (4.38), (4.39) and the first Bianchi identity we find

$$\begin{aligned}
\tilde{R}_{XX'Y} &= - \sum_{i=1}^r \left(\tilde{g}(\tilde{R}_{YX}X', \tilde{\varphi}e_i)\tilde{\varphi}e_i + \tilde{g}(\tilde{R}_{X'Y}X, \tilde{\varphi}e_i)\tilde{\varphi}e_i \right) \\
&\quad + \sum_{i=r+1}^n \left(\tilde{g}(\tilde{R}_{YX}X', \tilde{\varphi}e_i)\tilde{\varphi}e_i + \tilde{g}(\tilde{R}_{X'Y}X, \tilde{\varphi}e_i)\tilde{\varphi}e_i \right) \\
&= - \sum_{i=1}^r \left(\tilde{g}(\tilde{R}_{XY}\tilde{\varphi}e_i, X')\tilde{\varphi}e_i - \tilde{g}(\tilde{R}_{X'Y}\tilde{\varphi}e_i, X)\tilde{\varphi}e_i \right) \\
&\quad + \sum_{i=r+1}^n \left(\tilde{g}(\tilde{R}_{XY}\tilde{\varphi}e_i, X')\tilde{\varphi}e_i - \tilde{g}(\tilde{R}_{X'Y}\tilde{\varphi}e_i, X)\tilde{\varphi}e_i \right) \\
&= - \sum_{i=1}^r (\tilde{\kappa}\tilde{g}(X, \tilde{\varphi}^2e_i)\tilde{g}(\tilde{\varphi}Y, X')\tilde{\varphi}e_i - \tilde{\mu}\tilde{g}(X, \tilde{\varphi}Y)\tilde{g}(\tilde{\varphi}^2e_i, X')\tilde{\varphi}e_i \\
&\quad - \tilde{\kappa}\tilde{g}(X', \tilde{\varphi}^2e_i)\tilde{g}(\tilde{\varphi}Y, X)\tilde{\varphi}e_i + \tilde{\mu}\tilde{g}(X', \tilde{\varphi}Y)\tilde{g}(\tilde{\varphi}^2e_i, X)\tilde{\varphi}e_i) \\
&\quad + \sum_{i=r+1}^n (\tilde{\kappa}\tilde{g}(X, \tilde{\varphi}^2e_i)\tilde{g}(\tilde{\varphi}Y, X')\tilde{\varphi}e_i - \tilde{\mu}\tilde{g}(X, \tilde{\varphi}Y)\tilde{g}(\tilde{\varphi}^2e_i, X')\tilde{\varphi}e_i \\
&\quad - \tilde{\kappa}\tilde{g}(X', \tilde{\varphi}^2e_i)\tilde{g}(\tilde{\varphi}Y, X)\tilde{\varphi}e_i + \tilde{\mu}\tilde{g}(X', \tilde{\varphi}Y)\tilde{g}(\tilde{\varphi}^2e_i, X)\tilde{\varphi}e_i) \\
&= \tilde{\kappa}\tilde{g}(\tilde{\varphi}Y, X')\tilde{\varphi}X - \tilde{\mu}\tilde{g}(X, \tilde{\varphi}Y)\tilde{\varphi}X' - \tilde{\kappa}\tilde{g}(\tilde{\varphi}Y, X)\tilde{\varphi}X' + \tilde{\mu}\tilde{g}(\tilde{\varphi}Y, X')\tilde{\varphi}X \\
&= (\tilde{\kappa} + \tilde{\mu})(-\tilde{g}(\tilde{\varphi}X', Y)\tilde{\varphi}X + \tilde{g}(\tilde{\varphi}X, Y)\tilde{\varphi}X').
\end{aligned}$$

Thus (4.33) is proved. Now let us prove (4.35). We have

$$\begin{aligned}
\tilde{R}_{XY}Y' &= \tilde{g}(\tilde{R}_{XY}Y', \xi)\xi - \sum_{i=1}^r \tilde{g}(\tilde{R}_{XY}Y', e_i)e_i + \sum_{i=r+1}^n \tilde{g}(\tilde{R}_{XY}Y', e_i)e_i \\
(4.40) \quad &\quad + \sum_{i=1}^r \tilde{g}(\tilde{R}_{XY}Y', \tilde{\varphi}e_i)\tilde{\varphi}e_i - \sum_{i=r+1}^n \tilde{g}(\tilde{R}_{XY}Y', \tilde{\varphi}e_i)\tilde{\varphi}e_i.
\end{aligned}$$

Arguing as before we have that $\tilde{g}(\tilde{R}_{XY}Y', \xi) = \tilde{g}(\tilde{R}_{XY}Y', e_i) = 0$ for each $i \in \{1, \dots, n\}$. On the other hand, if $X \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$ and $Y, Z \in \Gamma(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}))$, then applying (4.29) we get

$$\tilde{R}_{XY}\tilde{h}Z - \tilde{h}\tilde{R}_{XY}Z = -(\tilde{\lambda}\tilde{R}_{XY}Z + \tilde{h}\tilde{R}_{XY}Z) = -2\tilde{\lambda}(\tilde{\kappa}\tilde{g}(X, \tilde{\varphi}Z)\tilde{\varphi}Y - \tilde{\mu}\tilde{g}(X, \tilde{\varphi}Y)\tilde{\varphi}Z)$$

and, taking the inner product with $W \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$, we have

$$(4.41) \quad \tilde{g}(\tilde{R}_{XY}Z, W) = \tilde{\kappa}\tilde{g}(X, \tilde{\varphi}Z)\tilde{g}(\tilde{\varphi}Y, W) - \tilde{\mu}\tilde{g}(X, \tilde{\varphi}Y)\tilde{g}(\tilde{\varphi}Z, W)$$

for any $X, W \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$ and $Y, Z \in \Gamma(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}))$. Using (4.40), (4.41) and the first Bianchi identity we get

$$\begin{aligned}
\tilde{R}_{XY}Y' &= \sum_{i=1}^r \left(\tilde{g}(\tilde{R}_{Y'X}Y, e_i)e_i + \tilde{g}(\tilde{R}_{Y'Y}X, e_i)e_i \right) - \sum_{i=r+1}^n \left(\tilde{g}(\tilde{R}_{Y'X}Y, e_i)e_i + \tilde{g}(\tilde{R}_{Y'Y}X, e_i)e_i \right) \\
&= \sum_{i=1}^r \left(\tilde{g}(\tilde{R}_{Xe_i}Y, Y')e_i - \tilde{g}(\tilde{R}_{XY'}Y, e_i)e_i \right) - \sum_{i=r+1}^n \left(\tilde{g}(\tilde{R}_{Xe_i}Y, Y')e_i - \tilde{g}(\tilde{R}_{XY'}Y, e_i)e_i \right) \\
&= \sum_{i=1}^r \left(-\tilde{\kappa}\tilde{g}(X, \tilde{\varphi}Y)\tilde{g}(\tilde{\varphi}Y', e_i) + \tilde{\mu}\tilde{g}(X, \tilde{\varphi}Y')\tilde{g}(\tilde{\varphi}Y, e_i) \right) e_i \\
&\quad + \sum_{i=r+1}^n \left(\tilde{\kappa}\tilde{g}(X, \tilde{\varphi}Y)\tilde{g}(\tilde{\varphi}Y', e_i) - \tilde{\mu}\tilde{g}(X, \tilde{\varphi}Y')\tilde{g}(\tilde{\varphi}Y, e_i) \right) e_i \\
&\quad + \sum_{i=1}^r \left(\tilde{\kappa} + \tilde{\mu} \right) \left(\tilde{g}(\tilde{\varphi}X, Y)\tilde{g}(\tilde{\varphi}e_i, Y') - \tilde{g}(\tilde{\varphi}e_i, Y)\tilde{g}(\tilde{\varphi}X, Y') \right) e_i \\
&\quad - \sum_{i=r+1}^n \left(\tilde{\kappa} + \tilde{\mu} \right) \left(\tilde{g}(\tilde{\varphi}X, Y)\tilde{g}(\tilde{\varphi}e_i, Y') - \tilde{g}(\tilde{\varphi}e_i, Y)\tilde{g}(\tilde{\varphi}X, Y') \right) e_i \\
&= \tilde{\kappa}\tilde{g}(X, \tilde{\varphi}Y)\tilde{\varphi}Y' - \tilde{\mu}\tilde{g}(X, \tilde{\varphi}Y')\tilde{\varphi}Y + (\tilde{\kappa} + \tilde{\mu}) \left(\tilde{g}(\tilde{\varphi}X, Y)\tilde{\varphi}Y' - \tilde{g}(\tilde{\varphi}X, Y')\tilde{\varphi}Y \right) \\
&= -\tilde{\kappa}\tilde{g}(\tilde{\varphi}X, Y')\tilde{\varphi}Y + \tilde{\mu}\tilde{g}(\tilde{\varphi}X, Y)\tilde{\varphi}Y'.
\end{aligned}$$

Finally, we show (4.32). By using (3.30) one obtains

$$(4.42) \quad \tilde{R}_{XX'}\tilde{\varphi}X'' - \tilde{\varphi}\tilde{R}_{XX'}X'' = \tilde{g}(X' - \tilde{h}X', X'')(\tilde{\varphi}X - \tilde{\varphi}\tilde{h}X) - \tilde{g}(X - \tilde{h}X, X'')(\tilde{\varphi}X' - \tilde{\varphi}\tilde{h}X').$$

Then by applying $\tilde{\varphi}$ to (4.42) we get

$$\begin{aligned}
\tilde{\varphi}\tilde{R}_{XX'}\tilde{\varphi}X'' - \tilde{R}_{XX'}X'' &= \tilde{g}(X' - \tilde{h}X', X'')(X - \tilde{h}X) - \tilde{g}(X - \tilde{h}X, X'')(X' - \tilde{h}X') \\
&= (1 - \tilde{\lambda})^2\tilde{g}(X', X'')X - (1 - \tilde{\lambda})^2\tilde{g}(X, X'')X'.
\end{aligned}$$

So that, by using (4.33), one has

$$\begin{aligned}
\tilde{R}_{XX'}X'' &= \tilde{\varphi}\tilde{R}_{XX'}\tilde{\varphi}X'' - (1 - \tilde{\lambda})^2\tilde{g}(X', X'')X + (1 - \tilde{\lambda})^2\tilde{g}(X, X'')X' \\
&= (\tilde{\kappa} + \tilde{\mu})(\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}X'')X' - \tilde{g}(\tilde{\varphi}X', \tilde{\varphi}X'')X) + (1 - \tilde{\lambda})^2(\tilde{g}(X, X'')X' - \tilde{g}(X', X'')X) \\
&= (2(\tilde{\lambda} - 1) + \tilde{\mu})(\tilde{g}(X', X'')X - \tilde{g}(X, X'')X').
\end{aligned}$$

The proofs of remaining cases are similar. \square

Using Theorem 4.11 one can easily prove the following corollaries.

Corollary 4.12. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} > -1$. Then its Riemannian curvature tensor \tilde{R} is given by following formula*

$$\begin{aligned}
\tilde{g}(\tilde{R}_{XY}Z, W) &= \left(-1 + \frac{\tilde{\mu}}{2}\right) (\tilde{g}(Y, Z)\tilde{g}(X, W) - \tilde{g}(X, Z)\tilde{g}(Y, W)) \\
&\quad + \tilde{g}(Y, Z)\tilde{g}(\tilde{h}X, W) - \tilde{g}(X, Z)\tilde{g}(\tilde{h}Y, W) \\
&\quad - \tilde{g}(Y, W)\tilde{g}(\tilde{h}X, Z) + \tilde{g}(X, W)\tilde{g}(\tilde{h}Y, Z) \\
&\quad + \frac{-1 + \frac{\tilde{\mu}}{2}}{\tilde{\kappa} + 1} \left(\tilde{g}(\tilde{h}Y, Z)\tilde{g}(\tilde{h}X, W) - \tilde{g}(\tilde{h}X, Z)\tilde{g}(\tilde{h}Y, W)\right) \\
&\quad - \frac{\tilde{\mu}}{2} (\tilde{g}(\tilde{\varphi}Y, Z)\tilde{g}(\tilde{\varphi}X, W) - \tilde{g}(\tilde{\varphi}X, Z)\tilde{g}(\tilde{\varphi}Y, W)) \\
&\quad + \frac{-\tilde{\kappa} - \frac{\tilde{\mu}}{2}}{\tilde{\kappa} + 1} \left(\tilde{g}(\tilde{\varphi}\tilde{h}Y, Z)\tilde{g}(\tilde{\varphi}\tilde{h}X, W) - \tilde{g}(\tilde{\varphi}\tilde{h}Y, W)\tilde{g}(\tilde{\varphi}\tilde{h}X, Z)\right) \\
&\quad + \tilde{\mu}\tilde{g}(\tilde{\varphi}X, Y)\tilde{g}(\tilde{\varphi}Z, W) \\
&\quad + \eta(X)\eta(W) \left(\left(\tilde{\kappa} + 1 - \frac{\tilde{\mu}}{2}\right)\tilde{g}(Y, Z) + (\tilde{\mu} - 1)\tilde{g}(\tilde{h}Y, Z)\right) \\
&\quad - \eta(X)\eta(Z) \left(\left(\tilde{\kappa} + 1 - \frac{\tilde{\mu}}{2}\right)\tilde{g}(Y, W) + (\tilde{\mu} - 1)\tilde{g}(\tilde{h}Y, W)\right) \\
&\quad + \eta(Y)\eta(Z) \left(\left(\tilde{\kappa} + 1 - \frac{\tilde{\mu}}{2}\right)\tilde{g}(X, W) + (\tilde{\mu} - 1)\tilde{g}(\tilde{h}X, W)\right) \\
&\quad - \eta(Y)\eta(W) \left(\left(\tilde{\kappa} + 1 - \frac{\tilde{\mu}}{2}\right)\tilde{g}(X, Z) + (\tilde{\mu} - 1)\tilde{g}(\tilde{h}X, Z)\right)
\end{aligned} \tag{4.43}$$

for all vector fields X, Y, Z, W on M .

Proof. We can decompose an arbitrary vector field X on M uniquely as $X = X_{\tilde{\lambda}} + X_{-\tilde{\lambda}} + \eta(X)\xi$, where $X_{\tilde{\lambda}} \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$ and $X_{-\tilde{\lambda}} \in \Gamma(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}))$. We then write $\tilde{R}_{XY}Z$ as a sum of terms of the form $\tilde{R}_{X_{\pm\tilde{\lambda}}Y_{\pm\tilde{\lambda}}}Z_{\pm\tilde{\lambda}}, \tilde{R}_{XY}\xi, \tilde{R}_{X\xi}Z$. Then by Theorem 4.11 and (3.7), and taking into account that, in fact

$$X_{\tilde{\lambda}} = \frac{1}{2}(X - \eta(X)\xi + \frac{1}{\tilde{\lambda}}\tilde{h}X), \quad X_{-\tilde{\lambda}} = \frac{1}{2}(X - \eta(X)\xi - \frac{1}{\tilde{\lambda}}\tilde{h}X),$$

we obtain (4.43). \square

Corollary 4.13. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} > -1$. Then for any $Z \in \Gamma(\mathcal{D})$ the ξ -sectional curvature $\tilde{K}(Z, \xi)$ is given by*

$$\tilde{K}(Z, \xi) = \tilde{\kappa} + \tilde{\mu} \frac{\tilde{g}(\tilde{h}Z, Z)}{\tilde{g}(Z, Z)} = \begin{cases} \tilde{\kappa} + \tilde{\lambda}\tilde{\mu}, & \text{if } Z \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda})); \\ \tilde{\kappa} - \tilde{\lambda}\tilde{\mu}, & \text{if } Z \in \Gamma(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})). \end{cases}$$

Moreover, the sectional curvature of plane sections normal to ξ is given by

$$K(X, X') = 2(\tilde{\lambda} - 1) + \tilde{\mu}, \quad K(Y, Y') = -2(\tilde{\lambda} + 1) + \tilde{\mu},$$

$$K(X, Y) = (\tilde{\kappa} - \tilde{\mu}) \frac{\tilde{g}(X, \tilde{\varphi}Y)^2}{\tilde{g}(X, X)\tilde{g}(Y, Y)},$$

for any $X, X' \in \Gamma(\mathcal{D}_{\tilde{h}}(\tilde{\lambda}))$, $Y, Y' \in \Gamma(\mathcal{D}_{\tilde{h}}(-\tilde{\lambda}))$.

Corollary 4.14. *In any $(2n+1)$ -dimensional paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ such that $\tilde{\kappa} > -1$, the Ricci operator \tilde{Q} is given by*

$$(4.44) \quad \tilde{Q} = (2(1 - n) + n\tilde{\mu})I + (2(n - 1) + \tilde{\mu})\tilde{h} + (2(n - 1) + n(2\tilde{\kappa} - \tilde{\mu}))\eta \otimes \xi$$

In particular, (M, \tilde{g}) is η -Einstein if and only if $\tilde{\mu} = 2(1 - n)$, Einstein if and only if $\tilde{\kappa} = \tilde{\mu} = 0$ and $n = 1$ (in this case the manifold is Ricci-flat).

In particular it follows that in dimension 3 any paracontact $(\tilde{\kappa}, 0)$ -manifold with $\tilde{\kappa} > -1$ is η -Einstein. Notice that, in dimension greater than 3 no paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} > -1$ can be Einstein, since one would find $\tilde{\kappa} = \frac{1-n^2}{n}$ and only for $n = 1$ one has that $\tilde{\kappa} > -1$.

5. PARACONTACT $(\tilde{\kappa}, \tilde{\mu})$ -MANIFOLDS WITH $\tilde{\kappa} < -1$

In this section we deal with paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifolds such that $\tilde{\kappa} < -1$. In this case, as stated in Corollary 3.10, $\tilde{\varphi}\tilde{h}$ is diagonalizable with eigenvectors $0, \pm\tilde{\lambda}$, where $\tilde{\lambda} := \sqrt{-1 - \tilde{\kappa}}$. As for the case $\tilde{\kappa} > -1$ we start by proving that the distributions defined by the eigenspaces of $\tilde{\varphi}\tilde{h}$ define two mutually orthogonal Legendre foliations. The main difference with the case $\tilde{\kappa} > -1$ and, more in general, with the theory of contact metric (κ, μ) -spaces, is that they are not totally geodesic, but they are totally umbilical.

Theorem 5.1. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} < -1$. Then the eigendistributions $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})$ and $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda})$ of the operator $\tilde{\varphi}\tilde{h}$ are integrable and define two mutually orthogonal Legendre foliations of M with totally umbilical leaves. Moreover, for any $X, Y \in \mathcal{D}_{\tilde{\varphi}\tilde{h}}(\pm\tilde{\lambda})$, $\tilde{\nabla}_X Y \in \mathcal{D}_{\tilde{\varphi}\tilde{h}}(\pm\tilde{\lambda}) \oplus \mathbb{R}\xi$.*

Proof. From (3.4), (3.5), (3.8) we get the formula

$$(5.1) \quad (\tilde{\nabla}_X \tilde{\varphi}\tilde{h})Y - (\tilde{\nabla}_Y \tilde{\varphi}\tilde{h})X = -(1 + \tilde{\kappa})(\eta(X)Y - \eta(Y)X) + (1 - \tilde{\mu})(\eta(X)\tilde{h}Y - \eta(Y)\tilde{h}X)$$

which holds for any paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold. From (5.2) it follows that, for any $X, Y, Z \in \Gamma(\mathcal{D})$, $\tilde{g}((\tilde{\nabla}_X \tilde{\varphi}\tilde{h})\tilde{\varphi}Y - (\tilde{\nabla}_{\tilde{\varphi}Y} \tilde{\varphi}\tilde{h})X, Z) = 0$, that is

$$(5.2) \quad \tilde{g}(\tilde{\nabla}_X \tilde{\varphi}\tilde{h}\tilde{\varphi}Y - \tilde{\varphi}\tilde{h}\tilde{\nabla}_X \tilde{\varphi}Y - \tilde{\nabla}_{\tilde{\varphi}Y} \tilde{\varphi}\tilde{h}X + \tilde{\varphi}\tilde{h}\tilde{\nabla}_{\tilde{\varphi}Y} X, Z) = 0.$$

Now, let us assume that $X, Y, Z \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}))$. Then (5.2) reduces to $0 = -\tilde{\lambda}\tilde{g}(\tilde{\nabla}_X \tilde{\varphi}Y, Z) - \tilde{\lambda}\tilde{g}(\tilde{\nabla}_X \tilde{\varphi}Y, Z) - \tilde{\lambda}\tilde{g}(\tilde{\nabla}_{\tilde{\varphi}Y} X, Z) + \tilde{\lambda}\tilde{g}(\tilde{\nabla}_{\tilde{\varphi}Y} X, Z) = -2\tilde{\lambda}\tilde{g}(\tilde{\nabla}_X \tilde{\varphi}Y, Z)$. Thus $\tilde{\nabla}_X Z \in \mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}) \oplus \mathbb{R}\xi$. In particular, since $\tilde{g}([X, Y], \xi) = \tilde{g}(X, \tilde{\nabla}_Y \xi) - \tilde{g}(Y, \tilde{\nabla}_X \xi) = \tilde{\lambda}\tilde{g}(Y, X) - \tilde{\lambda}\tilde{g}(X, Y) = 0$ for all $X, Y \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}))$, we have that $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})$ defines a foliation on M . Moreover, $\dim(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})) = n$ due to [11, Proposition 3.2]. Hence, being an n -dimensional integrable subbundle of the contact distribution, $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})$ is a Legendre foliation of M . Similar arguments work also for $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda})$. In order to complete the proof, let us consider $X \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}))$ and $Y \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda}))$. Then, for any $Z \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}))$, $\tilde{g}(\tilde{\nabla}_X Y, Z) = -\tilde{g}(Y, \tilde{\nabla}_X Z) = 0$, since $\tilde{\nabla}_X Z \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}) \oplus \mathbb{R}\xi)$. Hence $\tilde{\nabla}_X Y \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda}) \oplus \mathbb{R}\xi)$. In the same way one proves that $\tilde{\nabla}_Y X \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}) \oplus \mathbb{R}\xi)$. Finally we prove that the leaves of $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})$ and $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda})$ are totally umbilical. Since for any $X, X' \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}))$ $\tilde{\nabla}_X X' \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}) \oplus \mathbb{R}\xi)$, $B(X, X')$ is a section of $\mathbb{R}\xi$, where B denotes the second fundamental form. Actually $B(X, X') = -\tilde{\lambda}\tilde{g}(X, X')\xi$. Indeed by using (2.6) one has

$$\tilde{g}(B(X, X'), \xi) = \tilde{g}(\tilde{\nabla}_X X', \xi) = -\tilde{g}(X', \tilde{\nabla}_X \xi) = \tilde{g}(X', \tilde{\varphi}X) - \tilde{\lambda}\tilde{g}(X, X') = -\tilde{\lambda}\tilde{g}(X, X').$$

Then the mean curvature vector field is given by $H = -\tilde{\lambda}\xi$. Hence $B(X, X') = H\tilde{g}(X, X')$. The proof for the other foliation is similar. \square

Remark 5.2. Notice that the foliations $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})$ and $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda})$ are not totally geodesic. In fact a straightforward computation shows that, for all $X, Y \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\pm\tilde{\lambda}))$, $\tilde{g}(\tilde{\nabla}_X Y, \xi) = -\tilde{\lambda}\tilde{g}(X, \tilde{\varphi}Y)$, in general different from zero.

Now we find the explicit expressions of the Pang invariants of the Legendre foliations $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})$ and $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda})$. The proof is similar to that of Theorem 4.2 hence we omit it.

Theorem 5.3. *The Pang invariants of the Legendre foliations $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})$ and $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda})$ are given by*

$$(5.3) \quad \Pi_{\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})} = (\tilde{\mu} - 2)\tilde{g}|_{\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}) \times \mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})}$$

$$(5.4) \quad \Pi_{\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda})} = (\tilde{\mu} - 2)\tilde{g}|_{\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda}) \times \mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda})}.$$

Proposition 5.4. *In any paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} < -1$ one has*

$$(5.5) \quad (\tilde{\nabla}_X \tilde{\varphi}\tilde{h})Y = ((1 + \tilde{\kappa})\tilde{g}(X, Y) - \tilde{g}(\tilde{h}X, Y))\xi + \eta(Y)\tilde{h}(\tilde{h}X - X) - \tilde{\mu}\eta(X)\tilde{h}Y.$$

Proof. Let $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \xi\}$ be a $\tilde{\varphi}$ -basis as in Lemma 3.11. By using Theorem 5.1 we have, for any $X, Y \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}))$,

$$\begin{aligned} \tilde{\varphi}\tilde{h}\tilde{\nabla}_X Y &= \tilde{\varphi}\tilde{h} \left(-\sum_{i=1}^r \tilde{g}(\tilde{\nabla}_X Y, X_i)X_i + \sum_{i=r+1}^n \tilde{g}(\tilde{\nabla}_X Y, X_i)X_i + \tilde{g}(\tilde{\nabla}_X Y, \xi)\xi \right) \\ &= -\tilde{\lambda} \sum_{i=1}^n \tilde{g}(\tilde{\nabla}_X Y, X_i)X_i + \tilde{\lambda} \sum_{i=r+1}^n \tilde{g}(\tilde{\nabla}_X Y, X_i)X_i \\ &= \tilde{\lambda}\tilde{\nabla}_X Y - \tilde{\lambda}\tilde{g}(\tilde{\nabla}_X Y, \xi)\xi \\ &= \tilde{\nabla}_X \tilde{\varphi}\tilde{h}Y - \tilde{\lambda}\tilde{g}(\tilde{\nabla}_X Y, \xi)\xi. \end{aligned}$$

It follows that

$$(5.6) \quad (\tilde{\nabla}_X \tilde{\varphi}\tilde{h})Y = -\tilde{\lambda}\tilde{g}(Y, \tilde{\nabla}_X \xi)\xi = -\tilde{\lambda}\tilde{g}(Y, -\tilde{\varphi}X + \tilde{\varphi}\tilde{h}X)\xi = -\tilde{\lambda}^2\tilde{g}(X, Y)\xi.$$

Now, let us consider $X \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}))$ and $Y \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda}))$. Arguing as in the previous case, one finds

$$(5.7) \quad (\tilde{\nabla}_X \tilde{\varphi}\tilde{h})Y = (\tilde{\nabla}_Y \tilde{\varphi}\tilde{h})X = -\tilde{\lambda}\tilde{g}(X, \tilde{\varphi}Y)\xi.$$

Finally, for any $X, Y \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda}))$ one has

$$(5.8) \quad (\tilde{\nabla}_X \tilde{\varphi}\tilde{h})Y = -\tilde{\lambda}^2\tilde{g}(X, Y)\xi.$$

Then (5.5) follows from (3.6), (5.6), (5.7) and (5.8). \square

Corollary 5.5. *In any paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} \neq -1$ one has*

$$(5.9) \quad (\tilde{\nabla}_X \tilde{h})Y = -((1 + \tilde{\kappa})\tilde{g}(X, \tilde{\varphi}Y) + \tilde{g}(X, \tilde{\varphi}\tilde{h}Y))\xi + \eta(Y)\tilde{\varphi}\tilde{h}(\tilde{h}X - X) - \tilde{\mu}\eta(X)\tilde{\varphi}\tilde{h}Y.$$

Proof. If $\tilde{\kappa} > -1$, (5.9) is just (4.6) and there is nothing to prove. Next, if $\tilde{\kappa} < -1$, as $(\tilde{\nabla}_X \tilde{h})Y = (\tilde{\nabla}_X \tilde{\varphi})\tilde{\varphi}\tilde{h}Y + \tilde{\varphi}(\tilde{\nabla}_X \tilde{\varphi}\tilde{h})Y$, the assertion follows directly from (3.4) and (5.5). \square

Even if the Legendre foliations $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\pm\tilde{\lambda})$ are not totally geodesic and thus many properties are missing compared to the case $\tilde{\kappa} > -1$, also in this case we can find an interesting relationship with contact Riemannian structures. We have in fact the following result.

Theorem 5.6. *Any positive or negative definite paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} < -1$ carries a canonical contact Riemannian structure (ϕ, ξ, η, g) given by*

$$(5.10) \quad \phi := \pm \frac{1}{\sqrt{-1 - \tilde{\kappa}}} \tilde{h}, \quad g := -d\eta(\cdot, \phi) + \eta \otimes \eta,$$

where the sign \pm depends on the positive or negative definiteness of the paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold. Moreover, (ϕ, ξ, η, g) is a contact metric (κ, μ) -structure, where

$$\kappa = \tilde{\kappa} + 2 - \left(1 - \frac{\tilde{\mu}}{2}\right)^2, \quad \mu = 2.$$

Proof. Let us define a $(1, 1)$ -tensor field ϕ and a tensor g of type $(0, 2)$ by (5.10). First of all, using (3.2), one can easily prove that $\phi^2 = -I + \eta \otimes \xi$. Next we prove that g is a Riemannian metric. By using the symmetry of the operator $\tilde{\varphi}\tilde{h}$ with respect to \tilde{g} , one has, for any $X, Y \in \Gamma(TM)$,

$$\begin{aligned} g(X, Y) &= \mp \frac{1}{\sqrt{-1-\tilde{\kappa}}} d\eta(X, \tilde{h}Y) + \eta(X)\eta(Y) \\ &= \mp \frac{1}{\sqrt{-1-\tilde{\kappa}}} \tilde{g}(X, \tilde{\varphi}\tilde{h}Y) + \eta(X)\eta(Y) \\ &= \mp \frac{1}{\sqrt{-1-\tilde{\kappa}}} \tilde{g}(\tilde{\varphi}\tilde{h}X, Y) + \eta(X)\eta(Y) \\ &= g(Y, X), \end{aligned}$$

so that g is symmetric. In order to prove that it is also positive definite, let us consider a $\tilde{\varphi}$ -basis $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \xi\}$ as in Lemma 3.11. Then we have that $g(\xi, \xi) = 1$, $\tilde{g}(X_i, X_i) = \mp \frac{1}{\sqrt{-1-\tilde{\kappa}}} \tilde{g}(X_i, \tilde{\varphi}\tilde{h}X_i) = \mp \frac{1}{\sqrt{-1-\tilde{\kappa}}} \tilde{g}(X_i, X_i) = \mp \tilde{g}(X_i, X_i) = (\mp 1)(\mp 1) = 1$ and $\tilde{g}(Y_i, Y_i) = \mp \frac{1}{\sqrt{-1-\tilde{\kappa}}} \tilde{g}(Y_i, \tilde{\varphi}\tilde{h}Y_i) = \pm \tilde{g}(Y_i, Y_i) = (\pm 1)(\pm 1) = 1$. Finally one can straightforwardly check that $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ and $g(X, \phi Y) = d\eta(X, Y)$. Thus (ϕ, ξ, η, g) is a contact Riemannian structure. In order to prove that it satisfies a (κ, μ) -nullity condition, let us compute the operator $h := \frac{1}{2}\mathcal{L}_\xi\phi = \pm \frac{1}{2\sqrt{-1-\tilde{\kappa}}}\mathcal{L}_\xi\tilde{h}$. Notice that, for all $X \in \Gamma(TM)$,

$$\begin{aligned} (\mathcal{L}_\xi\tilde{h})X &= [\xi, \tilde{h}X] - \tilde{h}[\xi, X] \\ (5.11) \quad &= \tilde{\nabla}_\xi\tilde{h}X - \tilde{\nabla}_{\tilde{h}X}\xi - \tilde{h}\tilde{\nabla}_\xi X + \tilde{h}\tilde{\nabla}_X\xi \\ &= (\tilde{\nabla}_\xi\tilde{h})X - (-\tilde{\varphi}\tilde{h}X + \tilde{\varphi}\tilde{h}^2X) + \tilde{h}(-\tilde{\varphi}X + \tilde{\varphi}\tilde{h}X) \\ &= (2 - \tilde{\mu})\tilde{\varphi}\tilde{h}X - 2(1 + \tilde{\kappa})\tilde{\varphi}X. \end{aligned}$$

Consequently

$$h = \pm \left(\frac{1 - \frac{\tilde{\mu}}{2}}{\sqrt{-1-\tilde{\kappa}}} \tilde{\varphi}\tilde{h} + \sqrt{-1-\tilde{\kappa}}\tilde{\varphi} \right).$$

We prove that h is diagonalizable. Note that the matrix of h with respect to the $\tilde{\varphi}$ -basis $\{X_i, Y_i, \xi\}$ is given by

$$\pm \begin{pmatrix} 1 - \frac{\tilde{\mu}}{2} & \cdots & 0 & \sqrt{-1-\tilde{\kappa}} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 - \frac{\tilde{\mu}}{2} & 0 & \cdots & \sqrt{-1-\tilde{\kappa}} & 0 \\ \sqrt{-1-\tilde{\kappa}} & \cdots & 0 & -1 + \frac{\tilde{\mu}}{2} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \sqrt{-1-\tilde{\kappa}} & 0 & \cdots & -1 + \frac{\tilde{\mu}}{2} & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Thus the characteristic polynomial is given by $P(\lambda) = \mp \lambda \left(\lambda^2 - \left(1 - \frac{\tilde{\mu}}{2}\right)^2 + \tilde{\kappa} + 1 \right)^n$ and h admits the eigenvalues 0 and $\pm \sqrt{\left(1 - \frac{\tilde{\mu}}{2}\right)^2 - 1 - \tilde{\kappa}}$. After a long computation, one can prove that the corresponding eigendistributions are $\mathcal{D}_h(0) = \mathbb{R}\xi$, and

$$\mathcal{D}_h(\lambda) = \text{span}\{X_1 + \alpha Y_1, \dots, X_n + \alpha Y_n\}, \quad \mathcal{D}_h(-\lambda) = \text{span}\{X_1 - \beta Y_1, \dots, X_n - \beta Y_n\},$$

where $\alpha := \frac{1}{\lambda} \sqrt{\left(1 - \frac{\tilde{\mu}}{2}\right)^2 - 1 - \tilde{\kappa} - \frac{1}{\lambda} \left(1 - \frac{\tilde{\mu}}{2}\right)}$ and $\beta := \frac{1}{\lambda} \sqrt{\left(1 - \frac{\tilde{\mu}}{2}\right)^2 - 1 - \tilde{\kappa} + \frac{1}{\lambda} \left(1 - \frac{\tilde{\mu}}{2}\right)}$. Hence one can easily conclude that h is diagonalizable. Next, we prove that $\mathcal{D}_h(\lambda)$ and $\mathcal{D}_h(-\lambda)$ are Legendre foliations. Taking Theorem 5.1 into account, we can write $\tilde{\nabla}_{X_i} X_j = \sum_{k=1}^n f_{ij}^k X_k + \tilde{g}(\tilde{\nabla}_{X_i} X_j, \xi) \xi$ and $\tilde{\nabla}_{Y_i} X_j = \sum_{k=1}^n g_{ij}^k X_k + \tilde{g}(\tilde{\nabla}_{Y_i} X_j, \xi) \xi$, for some functions f_{ij}^k, g_{ij}^k . Then one finds

$$\tilde{\nabla}_{X_i} Y_j = \tilde{\nabla}_{X_i} \tilde{\varphi} X_j = \tilde{\varphi} \tilde{\nabla}_{X_i} X_j - \tilde{g}(X_i - \tilde{h} X_i, X_j) \xi = \sum_{k=1}^n f_{ij}^k Y_k - \delta_{ij} \xi$$

and, analogously, $\tilde{\nabla}_{Y_i} Y_j = \sum_{k=1}^n g_{ij}^k Y_k - \tilde{\lambda} \delta_{ij} \xi$. Notice that $\tilde{g}(\tilde{\nabla}_{X_i} X_j, \xi) = -\tilde{g}(X_j, \tilde{\nabla}_{X_i} \xi) = -\tilde{g}(X_j, -\tilde{\varphi} X_i + \tilde{\varphi} \tilde{h} X_i) = -\tilde{\lambda} \delta_{ij}$ and $\tilde{g}(\tilde{\nabla}_{X_i} X_j, \xi) = \delta_{ij}$. After a straightforward computation one can get

$$\tilde{\nabla}_{X_i + \alpha Y_i} (X_j + \alpha Y_j) = \sum_{k=1}^n ((f_{ij}^k + \alpha g_{ij}^k)(X_k + \alpha Y_k)) - \tilde{\lambda}(1 - \alpha^2) \delta_{ij} \xi.$$

Then

$$\begin{aligned} [X_i + \alpha Y_i, X_j + \alpha Y_j] &= \tilde{\nabla}_{X_i + \alpha Y_i} (X_j + \alpha Y_j) - \tilde{\nabla}_{X_j + \alpha Y_j} (X_i + \alpha Y_i) \\ &= \sum_{k=1}^n (f_{ij}^k - f_{ji}^k + \alpha(g_{ik}^k - g_{ji}^k))(X_k + \alpha Y_k), \end{aligned}$$

consequently $\mathcal{D}_h(\lambda)$ is involutive. In a similar way one proves the integrability of $\mathcal{D}_h(-\lambda)$. Moreover, for any $X \in \Gamma(\mathcal{D}_h(\pm\lambda))$, $\eta(X) = \pm \frac{1}{\lambda} \eta(hX) = 0$. Being n -dimensional integrable subbundles of the contact distribution, $\mathcal{D}_h(\lambda)$ and $\mathcal{D}_h(-\lambda)$ are Legendre foliations. In order to prove that $(M, \varphi, \xi, \eta, g)$ is a contact metric (κ, μ) -space, we show that the bi-Legendrian structure $(\mathcal{D}_h(\lambda), \mathcal{D}_h(-\lambda))$ satisfies the hypotheses of Theorem 2.5. First, due to the property $\phi h = -h\phi$, we have that $\mathcal{D}_h(\lambda)$ and $\mathcal{D}_h(-\lambda)$ are *conjugate* Legendre foliations, that is $\phi \mathcal{D}_h(\pm\lambda) = \mathcal{D}_h(\mp\lambda)$. Let us consider the bi-Legendrian connection ∇^{bl} associated to $(\mathcal{D}_h(\lambda), \mathcal{D}_h(-\lambda))$. By definition of bi-Legendrian connection, ∇^{bl} satisfies (i) and (iii) of Theorem 2.5. Moreover, $\nabla^{bl} \eta = \nabla^{bl} d\eta = 0$ and, since ∇^{bl} preserves $\mathcal{D}_h(\pm\lambda)$, also $\nabla^{bl} h = 0$. It remains to prove that ∇^{bl} preserves the tensor field ϕ and is a metric connection. Notice that, by virtue of [9, Proposition 2.9], in order to prove that $\nabla^{bl} \phi = 0$ and $\nabla^{bl} g = 0$, it is enough to show that $\mathcal{D}_h(\pm\lambda)$ are totally geodesic foliations (with respect to g). Let ∇ be the Levi-Civita connection of g and $X, X' \in \Gamma(\mathcal{D}_h(\lambda)), Y \in \Gamma(\mathcal{D}_h(-\lambda))$. As $g = \mp \frac{1}{\lambda} \tilde{g}(\cdot, \tilde{\varphi} h \cdot) + \eta \otimes \eta$, after a very long computation one can get

$$\begin{aligned} (5.12) \quad g(\nabla_X X', Y) &= -\frac{2\lambda}{2 - \tilde{\mu}} \tilde{g}(\tilde{\nabla}_X X', Y) - \frac{3\tilde{\lambda}}{2 - \tilde{\mu}} d^2 \eta(X, X', Y) \\ &\quad - \frac{1}{2\lambda} (X(d\eta(X', \tilde{h}Y)) + X'(d\eta(X, \tilde{h}Y)) + d\eta([X, X'], \tilde{h}Y)) \end{aligned}$$

Since \tilde{h} maps $\mathcal{D}_h(\pm\lambda)$ onto $\mathcal{D}_h(\mp\lambda)$ and due to Theorem 5.1, (5.12) implies that $\tilde{g}(\nabla_X X', Y) = 0$. Moreover, $g(\nabla_X X', \xi) = -g(X', \nabla_X \xi) = -g(X', -\phi X - \phi h X) = (1 + \lambda)g(X', \phi X) = 0$. Thus $\mathcal{D}_h(\lambda)$ is totally geodesic and in a similar way one can prove that also $\mathcal{D}_h(-\lambda)$ is totally geodesic. It follows that the bi-Legendrian connection ∇^{bl} preserves g and ϕ . Therefore all the assumptions of Theorem 2.5 are satisfied and we conclude that (ϕ, ξ, η, g) is a contact metric (κ, μ) -structure. It remains to find explicitly κ and μ . We have immediately that $\kappa = 1 - \lambda^2 = \tilde{\kappa} + 2 - \left(1 - \frac{\tilde{\mu}}{2}\right)^2$. In order to find μ , notice that by [5, (3.19)], one has

$$(5.13) \quad (\nabla_\xi h)X = \mu h \phi X = \mu \left(\left(\frac{\tilde{\mu}}{2} - 1 \right) \tilde{\varphi} X + \tilde{\varphi} \tilde{h} X \right),$$

for all $X \in \Gamma(TM)$. On the other hand, by using (2.3) and the relation $h^2 = (\kappa - 1)\phi$ (cf. [5]), we have

$$\begin{aligned}
(\nabla_\xi h)X &= \nabla_{hX}\xi + [\xi, hX] - h\nabla_X\xi - h[\xi, X] \\
(5.14) \qquad &= -2\phi hX - 2\phi h^2X + (\mathcal{L}_\xi h)X \\
&= -2\phi hX - 2(1 - \kappa)\phi X + (\mathcal{L}_\xi h)X.
\end{aligned}$$

But, due to (5.11),

$$\begin{aligned}
(\mathcal{L}_\xi h)X &= \pm \left(\frac{2 - \tilde{\mu}}{2\sqrt{-1 - \tilde{\kappa}}} (\mathcal{L}_\xi \tilde{\varphi})\tilde{h}X + \frac{2 - \tilde{\mu}}{2\sqrt{-1 - \tilde{\kappa}}} \tilde{\varphi}(\mathcal{L}_\xi \tilde{h})X + \tilde{\lambda}(\mathcal{L}_\xi \tilde{\varphi})X \right) \\
(5.15) \qquad &= \pm \left(\frac{2 - \tilde{\mu}}{\sqrt{-1 - \tilde{\kappa}}} \tilde{h}^2 + \frac{(2 - \tilde{\mu})^2}{2\sqrt{-1 - \tilde{\kappa}}} \tilde{\varphi}^2 \tilde{h}X + \sqrt{-1 - \tilde{\kappa}}(2 - \tilde{\mu})\tilde{\varphi}^2 X + 2\tilde{\lambda}\tilde{h}X \right) \\
&= \pm \left(\frac{(2 - \tilde{\mu})^2}{2\sqrt{-1 - \tilde{\kappa}}} + 2\sqrt{-1 - \tilde{\kappa}} \right) \tilde{h}X.
\end{aligned}$$

Thus by (5.15) and (5.14) we get

$$(5.16) \qquad (\nabla_\xi h)X = -2\phi hX - 2(1 - \kappa)\phi X \pm \left(\frac{(2 - \tilde{\mu})^2}{\sqrt{-1 - \tilde{\kappa}}} + 2\sqrt{-1 - \tilde{\kappa}} \right) \tilde{h}X,$$

where the sign \pm depends on the positive or negative definiteness of the paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$. Thus comparing (5.13) with (5.16) we obtain $\mu = 2$ both in the positive and in the negative definite case. \square

Remark 5.7. Notice that, since $\tilde{\kappa} < -1$, in this case we can not perform a construction analogous to that one described in Remark 4.9.

We now pass to study the curvature properties of a paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold with $\tilde{\kappa} < -1$. We start observing that (4.29) holds also in the case $\tilde{\kappa} < -1$ since its proof only needs the expression of the covariant derivative of \tilde{h} , which is the same in the cases $\tilde{\kappa} < -1$ and $\tilde{\kappa} > -1$ (cf. (4.6) and (5.9)). Moreover, by combining (3.29) with (5.9) we get

$$\begin{aligned}
\tilde{R}_{XY}\tilde{\varphi}Z - \tilde{\varphi}\tilde{R}_{XY}Z &= (\eta(Y)\tilde{g}((1 + \tilde{\kappa})\tilde{\varphi}X + (\tilde{\mu} - 1)\tilde{\varphi}\tilde{h}X, Z) - \eta(X)\tilde{g}((1 + \tilde{\kappa})\tilde{\varphi}Y \\
&\quad + (\tilde{\mu} - 1)\tilde{\varphi}\tilde{h}Y, Z))\xi - \eta(Y)\eta(Z)((1 + \tilde{\kappa})\tilde{\varphi}X + (\tilde{\mu} - 1)\tilde{\varphi}\tilde{h}X) \\
(5.17) \qquad &\quad + \eta(X)\eta(Z)((1 + \tilde{\kappa})\tilde{\varphi}Y + (\tilde{\mu} - 1)\tilde{\varphi}\tilde{h}Y) \\
&\quad + \tilde{g}(Y - \tilde{h}Y, Z)\tilde{\varphi}(X - \tilde{h}X) - \tilde{g}(X - \tilde{h}X, Z)\tilde{\varphi}(Y - \tilde{h}Y) \\
&\quad - \tilde{g}(\tilde{\varphi}(X - \tilde{h}X), Z)(Y - \tilde{h}Y) + \tilde{g}(\tilde{\varphi}(Y - \tilde{h}Y), Z)(X - \tilde{X}).
\end{aligned}$$

From (4.29) and (5.17), using (3.2) and the properties of the operator \tilde{h} , it follows that

$$\begin{aligned}
(5.18) \quad \tilde{R}_{XY}\tilde{\varphi}\tilde{h}Z - \tilde{\varphi}\tilde{h}\tilde{R}_{XY}Z &= \tilde{R}_{XY}\tilde{\varphi}\tilde{h}Z - \tilde{\varphi}\tilde{R}_{XY}\tilde{h}Z + \tilde{\varphi}(\tilde{R}_{XY}\tilde{h}Z - \tilde{h}\tilde{R}_{XY}Z) \\
&= (\eta(Y)\tilde{g}((1+2\tilde{\kappa})\tilde{\varphi}X + (\tilde{\mu}-1)\tilde{\varphi}\tilde{h}X, \tilde{h}Z) \\
&\quad - \eta(X)\tilde{g}((1+2\tilde{\kappa})\tilde{\varphi}Y + (\tilde{\mu}-1)\tilde{\varphi}\tilde{h}Y, \tilde{h}Z))\xi \\
&\quad + (\tilde{g}(Y - \tilde{h}Y, \tilde{h}Z) - \tilde{\mu}(1+\tilde{\kappa})\eta(Y)\eta(Z))\tilde{\varphi}X \\
&\quad - (\tilde{g}(X - \tilde{h}X, \tilde{h}Z) - \tilde{\mu}(1+\tilde{\kappa})\eta(X)\eta(Z))\tilde{\varphi}Y \\
&\quad - (\tilde{g}(Y - \tilde{h}Y, \tilde{h}Z) + \tilde{\kappa}\eta(Y)\eta(Z))\tilde{\varphi}\tilde{h}X + (\tilde{g}(X - \tilde{h}X, \tilde{h}Z) + \tilde{\kappa}\eta(X)\eta(Z))\tilde{\varphi}\tilde{h}Y \\
&\quad - (1+\tilde{\kappa})\tilde{g}(Y, \tilde{\varphi}Z + \tilde{\varphi}\tilde{h}Z)X + (1+\tilde{\kappa})\tilde{g}(X, \tilde{\varphi}Z + \tilde{\varphi}\tilde{h}Z)Y \\
&\quad + \tilde{g}(Y, \tilde{\varphi}Z + \tilde{\varphi}\tilde{h}Z)\tilde{h}X - \tilde{g}(X, \tilde{\varphi}Z + \tilde{\varphi}\tilde{h}Z)\tilde{h}Y - 2\tilde{\mu}\tilde{g}(X, \tilde{\varphi}Y)\tilde{h}Z.
\end{aligned}$$

We can prove now that also in the case $\tilde{\kappa} < -1$ the $(\tilde{\kappa}, \tilde{\mu})$ -nullity condition determines the curvature tensor field completely.

Theorem 5.8. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} < -1$. Then the curvature tensor field of M satisfies the following relations:*

$$(5.19) \quad \tilde{R}_{XX'}X'' = (\tilde{\kappa} - 1 + \tilde{\mu})(\tilde{g}(X', X'')X - \tilde{g}(X, X'')X') + \tilde{\lambda}(\tilde{g}(X', X'')\tilde{\varphi}X - \tilde{g}(X, X'')\tilde{\varphi}X')$$

$$(5.20) \quad \tilde{R}_{XX'}Y = -\tilde{\lambda}(\tilde{g}(X', \tilde{\varphi}Y)X - \tilde{g}(X, \tilde{\varphi}Y)X') - (1 - \tilde{\mu})(\tilde{g}(X', \tilde{\varphi}Y)\tilde{\varphi}X - \tilde{g}(X, \tilde{\varphi}Y)\tilde{\varphi}X')$$

$$(5.21) \quad \tilde{R}_{XY}X' = -\tilde{\lambda}\tilde{g}(X', \tilde{\varphi}Y)X - \tilde{g}(X', \tilde{\varphi}Y)\tilde{\varphi}X - \tilde{\lambda}^2\tilde{g}(X, X')Y + \tilde{\lambda}\tilde{g}(X, X')\tilde{\varphi}Y - \tilde{\mu}\tilde{g}(X, \tilde{\varphi}Y)\tilde{\varphi}X'$$

$$(5.22) \quad \tilde{R}_{XY}Y' = \tilde{\lambda}^2\tilde{g}(Y, Y')X + \tilde{\lambda}\tilde{g}(Y, Y')\tilde{\varphi}X + \tilde{\lambda}\tilde{g}(X, \tilde{\varphi}Y')Y - \tilde{g}(X, \tilde{\varphi}Y')\tilde{\varphi}Y - \tilde{\mu}\tilde{g}(X, \tilde{\varphi}Y)\tilde{\varphi}Y'$$

$$(5.23) \quad \tilde{R}_{YX'}X = -\tilde{\lambda}(\tilde{g}(X, \tilde{\varphi}Y')Y - \tilde{g}(X, \tilde{\varphi}Y)Y') + (1 - \tilde{\mu})(\tilde{g}(X, \tilde{\varphi}Y')\tilde{\varphi}Y - \tilde{g}(X, \tilde{\varphi}Y)\tilde{\varphi}Y')$$

$$(5.24) \quad \tilde{R}_{YX'}Y'' = (\tilde{\kappa} - 1 + \tilde{\mu})(\tilde{g}(Y', Y'')Y - \tilde{g}(Y, Y'')Y') - \tilde{\lambda}(\tilde{g}(Y', Y'')\tilde{\varphi}Y - \tilde{g}(Y, Y'')\tilde{\varphi}Y')$$

for any $X, X', X'' \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}))$ and $Y, Y', Y'' \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda}))$.

Proof. We prove (5.19) and (5.20), the remaining relations being analogous. First notice that, since $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})$ and $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda})$ are not totally geodesic, $\tilde{R}_{XX'}Y$ has components along both $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})$ and $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda})$. Moreover it has no components along $\mathbb{R}\xi$ because

$$\begin{aligned}
\tilde{g}(\tilde{R}_{XX'}Y, \xi) &= -\tilde{R}(\xi, Y, X, X') = -\tilde{R}(Y, \xi, X, X') = -\tilde{g}(\tilde{R}_{XX'}\xi, Y) \\
&= -\tilde{g}(\tilde{\kappa}(\eta(X')X - \eta(X)X') + \tilde{\mu}(\eta(X')\tilde{h}X - \eta(X)\tilde{h}X'), Y) = 0.
\end{aligned}$$

Due to the fact that $\tilde{h}\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\pm\tilde{\lambda}) = \mathcal{D}_{\tilde{\varphi}\tilde{h}}(\mp\tilde{\lambda})$, (5.18) implies

$$\begin{aligned}
\tilde{\lambda}\tilde{R}_{XX'}Y + \tilde{\varphi}\tilde{h}\tilde{R}_{XX'}Y &= -\tilde{g}(X', \tilde{h}Y)\tilde{\varphi}X + \tilde{g}(X, \tilde{h}Y)\tilde{\varphi}X' + \tilde{g}(X', \tilde{\lambda}\tilde{h}Y + (1+\tilde{\kappa})\tilde{\varphi}Y)X \\
&\quad - \tilde{g}(X, \tilde{\lambda}\tilde{h}Y + (1+\tilde{\kappa})\tilde{\varphi}Y)X' - \tilde{g}(X', \tilde{\varphi}Y)\tilde{h}X + \tilde{g}(X, \tilde{\varphi}Y)\tilde{h}X'.
\end{aligned}$$

Taking the inner product with any $U \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}))$ and using the symmetry of the operator $\tilde{\varphi}\tilde{h}$, we have

$$2\tilde{\lambda}\tilde{g}(\tilde{R}_{XX'}Y, U) = \tilde{g}(X', \tilde{\lambda}\tilde{h}Y + (1+\tilde{\kappa})\tilde{\varphi}Y)\tilde{g}(X, U) - \tilde{g}(X, \tilde{\lambda}\tilde{h}Y + (1+\tilde{\kappa})\tilde{\varphi}Y)\tilde{g}(X', U).$$

Now notice that $\tilde{h}Y = -\frac{1}{\tilde{\lambda}}\tilde{h}\tilde{\varphi}\tilde{h}Y = \frac{1}{\tilde{\lambda}}\tilde{\varphi}\tilde{h}^2Y = -\tilde{\lambda}\tilde{\varphi}Y$. Hence the previous equations yields

$$(5.25) \quad \tilde{g}(\tilde{R}_{XX'}Y, U) = \tilde{\lambda}\tilde{g}(X, \tilde{\varphi}Y)\tilde{g}(X', U) - \tilde{\lambda}\tilde{g}(X', \tilde{\varphi}Y)\tilde{g}(X, U).$$

Arguing in a similar way one can prove that

$$(5.26) \quad \tilde{g}(\tilde{R}_{XY}X', V) = \tilde{g}(X', \tilde{\varphi}Y)\tilde{g}(X, \tilde{\varphi}V) + (1+\tilde{\kappa})\tilde{g}(X, X')\tilde{g}(Y, V) + \tilde{\mu}\tilde{g}(X, \tilde{\varphi}Y)\tilde{g}(X', \tilde{\varphi}V)$$

for all $V \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda}))$. Now let us consider a $\tilde{\varphi}$ -basis $\{e_i, \tilde{\varphi}e_i, \xi\}$, $i \in \{1, \dots, n\}$, as in Lemma 3.11. Note that by (5.26) we have

$$\begin{aligned}
\tilde{g}(\tilde{R}_{XX'}Y, \tilde{\varphi}e_i) &= -\tilde{g}(\tilde{R}_{X'Y}X, \tilde{\varphi}e_i) + \tilde{g}(\tilde{R}_{XY}X', \tilde{\varphi}e_i) \\
&= -\tilde{g}(Y, \tilde{\varphi}X)\tilde{g}(\tilde{\varphi}X', \tilde{\varphi}e_i) - (1 + \tilde{\kappa})\tilde{g}(X, X')\tilde{g}(Y, \tilde{\varphi}e_i) + \tilde{\mu}\tilde{g}(X', \tilde{\varphi}Y)\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}e_i) \\
&\quad + \tilde{g}(Y, \tilde{\varphi}X')\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}e_i) + (1 + \tilde{\kappa})\tilde{g}(X, X')\tilde{g}(Y, \tilde{\varphi}e_i) - \tilde{\mu}\tilde{g}(X, \tilde{\varphi}Y)\tilde{g}(\tilde{\varphi}X', \tilde{\varphi}e_i) \\
(5.27) \quad &= (\tilde{\mu} - 1)\tilde{g}(X, \tilde{\varphi}Y)\tilde{g}(X', e_i) - (\tilde{\mu} - 1)\tilde{g}(X', \tilde{\varphi}Y)\tilde{g}(X, e_i).
\end{aligned}$$

Then, by using (5.25) and (5.27) we get

$$\begin{aligned}
\tilde{R}_{XX'}Y &= \sum_{i=1}^r \tilde{g}(\tilde{R}_{XX'}Y, e_i)e_i - \sum_{i=r+1}^n \tilde{g}(\tilde{R}_{XX'}Y, e_i)e_i - \sum_{i=1}^r \tilde{g}(\tilde{R}_{XX'}Y, \tilde{\varphi}e_i)\tilde{\varphi}e_i + \sum_{i=r+1}^n \tilde{g}(\tilde{R}_{XX'}Y, \tilde{\varphi}e_i)\tilde{\varphi}e_i \\
&= -\tilde{\lambda}\tilde{g}(X', \tilde{\varphi}Y)X + \tilde{\lambda}\tilde{g}(X, \tilde{\varphi}Y)X' - (1 - \tilde{\mu})\tilde{g}(X', \tilde{\varphi}Y)\tilde{\varphi}X + (1 - \tilde{\mu})\tilde{g}(X, \tilde{\varphi}Y)\tilde{\varphi}X'.
\end{aligned}$$

In order to prove (5.19), we use (5.17) and we get, after a long computation,

$$\begin{aligned}
\tilde{R}_{XX'}X'' &= \tilde{g}(\tilde{R}_{XX'}X'', \xi)\xi + \tilde{\varphi}\tilde{R}_{XX'}\tilde{\varphi}X'' - \tilde{\varphi}(\tilde{g}(X' - \tilde{h}X', X'')\tilde{\varphi}(X - \tilde{h}X) \\
&\quad - \tilde{g}(X - \tilde{h}X, X'')\tilde{\varphi}(X' - \tilde{h}X') - \tilde{g}(\tilde{\varphi}(X - \tilde{h}X), X'')(X' - \tilde{h}X') \\
&\quad + \tilde{g}(\tilde{\varphi}(X' - \tilde{h}X'), X'')(X - \tilde{h}X)) \\
&= (\tilde{\kappa} - 1 + \tilde{\mu})\tilde{g}(X', X'')X + \tilde{g}(X', X'')\tilde{h}X - (\tilde{\kappa} - 1 + \tilde{\mu})\tilde{g}(X, X'')X' - \tilde{g}(X, X'')\tilde{h}X'.
\end{aligned}$$

□

Corollary 5.9. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} < -1$. Then its Riemannian curvature tensor \tilde{R} is given by following formula*

$$\begin{aligned}
\tilde{g}(\tilde{R}_{XY}Z, W) &= \left(-1 + \frac{\tilde{\mu}}{2}\right) (\tilde{g}(Y, Z)\tilde{g}(X, W) - \tilde{g}(X, Z)\tilde{g}(Y, W)) \\
&\quad + \tilde{g}(Y, Z)\tilde{g}(\tilde{h}X, W) - \tilde{g}(X, Z)\tilde{g}(\tilde{h}Y, W) \\
&\quad - \tilde{g}(Y, W)\tilde{g}(\tilde{h}X, Z) + \tilde{g}(X, W)\tilde{g}(\tilde{h}Y, Z) \\
&\quad + \frac{-1 + \frac{\tilde{\mu}}{2}}{\tilde{\kappa} + 1} \left(\tilde{g}(\tilde{h}Y, Z)\tilde{g}(\tilde{h}X, W) - \tilde{g}(\tilde{h}X, Z)\tilde{g}(\tilde{h}Y, W)\right) \\
&\quad - \frac{\tilde{\mu}}{2} (\tilde{g}(\tilde{\varphi}Y, Z)\tilde{g}(\tilde{\varphi}X, W) - \tilde{g}(\tilde{\varphi}X, Z)\tilde{g}(\tilde{\varphi}Y, W)) \\
(5.28) \quad &\quad + \frac{-\tilde{\kappa} - \frac{\tilde{\mu}}{2}}{\tilde{\kappa} + 1} \left(\tilde{g}(\tilde{\varphi}\tilde{h}Y, Z)\tilde{g}(\tilde{\varphi}\tilde{h}X, W) - \tilde{g}(\tilde{\varphi}\tilde{h}Y, W)\tilde{g}(\tilde{\varphi}\tilde{h}X, Z)\right) \\
&\quad + \tilde{\mu}\tilde{g}(\tilde{\varphi}X, Y)\tilde{g}(\tilde{\varphi}Z, W) \\
&\quad + \eta(X)\eta(W) \left((\tilde{\kappa} + 1 - \frac{\tilde{\mu}}{2})\tilde{g}(Y, Z) + (\tilde{\mu} - 1)\tilde{g}(\tilde{h}Y, Z)\right) \\
&\quad - \eta(X)\eta(Z) \left((\tilde{\kappa} + 1 - \frac{\tilde{\mu}}{2})\tilde{g}(Y, W) + (\tilde{\mu} - 1)\tilde{g}(\tilde{h}Y, W)\right) \\
&\quad + \eta(Y)\eta(Z) \left((\tilde{\kappa} + 1 - \frac{\tilde{\mu}}{2})\tilde{g}(X, W) + (\tilde{\mu} - 1)\tilde{g}(\tilde{h}X, W)\right) \\
&\quad - \eta(Y)\eta(W) \left((\tilde{\kappa} + 1 - \frac{\tilde{\mu}}{2})\tilde{g}(X, Z) + (\tilde{\mu} - 1)\tilde{g}(\tilde{h}X, Z)\right)
\end{aligned}$$

for all vector fields X, Y, Z, W on M .

Proof. We can decompose an arbitrary vector field X on M uniquely as $X = X_{\tilde{\lambda}} + X_{-\tilde{\lambda}} + \eta(X)\xi$, where $X_{\tilde{\lambda}} \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}))$ and $X_{-\tilde{\lambda}} \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda}))$. We then write $\tilde{R}_{XY}Z$ as a sum of terms of the form $\tilde{R}_{X_{\pm\tilde{\lambda}}Y_{\pm\tilde{\lambda}}}Z_{\pm\tilde{\lambda}}$, $\tilde{R}_{XY}\xi$, $\tilde{R}_{X\xi}Z$. Then by Theorem 5.8, and taking into account that, in fact

$$X_{\tilde{\lambda}} = \frac{1}{2}(X - \eta(X)\xi + \frac{1}{\sqrt{-1 - \tilde{\kappa}}}\tilde{\varphi}\tilde{h}X), \quad X_{-\tilde{\lambda}} = \frac{1}{2}(X - \eta(X)\xi - \frac{1}{\sqrt{-1 - \tilde{\kappa}}}\tilde{\varphi}\tilde{h}X),$$

we obtain (5.28). \square

Remark 5.10. We point out that the surprising fact that formula (5.28) is the same as (4.43), though the cases $\tilde{\kappa} < -1$ and $\tilde{\kappa} > -1$ are geometrically very different from each other.

Corollary 5.11. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} < -1$. Then for any $X \in \Gamma(\mathcal{D})$ the ξ -sectional curvature $\tilde{K}(X, \xi)$ is constant and is given by $\tilde{K}(X, \xi) = \tilde{\kappa}$. Moreover, the sectional curvature of plane sections normal to ξ is given by*

$$\begin{aligned} \tilde{K}(X, X') &= \tilde{K}(Y, Y') = \tilde{\kappa} - 1 + \tilde{\mu} \\ \tilde{K}(X, Y) &= \tilde{\lambda}^2 - (\tilde{\mu} + 1) \frac{\tilde{g}(X, \tilde{\varphi}Y)^2}{\tilde{g}(X, X)\tilde{g}(Y, Y)} \end{aligned}$$

for any $X, X' \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda}))$ and $Y, Y' \in \Gamma(\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda}))$.

Using Theorem 5.8, (3.3) and (3.37), one can easily prove the following result.

Corollary 5.12. *In any $(2n+1)$ -dimensional paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ such that $\tilde{\kappa} < -1$, the Ricci operator \tilde{Q} is given by*

$$\tilde{Q} = (2(1 - n) + n\tilde{\mu})I + (2(n + 1) + \tilde{\mu})\tilde{h} + (2(n - 1) + n(2\tilde{\kappa} - \tilde{\mu}))\eta \otimes \xi.$$

In particular (M, \tilde{g}) is η -Einstein if and only if $\tilde{\mu} = 2(1 - n)$, Einstein if and only if $\tilde{\kappa} = \frac{1-n^2}{n}$ and $\tilde{\mu} = 2(1 - n)$.

Remark 5.13. We point out that, according to Corollary 5.12, if $\tilde{\kappa} < -1$, there exist Einstein paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifolds also in dimension greater than 3. This is a relevant difference with respect to the case $\tilde{\kappa} > -1$ (cf. Corollary 4.14) and, moreover, with respect to the contact metric case.

We conclude with an example of paracontact $(\tilde{\kappa}, \tilde{\mu})$ -manifold such that $\tilde{\kappa} < -1$.

Example 5.14. *Let \mathfrak{g} be the Lie algebra with basis $\{e_1, e_2, e_3, e_4, e_5\}$ and non-zero Lie brackets*

$$\begin{aligned} [e_1, e_5] &= \alpha\beta e_1 + \alpha\beta e_2, & [e_2, e_5] &= \alpha\beta e_1 + \alpha\beta e_2, \\ [e_3, e_5] &= -\alpha\beta e_3 + \alpha\beta e_4, & [e_4, e_5] &= \alpha\beta e_3 - \alpha\beta e_4, \\ [e_1, e_2] &= \alpha e_1 + \alpha e_2, & [e_1, e_3] &= \beta e_2 + \alpha e_4 - 2e_5, & [e_1, e_4] &= \beta e_2 + \alpha e_3, \\ [e_2, e_3] &= \beta e_1 - \alpha e_4, & [e_2, e_4] &= \beta e_1 - \alpha e_3 + 2e_5, & [e_3, e_4] &= -\beta e_3 + \beta e_4 \end{aligned}$$

where α, β are non-zero real numbers such that $\alpha\beta > 0$. Let G be a Lie group whose Lie algebra is \mathfrak{g} . Define on G a left invariant paracontact metric structure $(\tilde{\varphi}, \xi, \eta, \tilde{g})$ by imposing that, at the identity, $\tilde{g}(e_1, e_1) = \tilde{g}(e_4, e_4) = -\tilde{g}(e_2, e_2) = -\tilde{g}(e_3, e_3) = \tilde{g}(e_5, e_5) = 1$, $\tilde{g}(e_i, e_j) = 0$ for any $i \neq j$, and $\tilde{\varphi}e_1 = e_3$, $\tilde{\varphi}e_2 = e_4$, $\tilde{\varphi}e_3 = e_1$, $\tilde{\varphi}e_4 = e_2$, $\tilde{\varphi}e_5 = 0$, $\xi = e_5$ and $\eta = g(\cdot, e_5)$. A very long but straightforward

computation shows that

$$\begin{aligned}\tilde{\nabla}_{e_1}\xi &= \alpha\beta e_1 - \tilde{\varphi}e_1, & \tilde{\nabla}_{e_2}\xi &= \alpha\beta e_2 - \tilde{\varphi}e_2, & \tilde{\nabla}_{\tilde{\varphi}e_1}\xi &= -e_1 - \alpha\beta\tilde{\varphi}e_1, & \tilde{\nabla}_{\tilde{\varphi}e_2}\xi &= -e_2 - \alpha\beta\tilde{\varphi}e_2, \\ \tilde{\nabla}_{\xi}e_1 &= -\alpha\beta e_2 - \tilde{\varphi}e_1, & \tilde{\nabla}_{\xi}e_2 &= -\alpha\beta e_1 - \tilde{\varphi}e_2, & \tilde{\nabla}_{\xi}\tilde{\varphi}e_1 &= -e_1 - \alpha\beta\tilde{\varphi}e_2, & \tilde{\nabla}_{\xi}\tilde{\varphi}e_2 &= -e_2 - \alpha\beta\tilde{\varphi}e_1, \\ \tilde{\nabla}_{e_1}e_1 &= \alpha e_2 - \alpha\beta e_5, & \tilde{\nabla}_{e_1}e_2 &= \alpha e_1, & \tilde{\nabla}_{e_1}\tilde{\varphi}e_1 &= \alpha\tilde{\varphi}e_2 - e_5, & \tilde{\nabla}_{e_1}\tilde{\varphi}e_2 &= \alpha\tilde{\varphi}e_1, \\ \tilde{\nabla}_{e_2}e_1 &= -\alpha e_2, & \tilde{\nabla}_{e_2}e_2 &= -\alpha e_1 + \alpha\beta e_5, & \tilde{\nabla}_{e_2}\tilde{\varphi}e_1 &= -\alpha\tilde{\varphi}e_2, & \tilde{\nabla}_{e_2}\tilde{\varphi}e_2 &= -\alpha\tilde{\varphi}e_1 + e_5, \\ \tilde{\nabla}_{\tilde{\varphi}e_1}e_1 &= -\beta e_2 + e_5, & \tilde{\nabla}_{\tilde{\varphi}e_1}e_2 &= -\beta e_1, & \tilde{\nabla}_{\tilde{\varphi}e_1}\tilde{\varphi}e_1 &= -\beta\tilde{\varphi}e_2 - \alpha\beta e_5, & \tilde{\nabla}_{\tilde{\varphi}e_1}\tilde{\varphi}e_2 &= -\beta\tilde{\varphi}e_1, \\ \tilde{\nabla}_{\tilde{\varphi}e_2}e_1 &= -\beta e_2, & \tilde{\nabla}_{\tilde{\varphi}e_2}e_2 &= -\beta e_1 - e_5, & \tilde{\nabla}_{\tilde{\varphi}e_2}\tilde{\varphi}e_1 &= -\beta\tilde{\varphi}e_2, & \tilde{\nabla}_{\tilde{\varphi}e_2}\tilde{\varphi}e_2 &= -\beta\tilde{\varphi}e_1 + \alpha\beta e_5.\end{aligned}$$

where $\tilde{\lambda} = \alpha\beta$ and $\tilde{\mu} = 2$. Then one can prove that the curvature tensor field of the Levi-Civita connection of (G, \tilde{g}) satisfies the $(\tilde{\kappa}, \tilde{\mu})$ -nullity condition (3.1), with $\tilde{\kappa} = -1 - (\alpha\beta)^2$ and $\tilde{\mu} = 2$.

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