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On harmonicity in some Moufang-Klingenberg planes

Basri Çelik, Atilla Akpınar and Süleyman Çiftçi

Abstract

In this paper we study Moufang-Klingenberg planes $\mathbf{M}(\mathcal{A})$ defined over a local alternative ring $\mathcal{A}$ of dual numbers. We show that some collineations of $\mathbf{M}(\mathcal{A})$ preserve cross-ratio and thus establish a relation between harmonicity and harmonic position.

Key Words: Moufang-Klingenberg planes, local alternative ring, projective collineation, cross-ratio, harmonicity, harmonic position.

1. Introduction

As stated in [10, p. 55], a one-dimensional projectivity has two different analogues in two dimensions: one relating points to points and lines to lines, the other relating points to lines and lines to points. The former kind is a collineation, the latter a correlation. Although the general theory is due to von Staudt, and the names *collineation* and *correlation* to Möbius (1827), some special collineations were used much earlier, e.g. by Newton and La Hire. Moreover, the classical transformations of the Euclidean plane, viz. translations, rotations, reflexions, and dilations, all provide instances of collineations.

In the Euclidean plane, Desargues established the fundamental fact that cross-ratio (a concept originally introduced by Pappus of Alexandria c. 300 A.D) is invariant under projection [3, p. 133]. For this reason, cross-ratio is one of the most important concepts of projective geometry.

In this paper we are interested in the class (which we will denote by $\mathbf{M}(\mathcal{A})$) of Moufang-Klingenberg (MK) planes coordinatized by a local alternative ring $\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$ (where $\mathbf{A}$ is an alternative field, $\varepsilon \notin \mathbf{A}$ and $\varepsilon^2 = 0$) introduced by Blunck in [7]. We will show that some collineations, which were used in obtaining the important result that the collineations group of $\mathbf{M}(\mathcal{A})$ acts transitively on 4-gons [8, Theorem 3], preserve cross-ratio. As we stated in [1], by collineations preserving cross-ratio we will generalize the relation between harmonicity and harmonic position which is given for the points of only the line $[1, 0, 0]$ in [1, Theorem 16], to any line of $\mathbf{M}(\mathcal{A})$. For more information about some well-known properties of cross-ratio in the case of Moufang planes or MK-planes $\mathbf{M}(\mathcal{A})$, respectively, we refer the reader to [11, 4, 9] or [7, 1].

Section 2 includes some basic definitions and results from the literature.

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In Section 3 we will give four collineations of $\mathbf{M}(\mathcal{A})$ from [8] and we show that these collineations preserve cross-ratio.

In Section 4 we will show that the relation between harmonicity (which is an algebraic property) and harmonic position (which is a geometric property), is valid for any line in $\mathbf{M}(\mathcal{A})$, and is the main result of this paper.

2. Preliminaries

Let $\mathbf{M} = (\mathbf{P}, \mathbf{L}, \in, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \in)$ (points, lines, incidence) and an equivalence relation '$\sim$' (neighbour relation) on $\mathbf{P}$ and on $\mathbf{L}$, respectively. Then $\mathbf{M}$ is called a *projective Klingenberg plane* (PK-plane), if it satisfies the following axioms:

(PK1) If P, Q are non-neighbour points, then there is a unique line PQ through P and Q.

(PK2) If g, h are non-neighbour lines, then there is a unique point $g \cap h$ on both g and h.

(PK3) There is a projective plane $\mathbf{M}^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$ and an incidence structure epimorphism $\Psi : \mathbf{M} \to \mathbf{M}^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim Q, \ \Psi(g) = \Psi(h) \iff g \sim h$$

hold for all $P, Q \in \mathbf{P}$, $g, h \in \mathbf{L}$.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a *collineation* of $\mathbf{M}$.

A *Moufang-Klingenberg plane* (MK-plane) is a PK-plane $\mathbf{M}$ that generalizes a Moufang plane, and for which $\mathbf{M}^*$ is a Moufang plane (for the exact definition, see [2]).

An *alternative ring (field)* $\mathbf{R}$ is a not necessarily associative ring (field) that satisfies the alternative laws $a\,(ab) = a^2 b, (ba)\,a = ba^2, \ \forall a, b \in \mathbf{R}$. An alternative ring $\mathbf{R}$ with identity element 1 is called *local* if the set $\mathbf{I}$ of its non-unit elements is an ideal.

We are now ready to give consecutively two important lemmas related to alternative rings, which will be used in the proof of Theorem 4.

Lemma 1 *The subring generated by any two elements of an alternative ring is associative (cf. [13, Theorem 3.1]).*

Lemma 2 *The identities*

$$x\,(y\,(xz)) = (xyx)\,z; \quad ((yx)\,z)\,x = y\,(xzx)\,; \quad (xy)\,(zx) = x\,(yz)\,x,$$

which are known as Moufang identities, are satisfied in every alternative ring (cf. [12, p. 160]).

For any local alternative ring $\mathbf{R}$ one can construct MK-plane $\mathbf{M}\,(\mathbf{R})$, and vice versa. Detailed information about the coordinatization of MK-planes can be found in [2].

Let $\mathbf{A}$ be an alternative field and $\varepsilon \notin \mathbf{A}$. Consider $\mathcal{A} := \mathbf{A}\,(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$ with componentwise addition and multiplication as follows:

$$(a_1 + a_2\varepsilon)\,(b_1 + b_2\varepsilon) = a_1 b_1 + (a_1 b_2 + a_2 b_1)\,\varepsilon, \quad (a_i, b_i \in \mathbf{A}, \ i = 1, 2)\,.$$

Then $\mathcal{A}$ is a local alternative ring with ideal $\mathbf{I} = \mathbf{A}\varepsilon$ of non-units. The set of formal inverses of the non-units of $\mathcal{A}$ is denoted as $\mathbf{I}^{-1}$. Calculations with the elements of $\mathbf{I}^{-1}$ are defined as ([6]):

$$\begin{aligned}(a\varepsilon)^{-1} + t \quad &: \quad = (a\varepsilon)^{-1} := t + (a\varepsilon)^{-1};\ q\,(a\varepsilon)^{-1} := \left(aq^{-1}\varepsilon\right)^{-1} \\ (a\varepsilon)^{-1} q \quad &: \quad = \left(q^{-1}a\varepsilon\right)^{-1};\ \left((a\varepsilon)^{-1}\right)^{-1} := a\varepsilon,\end{aligned}$$

where $(a\varepsilon)^{-1} \in \mathbf{I}^{-1},\ t \in \mathcal{A},\ q \in \mathcal{A} \setminus \mathbf{I}$. (Other terms are not defined.) We will often need these calculations in the proof of Theorem 4. For more information about $\mathcal{A}$ and its relation to MK-planes, the reader is referred to the papers of Blunck [6, 7].

Throughout this paper we restrict ourselves to the MK-plane $\mathbf{M}(\mathcal{A})$ and we assume $char\mathbf{A} \neq \mathbf{2}$.

Blunck [7] gives the following algebraic definition of the cross-ratio for the points on the line $g := [1, 0, 0]$ in $\mathbf{M}(\mathcal{A})$:

$$\begin{aligned}(A, B; C, D) \quad &: \quad = (a, b; c, d) =< \left((a-d)^{-1}(b-d)\right)\left((b-c)^{-1}(a-c)\right) > \\ (Z, B; C, D) \quad &: \quad = \left(z^{-1}, b; c, d\right) =< \left((1-dz)^{-1}(b-d)\right)\left((b-c)^{-1}(1-cz)\right) > \\ (A, Z; C, D) \quad &: \quad = \left(a, z^{-1}; c, d\right) =< \left((a-d)^{-1}(1-dz)\right)\left((1-cz)^{-1}(a-c)\right) > \\ (A, B; Z, D) \quad &: \quad = \left(a, b; z^{-1}, d\right) =< \left((a-d)^{-1}(b-d)\right)\left((1-zb)^{-1}(1-za)\right) > \\ (A, B; C, Z) \quad &: \quad = \left(a, b; c, z^{-1}\right) =< \left((1-za)^{-1}(1-zb)\right)\left((b-c)^{-1}(a-c)\right) >,\end{aligned}$$

where $A = (0, a, 1),\ B = (0, b, 1),\ C = (0, c, 1),\ D = (0, d, 1),\ Z = (0, 1, z)$ are pairwise non-neighbour points of g and $< x >= \left\{y^{-1}xy \mid y \in \mathcal{A}\right\}$.

In [6, Theorem 2], it is shown that the transformations

$$\begin{aligned}t_u(x) \quad &= \quad x + u \text{ where } u \in \mathcal{A};\ r_u(x) = xu \text{ where } u \in \mathcal{A} \setminus \mathbf{I} \\ i(x) \quad &= \quad x^{-1};\ l_u(x) = ux = \left(ir_u^{-1}i\right)(x) \text{ where } u \in \mathcal{A} \setminus \mathbf{I}\end{aligned}$$

are defined on the line g preserve cross-ratios. This fact will assume a very important role in the proof of Theorem 4.

We give the following result from [1, Theorem 8]. This result states a simple way for calculation of the cross-ratio of the points on any line in $\mathbf{M}(\mathcal{A})$.

Theorem 3 *Let* $\{O, U, V, E\}$ *be the basis of* $\mathbf{M}(\mathcal{A})$*, where* $O = (0, 0, 1), U = (1, 0, 0), V = (0, 1, 0), E = (1, 1, 1)$ *(see [2, Section 4]). Then, according to types of lines, the cross-ratio of the points on the line* l *can be calculated as follows:*

If $A,\ B,\ C,\ D$ *and* Z *are the pairwise non-neighbour points*

(a) *of the line* $l = [m,1,k]$, *where* $A = (a, am+k, 1)$, $B = (b, bm+k, 1)$, $C = (c, cm+k, 1)$, $D = (d, dm+k, 1)$ *are not near the line* $UV = [0,0,1]$ *and* $Z = (1, m+zp, z)$ is near to UV,

(b) *of the line* $l = [1,n,p]$, *where* $A = (an+p, a, 1)$, $B = (bn+p, b, 1)$, $C = (cn+p, c, 1)$, $D = (dn+p, d, 1)$ *are not neighbour to* V *and* $Z = (n+zp, 1, z) \sim V$,

(c) *of the line* $l = [q,n,1]$, *where* $A = (1, a, q+an)$, $B = (1, b, q+bn)$, $C = (1, c, q+cn)$, $D = (1, d, q+dn)$ *are not neighbour to* V *and* $Z = (z, 1, zq+n) \sim V$,

then

$$\begin{aligned} (A,B;C,D) &= (a,b;c,d)\,;\ \ (Z,B;C,D) = \left(z^{-1},b;c,d\right); \\ (A,Z;C,D) &= \left(a,z^{-1};c,d\right);\ \ (A,B;Z,D) = \left(a,b;z^{-1},d\right); \\ (A,B;C,Z) &= \left(a,b;c,z^{-1}\right). \end{aligned}$$

In the next section, we deal with some collineations of $\mathbf{M}(\mathcal{A})$.

3. Some collineations preserving cross-ratio

In this section we would like to show that the following collineations [8, Lemma 3] preserve cross-ratios. In the next section we will only use that these collineations preserve harmonicity, a special case of cross-ratio. We start by giving the collineations, where $w, z, q, n \in \mathbf{A}$:

For any $u, v \in \mathcal{A}$, the map $\mathrm{T}_{u,v}$ transforms points and lines as

$$(x,y,1) \to (x+u, y+v, 1)\,,\ \ (1,y,z\varepsilon) \to (1, y+z(v-uy)\varepsilon, z\varepsilon)\,,$$

$$(w\varepsilon, 1, z\varepsilon) \to ((w+zu)\varepsilon, 1, z\varepsilon)\,;$$

$$[m,1,k] \to [m,1,k+v-um]\,,\ \ [1,n\varepsilon,p] \to [1,n\varepsilon,p+u-vn\varepsilon]\,,$$

$$[q\varepsilon, n\varepsilon, 1] \to [q\varepsilon, n\varepsilon, 1]\,.$$

The map I_1 transforms points and lines as

$$(x,y,1) \to \left(x^{-1}, x^{-1}y, 1\right) \quad if \quad x \notin \mathbf{I},\ \ (x,y,1) \to (1,y,x) \quad if \quad x \in \mathbf{I},$$

$$(1,y,z\varepsilon) \to (z\varepsilon, y, 1)\,,\ \ (w\varepsilon, 1, z\varepsilon) \to (z\varepsilon, 1, w\varepsilon)\,;$$

$$[m,1,k] \to [k,1,m]\,,\ \ [1,n\varepsilon,p] \to [p,n\varepsilon,1] \quad if \quad p \in \mathbf{I},$$

$$[1,n\varepsilon,p] \to \left[1, -(np^{-1})\varepsilon, p^{-1}\right] \quad if \quad p \notin \mathbf{I},\ \ [q\varepsilon, n\varepsilon, 1] \to [1, n\varepsilon, q\varepsilon]\,.$$

The map F transforms points and lines as

$$(x,y,1)\to(y,x,1)\,,\ \ (1,y,z\varepsilon)\to(y,1,z\varepsilon)\quad if\quad y\in\mathbf{I},$$
$$(1,y,z\varepsilon)\to\left(1,y^{-1},(y^{-1}z)\varepsilon\right)\quad if\quad y\notin\mathbf{I},\ \ (w\varepsilon,1,z\varepsilon)\to(1,w\varepsilon,z\varepsilon)\,;$$

$$[m,1,k]\to[1,m,k]\quad if\quad m\in\mathbf{I},\ \ [m,1,k]\to\left[m^{-1},1,-km^{-1}\right]\quad if\quad m\notin\mathbf{I},$$
$$[1,n\varepsilon,p]\to[n\varepsilon,1,p]\,,\ \ [q\varepsilon,n\varepsilon,1]\to[n\varepsilon,q\varepsilon,1]\,.$$

For any $u\in\mathcal{A}$, the map G_u transforms points and lines as

$$(x,y,1)\to(x,y-xu,1)\,,(1,y,z\varepsilon)\to(1,y-u,z\varepsilon)\,,(w\varepsilon,1,z\varepsilon)\to(w\varepsilon,1,z\varepsilon)\,;$$

$$[m,1,k]\to[m-u,1,k]\,,\ \ [1,n\varepsilon,p]\to[1,n\varepsilon,p+((pu)n)\varepsilon]\,,$$
$$[q\varepsilon,n\varepsilon,1]\to[(q+un)\varepsilon,n\varepsilon,1]\,.$$

By the following theorem we will show that the collineations $\mathrm{T}_{u,v}$, I_1, F and G_u preserve the cross-ratio. In the proof of this theorem we will use the following facts: the transformations t_u, r_u, i and l_u preserve cross-ratio, Lemma 1, Moufang identities, the inverse of an element $a=a_1+a_2\varepsilon$ of $\mathcal{A}$ is $a^{-1}=a_1^{-1}-a_1^{-1}a_2a_1^{-1}\varepsilon$ and finally the calculations by elements of $\mathbf{I}^{-1}$ given in section 2.

Theorem 4 *The collineations $T_{u,v}$, I_1, F and G_u preserve cross-ratio.*

Proof. Let A,B,C,D and Z be points with the property given in the statement of Theorem 3. Then it is obvious that

$$\begin{aligned}(A,B;C,D)&=(a,b;c,d)\,,\\(Z,B;C,D)&=\left(z^{-1},b;c,d\right),\\(A,Z;C,D)&=\left(a,z^{-1};c,d\right),\\(A,B;Z,D)&=\left(a,b;z^{-1},d\right),\\(A,B;C,Z)&=\left(a,b;c,z^{-1}\right)\end{aligned}\qquad(1)$$

where $z\in\mathbf{I}$. In this case we must find the effect of φ to the points of any line where φ is any one of the collineations $\mathrm{T}_{u,v}$, I_1, F and G_u. Remainder of the proof is conducted in the following four parts.

Part i) Let $\varphi=\mathrm{T}_{u,v}$. If $l=[m,1,k]$ then

$$\begin{aligned}\varphi(X)&=\varphi(x,xm+k,1)=(x+u,xm+k+v,1)\,,\\\varphi(Z)&=\varphi(1,m+zk,z)=(1,m+zk+z\,(v-u\,(m+zk))\,,z)\end{aligned}$$

and $\varphi(l)=[m,1,k+v-um]$. From (a) of Theorem 3, we have

$$\begin{aligned}(\varphi(A)\,,\varphi(B)\,;\varphi(C)\,,\varphi(D))&=(a+u,b+u;c+u,d+u)\overset{\sigma}{=}(a,b;c,d)\,,\\(\varphi(Z)\,,\varphi(B)\,;\varphi(C)\,,\varphi(D))&=\left(z^{-1},b+u;c+u,d+u\right)\overset{\sigma}{=}\left(z^{-1},b;c,d\right),\end{aligned}$$

where $\sigma = t_{-u}$ and $(x_1, x_2, x_3, x_4) \overset{\sigma}{=} (y_1, y_2, y_3, y_4)$ means $\sigma(x_i) = y_i$ for $i = 1, 2, 3, 4$.

If $l = [1, n, p]$ then

$$\begin{aligned}\varphi(X) &= \varphi(xn + p, x, 1) = (xn + p + u, x + v, 1),\\ \varphi(Z) &= \varphi(n + zp, 1, z) = (n + zp + zu, 1, z),\end{aligned}$$

and $\varphi(l) = [1, n, p + u - vn]$. From (b) of Theorem 3 we obtain

$$\begin{aligned}(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (a + v, b + v; c + v, d + v) \overset{\sigma}{=} (a, b; c, d),\\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left(z^{-1}, b + v; c + v, d + v\right) \overset{\sigma}{=} \left(z^{-1}, b; c, d\right),\end{aligned}$$

where $\sigma = t_{-v}$.

If $l = [q, n, 1]$ then

$$\begin{aligned}\varphi(X) &= \varphi(1, x, q + xn) = (1, x + (q + xn)(v - ux), q + xn),\\ \varphi(Z) &= \varphi(z, 1, zq + n) = (z + nu, 1, zq + n)\end{aligned}$$

and $\varphi(l) = [q, n, 1]$. From (c) of Theorem 3 we have

$$\begin{aligned}(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (a_1, b_1; c_1, d_1)\\ &\overset{\sigma}{=} (a + (an)v - q(ua),\\ &\quad b + (bn)v - q(ub);\\ &\quad c + (cn)v - q(uc),\\ &\quad d + (dn)v - q(ud)),\\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left((z + nu)^{-1}, b_1; c_1, d_1\right)\\ &\overset{\sigma}{=} (z^{-1},\\ &\quad b + (bn)v - q(ub);\\ &\quad c + (cn)v - q(uc),\\ &\quad d + (dn)v - q(ud))\end{aligned} \tag{2}$$

where $a_1 = a+(q + an)(v - ua)$, $b_1 = b+(q + bn)(v - ub)$, $c_1 = c+(q + cn)(v - uc)$, $d_1 = d+(q + dn)(v - ud)$ and $\sigma = i \circ t_{-nu} \circ i \circ t_{-qv}$. In the latter case, there are four subcases: $u \in \mathbf{I}$ or $u \notin \mathbf{I}$, while $v \in \mathbf{I}$ and $u \in \mathbf{I}$ or $u \notin \mathbf{I}$ while $v \notin \mathbf{I}$.

Case 1. Let $v \in \mathbf{I}$, $u \in \mathbf{I}$. Then $(xn)v = q(ux) = 0$ since n, $q \in \mathbf{I}$. So, the cross-ratios in (2) are, respectively,

$$\begin{aligned}(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (a, b; c, d)\\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left(z^{-1}, b; c, d\right).\end{aligned}$$

Case 2. Let $v \in \mathbf{I}$, $u \notin \mathbf{I}$. Then $(xn)v = 0$. So, the cross-ratios in (2) are, respectively,

$$(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) \overset{\sigma}{=} (a, b; c, d)$$

$$(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) \overset{\sigma}{=} \left(z^{-1}, b; c, d\right),$$

where $\sigma = l_{(u-uqu)^{-1}} \circ l_u$, (Note that $\left(zu^{-1}\right)(uqu) = 0$ since $z,\ q \in \mathbf{I}$.)

Case 3. Let $v \notin \mathbf{I},\ u \in \mathbf{I}$. Then $q(ux) = 0$. So, the cross-ratios in (2) are, respectively,

$$(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) \overset{\sigma}{=} (a, b; c, d)$$
$$(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) \overset{\sigma}{=} \left(z^{-1}, b; c, d\right),$$

where $\sigma = r_{(v^{-1}+n)^{-1}} \circ r_{v^{-1}}$. (Note that $n(vz) = 0$ since $n,\ z \in \mathbf{I}$.)

Case 4. Let $v \notin \mathbf{I},\ u \notin \mathbf{I}$. Then the cross-ratios in (2) are, respectively,

$$(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) \overset{\sigma}{=} (a - q(ua), b - q(ub); c - q(uc), d - q(ud))$$
$$(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) \overset{\sigma}{=} \left(z^{-1}, b - q(ub); c - q(uc), d - q(ud)\right),$$

where $\sigma = r_{(v^{-1}+n)^{-1}} \circ r_{v^{-1}}$. (Note that $n(vz) = 0$.) Finally, the remaining proof follows from Case 2.

Part ii) Let $\varphi = \mathrm{I}_1$. If $l = [m, 1, k]$, then

$$\begin{aligned} \varphi(X) &= \varphi(x, xm + k, 1) = \left(x^{-1}, x^{-1}(xm + k), 1\right) \text{ where } x \notin \mathbf{I}, \\ \varphi(X) &= \varphi(x, xm + k, 1) = (1, xm + k, x) \text{ where } x \in \mathbf{I}, \\ \varphi(Z) &= \varphi(1, m + zk, z) = (z, m + zk, 1) \end{aligned}$$

and $\varphi(l) = [k, 1, m]$. From (a) of Theorem 3 we have

$$\begin{aligned} (\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= \left(a^{-1}, b^{-1}; c^{-1}, d^{-1}\right) \overset{\sigma}{=} (a, b; c, d), \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left(z, b^{-1}; c^{-1}, d^{-1}\right) \overset{\sigma}{=} \left(z^{-1}, b; c, d\right) \end{aligned}$$

where $\sigma = i$.

If $l = [1, n, p]$, then

$$\begin{aligned} \varphi(X) &= \varphi(xn + p, x, 1) = \left((xn + p)^{-1}, (xn + p)^{-1} x, 1\right) \text{ where } xn + p \notin \mathbf{I}, \\ \varphi(X) &= \varphi(xn + p, x, 1) = (1, x, xn + p) \text{ where } xn + p \in \mathbf{I}, \\ \varphi(Z) &= \varphi(n + zp, 1, z) = (z, 1, n + zp), \end{aligned}$$

$\varphi(l) = \left[1, -np^{-1}, p^{-1}\right]$ where $p \notin \mathbf{I}$ and $\varphi(l) = [p, n, 1]$ where $p \in \mathbf{I}$. From (b) of Theorem 3, the cross-ratio of the points of $\left[1, -np^{-1}, p^{-1}\right]$ is

$$\begin{aligned} (\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= ((an + p)^{-1} a, (bn + p)^{-1} b; \\ &\quad (cn + p)^{-1} c, (dn + p)^{-1} d) \\ &\overset{\sigma}{=} (a, b; c, d), \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= ((n + zp)^{-1}, (bn + p)^{-1} b; \\ &\quad (cn + p)^{-1} c, (dn + p)^{-1} d) \\ &\overset{\sigma}{=} \left(z^{-1}, b; c, d\right) \end{aligned}$$

where $\sigma = i \circ r_{p^{-1}} \circ t_{-n} \circ i$. Similarly, from (c) of Theorem 3, if the cross-ratio of the points of $[p, n, 1]$ is calculated then we see that it has the desired property.

If $l = [q, n, 1]$ then

$$\begin{aligned}\varphi(X) &= \varphi(1, x, q + xn) = (q + xn, x, 1), \\ \varphi(Z) &= \varphi(z, 1, zq + n) = (zq + n, 1, z)\end{aligned}$$

and $\varphi(l) = [1, n, q]$. From (b) of Theorem 3 we immediately have

$$\begin{aligned}(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (a, b; c, d), \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left(z^{-1}, b; c, d\right).\end{aligned}$$

Part iii) Let $\varphi =$F. If $l = [m, 1, k]$, then

$$\begin{aligned}\varphi(X) &= \varphi(x, xm + k, 1) = (xm + k, x, 1), \\ \varphi(Z) &= \varphi(1, m + zk, z) = \left(1, (m + zk)^{-1}, (m + zk)^{-1} z\right) \text{ where } m + zk \notin \mathbf{I}, \\ \varphi(Z) &= \varphi(1, m + zk, z) = (m + zk, 1, z) \text{ where } m + zk \in \mathbf{I},\end{aligned}$$

$\varphi(l) = \left[m^{-1}, 1, -km^{-1}\right]$ where $m \notin \mathbf{I}$ and $\varphi(l) = [1, m, k]$ where $m \in \mathbf{I}$. From (a) of Theorem 3, the cross-ratio of the points of $\left[m^{-1}, 1, -km^{-1}\right]$ is

$$\begin{aligned}(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (am + k, bm + k; cm + k, dm + k) \\ &\overset{\sigma}{=} (a, b; c, d), \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left(z^{-1}(m + zk), bm + k; cm + k, dm + k\right) \\ &\overset{\sigma}{=} \left(z^{-1}, b; c, d\right),\end{aligned}$$

where $\sigma = r_{m^{-1}} \circ t_{-k}$. Similarly, from (b) of Theorem 3, if the cross-ratio of the points of $[1, m, k]$ is calculated then we see that it has the desired property.

If $l = [1, n, p]$ then

$$\begin{aligned}\varphi(X) &= \varphi(xn + p, x, 1) = (x, xn + p, 1), \\ \varphi(Z) &= \varphi(n + zp, 1, z) = (1, n + zp, z)\end{aligned}$$

and $\varphi(l) = [n, 1, p]$. From (a) of Theorem 3 we immediately have

$$\begin{aligned}(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (a, b; c, d), \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left(z^{-1}, b; c, d\right).\end{aligned}$$

If $l = [q, n, 1]$ then

$$\begin{aligned}\varphi(X) &= \varphi(1, x, q + xn) = \left(1, x^{-1}, x^{-1}(q + xn)\right) \text{ where } x \notin \mathbf{I}, \\ \varphi(X) &= \varphi(1, x, q + xn) = (x, 1, q + xn) \text{ where } x \in \mathbf{I}, \\ \varphi(Z) &= \varphi(z, 1, zq + n) = (1, z, zq + n)\end{aligned}$$

and $\varphi(l) = [n, q, 1]$. From (c) of Theorem 3 we obtain

$$\begin{aligned}
(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= \left(a^{-1}, b^{-1}; c^{-1}, d^{-1}\right) \stackrel{\sigma}{=} (a, b; c, d), \\
(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left(z, b^{-1}; c^{-1}, d^{-1}\right) \stackrel{\sigma}{=} \left(z^{-1}, b; c, d\right),
\end{aligned}$$

where $\sigma = i$.

Part iv) Let $\varphi = \mathrm{G}_u$. If $l = [m, 1, k]$, then

$$\begin{aligned}
\varphi(X) &= \varphi(x, xm + k, 1) = (x, xm + k - xu, 1), \\
\varphi(Z) &= \varphi(1, m + zk, z) = (1, m + zk - u, z)
\end{aligned}$$

and $\varphi(l) = [m - u, 1, k]$. From (a) of Theorem 3 we immediately have

$$\begin{aligned}
(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (a, b; c, d), \\
(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left(z^{-1}, b; c, d\right).
\end{aligned}$$

If $l = [1, n, p]$ then

$$\begin{aligned}
\varphi(X) &= \varphi(xn + p, x, 1) = (xn + p, x - (xn + p)u, 1), \\
\varphi(Z) &= \varphi(n + zp, 1, z) = (n + zp, 1, z)
\end{aligned}$$

and $\varphi(l) = [1, n, p + (pu)n]$. From (b) of Theorem 3, we obtain

$$\begin{aligned}
(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (a - (an + p)u, b - (bn + p)u; \\
&\quad\; c - (cn + p)u, d - (dn + p)u), \\
&\stackrel{\sigma}{=} (a, b; c, d), \\
(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= (z^{-1}, b - (bn + p)u; \\
&\quad\; c - (cn + p)u, d - (dn + p)u) \\
&\stackrel{\sigma}{=} \left(z^{-1}, b; c, d\right),
\end{aligned}$$

where $\sigma = t_{pu}$ and $nu = 0$ while $u \in \mathbf{I}$ or $\sigma = r_{(u^{-1}-n)^{-1}} \circ t_p \circ r_{u^{-1}}$ and $nz = 0$ while $u \notin \mathbf{I}$.

If $l = [q, n, 1]$, then

$$\begin{aligned}
\varphi(X) &= \varphi(1, x, q + xn) = (1, x - u, q + xn), \\
\varphi(Z) &= \varphi(z, 1, zq + n) = (z, 1, zq + n)
\end{aligned}$$

and $\varphi(l) = [q + un, n, 1]$. From (c) of Theorem 3 we have

$$\begin{aligned}
(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (a - u, b - u; c - u, d - u) \stackrel{\sigma}{=} (a, b; c, d), \\
(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left(z^{-1}, b - u; c - u, d - u\right) \stackrel{\sigma}{=} \left(z^{-1}, b; c, d\right)
\end{aligned}$$

where $\sigma = t_u$.

Consequently, by considering other all cases we get

$$\begin{aligned}
(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (a, b; c, d) \\
(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= \left(z^{-1}, b; c, d\right) \\
(\varphi(A), \varphi(Z); \varphi(C), \varphi(D)) &= \left(a, z^{-1}; c, d\right) \\
(\varphi(A), \varphi(B); \varphi(Z), \varphi(D)) &= \left(a, b; z^{-1}, d\right) \\
(\varphi(A), \varphi(B); \varphi(C), \varphi(Z)) &= \left(a, b; c, z^{-1}\right)
\end{aligned}$$

for every collineation φ. Combining the last result and the result of (1), the proof is completed. □

In the next section, we examine the relation between harmonicity and harmonic position in $\mathbf{M}(\mathcal{A})$

4. Harmonicity and points in harmonic position

In the final section we will show that the relation between harmonicity and harmonic position, given for the points of only the line g in [1, Theorem 16], is valid for any line in $\mathbf{M}(\mathcal{A})$. That is, we will generalize this relation to any line of $\mathbf{M}(\mathcal{A})$.

We begin by giving consecutively two definitions in $\mathbf{M}(\mathcal{A})$ from [8] and [1], respectively.

Definition 5 *In $\boldsymbol{M}(\mathcal{A})$, any pairwise non-neighbour four points is called an (ordered) 4-gon if no three of its elements are on neighbour lines.*

Definition 6 *Let l be a line in $\boldsymbol{M}(\mathcal{A})$. Let A, B, C, D be pairwise non-neighbour four points of l. Then A, B, C, D are called harmonic if $(A, B; C, D) = <-1>$ and A, B, C, D are called in harmonic position if there exists a 4-gon (P_1, P_2, Q_1, Q_2) such that $P_1P_2 \cap Q_1Q_2 = A$, $P_1Q_2 \cap P_2Q_1 = B$, $P_1Q_1 \cap l = C$, $P_2Q_2 \cap l = D$ (see Figure 1). We let $h(A, B, C, D)$ represent the statement: A, B, C, D are harmonic. We let $H(A, B, C, D)$ represent the statement: A, B, C, D are in harmonic position.*

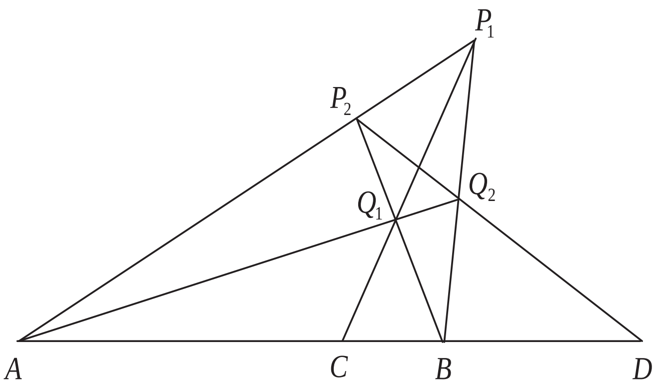

Figure 1

By the following theorem we will be able to generalize the relation between harmonicity and harmonic position which is given for the points of only the line $g = [1,0,0]$ in [1, Theorem 16], to any line of $\mathbf{M}(\mathcal{A})$.

Theorem 7 *There exists a collineation transforming one of any two lines to other. In other words,* $\mathbb{M}(\mathcal{A})$ *is line-transitivite.*

Proof. For the proof, it suffices to show that there exists a collineation Γ which transforms any line l of $\mathbb{M}(\mathcal{A})$ to $g = [1,0,0]$. In this case, since

$$[m,1,k] \stackrel{\mathrm{G}_m}{\to} [0,1,k] \stackrel{\mathrm{F}}{\to} [1,0,k] \stackrel{\mathrm{T}_{-k,0}}{\to} [1,0,0],$$

$$[1,n,p] \stackrel{\mathrm{F}}{\to} [n,1,p] \stackrel{\mathrm{G}_n}{\to} [0,1,p] \stackrel{\mathrm{F}}{\to} [1,0,p] \stackrel{\mathrm{T}_{-p,0}}{\to} [1,0,0],$$

and finally

$$[q,n,1] \stackrel{\mathrm{I}_1}{\to} [1,n,q] \stackrel{\mathrm{F}}{\to} [n,1,q] \stackrel{\mathrm{G}_n}{\to} [0,1,q] \stackrel{\mathrm{F}}{\to} [1,0,q] \stackrel{\mathrm{T}_{-q,0}}{\to} [1,0,0],$$

then the transformation $\Gamma : \mathbb{M}(\mathcal{A}) \to \mathbb{M}(\mathcal{A})$

$$\Gamma := \begin{cases} \mathrm{T}_{-k,0} \circ \mathrm{F} \circ \mathrm{G}_m; & \text{if } l = [m,1,k] \\ \mathrm{T}_{-p,0} \circ \mathrm{F} \circ \mathrm{G}_n \circ \mathrm{F}; & \text{if } l = [1,n,p] \\ \mathrm{T}_{-q,0} \circ \mathrm{F} \circ \mathrm{G}_n \circ \mathrm{F} \circ \mathrm{I}_1; & \text{if } l = [q,n,1] \end{cases}$$

is a collineation which transforms l to g. □

Now, we will give a result which is easily obtained by Theorem 7 and Theorem 4.

Corollary 8 *The transformation* Γ*, in the proof of Theorem 7, preserves cross-ratio.*

So, we can state the main result of this paper.

Corollary 9 *In* $\mathbf{M}(\mathcal{A})$*,* $H(A,B,C,D)$ *if and only if* $h(A,B,C,D)$ *where* A*,* B*,* C*,* D *are the points of any line* l*.*

Proof. Let l be any line in $\mathbf{M}(\mathcal{A})$. Let Γ be the collineation transforming l to $g = [1,0,0]$, which is given in the proof of Theorem 7. Then, the proof is obvious since the collineation Γ preserves cross-ratio by Corollary 8 and any collineation is an incidence structure automorphism which preserves and reflects the neighbour relation and hence maps a configuration as in Figure 1 to a configuration of the same type. □

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Basri ÇELİK, Atilla AKPINAR, Süleyman ÇİFTÇİ
Uludag University, Science and Art Faculty,
Department of Mathematics,
Gorukle, 16059, Bursa-TURKEY
e-mail: basri@uludag.edu.tr, aakpinar@uludag.edu.tr
e-mail: sciftci@uludag.edu.tr

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