Hacettepe Journal of Mathematics and Statistics
Volume 40 (1) (2011), 1 – 13

CROSS-RATIOS OF POINTS AND LINES IN SOME MOUFANG-KLINGENBERG PLANES

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Received 10 : 11 : 2008 : Accepted 18 : 06 : 2009

Abstract

This paper deals with Moufang-Klingenberg planes $\mathbf{M}(\mathcal{A})$ defined over a local alternative ring $\mathcal{A}$ of dual numbers. The cross-ratio of concurrent lines is defined in a special case, and extended to the whole plane $\mathbf{M}(\mathcal{A})$. So, some results related to cross-ratios of points of $\mathbf{M}(\mathcal{A})$ are carried over to lines of $\mathbf{M}(\mathcal{A})$. Namely, we show that four pairwise non-neighbour lines passing through the point U are in harmonic position if and only if they are harmonic.

Keywords: Moufang-Klingenberg planes, Local alternative ring, Cross-ratio.

2000 AMS Classification: 51 C 05, 51 A 35, 17 D 05.

1. Introduction

Coordinatizing rings are constructed to benefit from algebra in the examination of projective planes. There are very close relations between algebraic properties of the coordinatizing ring and geometric properties of the associated plane. If projective plane on which we study is a Pappian plane, a Desarguesian plane or a Moufang plane, then the coordinatizing ring is a field, a division ring (skew field) or an alternative field (alternative division ring), respectively [14, p. 154]. Besides, if the projective plane is a Moufang-Klingenberg (MK) plane then the coordinatizing ring is a local alternative ring [2, Theorem 3.10 and Theorem 4.1].

In the case of projective planes, numerical equations written in the Euclidean plane are not valid since metric concepts are not available. The only exception to this situation is the equation related to the cross-ratio. The cross-ratio of four collinear points A, B, C, D in Euclidean geometry is the number defined to be

$$(A, B; C, D) = \frac{AC}{BC} : \frac{AD}{BD}.$$

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In the Euclidean plane, Desargues established the fundamental fact that the *cross-ratio* (a concept originally introduced by Pappus of Alexandria c.300 B.C) is invariant under projection [3, p. 133]. This states that the cross-ratio is a projective concept, that is unchanged under projective transformations. The concept of cross-ratio (expressed as a ratio of ratios in [15, p. 122] or an hormanic ratio in [13, p. 150]) has great importance since it is the only numerical result which is a projective invariant. But, since the coordinatizing ring of a projective plane has different algebraic properties, a general definition of cross-ratio which is valid in all of projective planes could not be given. The closest one to the definition of the cross-ratio given in the Euclidean plane is that given for the Pappus plane, whose coordinatizing ring is a field: the cross-ratio of four collinear points $A=(a_1,a_2)$, $B=(b_1,b_2)$, $C=(c_1,c_2)$, $D=(d_1,d_2)$ is then given by

$$(A,B;C,D)=\frac{(a_1c_2-a_2c_1)\,(b_1d_2-b_2d_1)}{(a_1d_2-a_2d_1)\,(b_1c_2-b_2c_1)}.$$

If we take non-homogenous coordinates $a=\frac{a_1}{a_2}$, $b=\frac{b_1}{b_2}$, $c=\frac{c_1}{c_2}$, $d=\frac{d_1}{d_2}$ instead of $A=(a_1,a_2)$, $B=(b_1,b_2)$, $C=(c_1,c_2)$, $D=(d_1,d_2)$, respectively, then the above definition of cross-ratio can be stated as follows [13, p. 151]:

$$(A,B;C,D)=\frac{(a-c)\,(b-d)}{(a-d)\,(b-c)}=(a,b;c,d)\,.$$

(Note that if any one of the points A, B, C, D is ∞, the factors involving ∞ are cancelled [12, p. 81]). As for the Moufang (or Desarguesian) planes, which form a more general class than the Pappian planes, Ferrar [11] gives the following algebraic definition of the cross-ratio for the points on the line $[0,0]$,

$$(A,B;C,D)=(a,b;c,d):=\left\langle \left((a-d)^{-1}(b-d)\right)\left((b-c)^{-1}\,(a-c)\right)\right\rangle$$

where $A=(a,0)$, $B=(b,0)$, $C=(c,0)$, $D=(d,0)$. Here, $\langle x\rangle=\left\{y^{-1}xy \;\middle|\; y\in\mathbf{R}\right\}$, where $\mathbf{R}$ is the coordinatizing ring. According to the definition, if any one of the points A, B, C, D is ∞, the factors involving ∞ are cancelled. The definition of the cross-ratio given for the points on the line $[0,0]$ is extended to the whole plane, by considering the fact that perspectivities preserve cross-ratio [9, p. 25]. For more information about some well-known properties of cross-ratio in the case of Moufang planes, the reader can refer to the papers [11, 4, 8, 10].

One of the aims of this paper is to give the definition of the cross-ratio of concurrent lines (on the special point) in a certain class (which we will denote by $\mathbf{M}(\mathcal{A})$) of MK-planes, coordinatized by a local alternative ring $\mathcal{A}:=\mathbf{A}\,(\varepsilon)=\mathbf{A}+\mathbf{A}\varepsilon$ (an alternative field $\mathbf{A}$, $\varepsilon\notin\mathbf{A}$ and $\varepsilon^2=0$) introduced by Blunck in [7], and then is to extend this definition to the whole plane $\mathbf{M}(\mathcal{A})$. Some properties of the cross-ratios obtained in [1] for points of $\mathbf{M}(\mathcal{A})$ are investigated for lines of $\mathbf{M}(\mathcal{A})$, and so a relation between the harmonicity (which is an algebraic property of $\mathcal{A}$) and the harmonic position (which is a geometric property of $\mathbf{M}(\mathcal{A})$) is established, which is the other aim of this paper.

Section 2 includes some basic definitions and results from the literature.

In Section 3, the concept of cross-ratio in $\mathbf{M}(\mathcal{A})$ is mentioned. First, some results related to the cross-ratio of points of $\mathbf{M}(\mathcal{A})$ are given. Second, the cross-ratio of the lines passing through the point $U=(1,0,0)$ is defined in $\mathbf{M}(\mathcal{A})$. Third, using this definition, the cross-ratio of concurrent lines is extended to the whole plane $\mathbf{M}(\mathcal{A})$. Next, a simple way for the calculation of the cross-ratio of lines passing through the point P, according to the type of P, is given. This paper is concluded by constructing the relation between the harmonicity and the harmonic position, for lines passing through the point U.

2. Preliminaries

Let $\mathbf{M} = (\mathbf{P}, \mathbf{L}, \in, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \in)$ (points, lines, incidence), and an equivalence relation '$\sim$' (neighbour relation) on $\mathbf{P}$ and on $\mathbf{L}$, respectively. Then $\mathbf{M}$ is called a *projective Klingenberg plane* (PK-plane), if it satisfies the following axioms:

(PK1): If $P,\ Q$ are non-neighbour points, then there is a unique line PQ through P and Q.

(PK2): If $\mathbf{k},\ \mathbf{h}$ are non-neighbour lines, then there is a unique point $\mathbf{k} \cap \mathbf{h}$ on both $\mathbf{k}$ and $\mathbf{h}$.

(PK3): There is a projective plane $\mathbf{M}^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$ and an incidence structure epimorphism $\Psi : \mathbf{M} \to \mathbf{M}^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim Q,\ \Psi(\mathbf{k}) = \Psi(\mathbf{h}) \iff \mathbf{k} \sim \mathbf{h}$$

hold for all $P, Q \in \mathbf{P}$; $\mathbf{k}, \mathbf{h} \in \mathbf{L}$.

A point $P \in \mathbf{P}$ is called *near* a line $\mathbf{k} \in \mathbf{L}$ iff there exists a line $\mathbf{h} \sim \mathbf{k}$ such that $P \in \mathbf{h}$.

Let $\mathbf{h}, \mathbf{k} \in \mathbf{L}$, $C \in \mathbf{P}$ with C not near to $\mathbf{h}, \mathbf{k}$. Then the well-defined bijection $\sigma := \sigma_C(\mathbf{k}, \mathbf{h})$ mapping $\mathbf{h}$ to $\mathbf{k}$, by the rule $\sigma_C(X) = XC \cap \mathbf{k}$ is called a *perspectivity* from $\mathbf{h}$ to $\mathbf{k}$ with center C. Dually, let $P, Q \in \mathbf{P}$, $\mathbf{e} \in \mathbf{L}$ with $\mathbf{e}$ not near to P, Q. Then the well-defined bijection $\sigma := \sigma_{\mathbf{e}}(Q, P)$ which maps the lines passing through P to the lines passing through Q, by the rule $\sigma_{\mathbf{e}}(\mathbf{x}) = \mathbf{xe} \cup Q$ is called a *perspectivity* from P to Q with axis $\mathbf{e}$. A product of a finite number of perspectivities is called a *projectivity*.

A PK-plane $\mathbf{M} = (\mathbf{P}, \mathbf{L}, \in, \sim)$ is called a *Moufang-Klingenberg plane* (MK-plane), if it is $(C, \mathbf{a})$-transitive for all $C \in \mathbf{P}$, $\mathbf{a} \in \mathbf{L}$ with $C \in \mathbf{a}$, i.e. if all possible relations exist. For every MK-plane the canonical image $\mathbf{M}^*$ is a Moufang plane [7].

An *alternative ring* $\mathbf{R}$ is a not necessarily associative ring that satisfies the alternative laws $a(ab) = a^2 b,\ (ba)a = ba^2,\ \forall a, b \in \mathbf{R}$. An alternative ring $\mathbf{R}$ with identity element 1 is called *local* if the set $\mathbf{I}$ of its non-unit elements is an ideal.

We are now ready to give a lemma related to alternative rings, which is used in many calculations throughout this paper.

2.1. Lemma. [17, Theorem 3.1] *The subring generated by any two elements of an alternative ring is associative.* □

We summarize some basic concepts about the coordinatization of MK-planes from [5].

Let $\mathbf{R}$ be a local alternative ring. Then $\mathbf{M}(\mathbf{R}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$ is the incidence structure with neighbour relation defined as follows:

$$\mathbf{P} = \{(x, y, 1) |\ x, y \in \mathbf{R}\} \cup \{(1, y, z) |\ y \in \mathbf{R},\ z \in \mathbf{I}\} \cup \{(w, 1, z) |\ w, z \in \mathbf{I}\},$$
$$\mathbf{L} = \{[m, 1, p] \mid m, p \in \mathbf{R}\} \cup \{[1, n, p] \mid p \in \mathbf{R},\ n \in \mathbf{I}\} \cup \{[q, n, 1] \mid q, n \in \mathbf{I}\},$$

$$[m,1,p] = \{(x, xm + p, 1) |\, x \in \mathbf{R}\} \cup \{(1, zp + m, z) |\, z \in \mathbf{I}\},$$
$$[1, n, p] = \{(yn + p, y, 1) |\, y \in \mathbf{R}\} \cup \{(zp + n, 1, z) |\, z \in \mathbf{I}\},$$
$$[q, n, 1] = \{(1, y, yn + q) |\, y \in \mathbf{R}\} \cup \{(w, 1, wq + n) |\, w \in \mathbf{I}\}.$$

$$P = (x_1, x_2, x_3) \sim (y_1, y_2, y_3) = Q \iff x_i - y_i \in \mathbf{I}\ \ (i = 1, 2, 3)), \forall P, Q \in \mathbf{P};$$
$$\mathbf{k} = [x_1, x_2, x_3] \sim [y_1, y_2, y_3] = \mathbf{h} \Leftrightarrow x_i - y_i \in \mathbf{I}\ \ (i = 1, 2, 3)), \forall \mathbf{k}, \mathbf{h} \in \mathbf{L}.$$

For more detailed information about the coordinatization, see the papers [2, 5].

Now it is time to give the following lemma from [2].

2.2. Lemma. $\mathbf{M}(\mathbf{R})$ *is a MK-plane, and each MK-plane is isomorphic to some* $\mathbf{M}(\mathbf{R})$. □

Let $\mathbf{A}$ be an alternative field and $\varepsilon \notin \mathbf{A}$. Consider $\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$, with componentwise addition and multiplication as follows:

$$(a_1 + a_2\varepsilon)(b_1 + b_2\varepsilon) = a_1b_1 + (a_1b_2 + a_2b_1)\varepsilon, \ (a_i, b_i \in \mathbf{A}, i = 1, 2).$$

Then $\mathcal{A}$ is a local alternative ring with the ideal $\mathbf{I} = \mathbf{A}\varepsilon$ of non-units. The set of formal inverses of the non-units of $\mathcal{A}$ is denoted by $\mathbf{I}^{-1}$. Calculations with the elements of $\mathbf{I}^{-1}$ are defined as follows [6]:

$$(a\varepsilon)^{-1} + t := (a\varepsilon)^{-1} := t + (a\varepsilon)^{-1}, \ q(a\varepsilon)^{-1} := \left(aq^{-1}\varepsilon\right)^{-1},$$
$$(a\varepsilon)^{-1} q := \left(q^{-1}a\varepsilon\right)^{-1}, \ \left((a\varepsilon)^{-1}\right)^{-1} := a\varepsilon,$$

where $(a\varepsilon)^{-1} \in \mathbf{I}^{-1}$, $t \in \mathcal{A}$, $q \in \mathcal{A} \setminus \mathbf{I}$. (Other terms are not defined). For more information about $\mathcal{A}$ and its relation to MK-planes, the reader is referred to the papers [6, 7]. In [7], the centre $\mathbf{Z}(\mathcal{A})$ is defined to be the (commutative, associative) subring of $\mathcal{A}$ which is commuting and associating with all elements of $\mathcal{A}$. It is $\mathbf{Z}(\mathcal{A}) := \mathbf{Z}(\varepsilon) = \mathbf{Z} + \mathbf{Z}\varepsilon$, where $\mathbf{Z} = \{z \in \mathbf{A} | za = az, \ \forall a \in \mathbf{A}\}$ is the centre of $\mathbf{A}$. If $\mathbf{A}$ is not associative, then $\mathbf{A}$ is a Cayley division algebra over its centre $\mathbf{Z}$, see [16] or [18]. Throughout we assume $\mathrm{char}\mathbf{A} \neq \mathbf{2}$, and also we restrict ourselves to the MK-planes $\mathbf{M}(\mathcal{A})$.

Blunck [7] gives the following algebraic definition of the cross-ratio for points on the line $\mathbf{g} := [1, 0, 0]$ in $\mathbf{M}(\mathcal{A})$

$$(A, B; C, D) := (a, b; c, d) = \langle\left((a-d)^{-1}(b-d)\right)\left((b-c)^{-1}(a-c)\right)\rangle,$$
$$(K^{-1}, B; C, D) := \left(k^{-1}, b; c, d\right) = \langle\left((1-dk)^{-1}(b-d)\right)\left((b-c)^{-1}(1-ck)\right)\rangle,$$
$$(A, K^{-1}; C, D) := \left(a, k^{-1}; c, d\right) = \langle\left((a-d)^{-1}(1-dk)\right)\left((1-ck)^{-1}(a-c)\right)\rangle,$$
$$(A, B; K^{-1}, D) := \left(a, b; k^{-1}, d\right) = \langle\left((a-d)^{-1}(b-d)\right)\left((1-kb)^{-1}(1-ka)\right)\rangle,$$
$$(A, B; C, K^{-1}) := \left(a, b; c, k^{-1}\right) = \langle\left((1-ka)^{-1}(1-kb)\right)\left((b-c)^{-1}(a-c)\right)\rangle,$$

where $A = (0, a, 1)$, $B = (0, b, 1)$, $C = (0, c, 1)$, $D = (0, d, 1)$, $K^{-1} = (0, 1, k)$ are pairwise non-neighbour points. Here, $k \in \mathbf{I}$ and $\langle x \rangle = \{y^{-1}xy \mid y \in \mathcal{A}\}$.

By this definition, the following important result about the cross-ratio of points on any line in $\mathbf{M}(\mathcal{A})$ is obtained [1, Theorem 8].

2.3. Lemma. *Let* $\{O, U, V, E\}$ *be the basis of* $M(A)$ *where* $O = (0, 0, 1)$, $U = (1, 0, 0)$, $V = (0, 1, 0)$, $E = (1, 1, 1)$ *(see* [2, Section 4]*). Then, according to types of lines, the cross-ratio of points on the line* l *can be calculated as follows:*

If A, B, C, D *and* K^{-1} *are pairwise non-neighbour points:*

(a) *Of the line* $l = [m, 1, p]$, *where* $A = (a, am + p, 1)$, $B = (b, bm + p, 1)$, $C = (c, cm + p, 1)$, $D = (d, dm + p, 1)$ *are not near to the line* $UV = [0, 0, 1]$ *and* $K^{-1} = (1, m + kp, k)$ *is near to* UV,

(b) *Of the line* $l = [1, n, p]$, *where* $A = (an + p, a, 1)$, $B = (bn + p, b, 1)$, $C = (cn + p, c, 1)$, $D = (dn + p, d, 1)$ *are not neighbour to* V *and* $K^{-1} = (n + kp, 1, k) \sim V$,

(c) *Of the line* $l = [q, n, 1]$, *where* $A = (1, a, q + an)$, $B = (1, b, q + bn)$, $C = (1, c, q + cn)$, $D = (1, d, q + dn)$ *are not neighbour to* V *and* $K^{-1} = (k, 1, kq + n) \sim V$, *then*

$$\begin{aligned} (A,B;C,D) &= (a,b;c,d)\,, & \big(K^{-1},B;C,D\big) &= \big(k^{-1},b;c,d\big)\,,\\ \big(A,K^{-1};C,D\big) &= \big(a,k^{-1};c,d\big)\,, & \big(A,B;K^{-1},D\big) &= \big(a,b;k^{-1},d\big)\,,\\ \big(A,B;C,K^{-1}\big) &= \big(a,b;c,k^{-1}\big)\,. \end{aligned}$$ □

Moreover, we also know the following result from [1, Theorem 9]:

2.4. Lemma. *In $M(A)$, perspectivities preserve cross-ratios.*

3. Cross-ratio of concurrent lines in $\mathbf{M}(\mathcal{A})$

We denote by $\mathcal{L}_U$ the set of lines passing through the point $U=(1,0,0)$ in $\mathbf{M}(\mathcal{A})$. Also, we identify $\mathcal{L}_U$ with $\mathcal{A}\cup\mathbf{I}^{-1}$, where $\mathbf{x}=[0,1,x]\longleftrightarrow x$ and $\mathbf{k}^{-1}=[0,k,1]\longleftrightarrow k^{-1}$. So, on $\mathcal{A}\cup\mathbf{I}^{-1}$, we have a neighbour relation "$\sim$"defined algebraically by $x\sim y :\iff (x,y\in\mathcal{A}$ and $x-y\in\mathbf{I})$ or $(x,y\in\mathbf{I}^{-1})$, which coincides with our neighbour relation on $\mathcal{L}_U$.

Let $\mathbf{a}=[0,1,a]$, $\mathbf{b}=[0,1,b]$, $\mathbf{c}=[0,1,c]$, $\mathbf{d}=[0,1,d]$, $\mathbf{k}^{-1}=[0,k,1]\in\mathcal{L}_U$ be pairwise non-neighbour lines, where a, b, c, $d\in\mathcal{A}$ and $k\in\mathbf{I}$. Then the definition of the cross-ratio for the lines can be given, in the same way as in the above definition, as follows:

$$\begin{aligned} \lfloor\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d}\rfloor &:= (a,b;c,d)\,, & \big\lfloor\mathbf{k}^{-1},\mathbf{b};\mathbf{c},\mathbf{d}\big\rfloor &:= \big(k^{-1},b;c,d\big)\,,\\ \big\lfloor\mathbf{a},\mathbf{k}^{-1};\mathbf{c},\mathbf{d}\big\rfloor &:= \big(a,k^{-1};c,d\big) & \big\lfloor\mathbf{a},\mathbf{b};\mathbf{k}^{-1},\mathbf{d}\big\rfloor &:= \big(a,b;k^{-1},d\big)\,,\\ \big\lfloor\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{k}^{-1}\big\rfloor &:= \big(a,b;c,k^{-1}\big)\,. \end{aligned}$$

Calculations with the elements of $\mathbf{I}^{-1}$ have been given in Section 2.

3.1. Theorem. *The cross-ratio of four pairwise non-neighbour lines of L_U equals the cross-ratio of the intersection points of these lines and the line* $\mathbf{g}$.

Proof. Let $\mathbf{a}=[0,1,a]$, $\mathbf{b}=[0,1,b]$, $\mathbf{c}=[0,1,c]$, $\mathbf{d}=[0,1,d]$, $\mathbf{k}^{-1}=[0,k,1]\in\mathcal{L}_U$ be pairwise non-neighbour lines. Then $\mathbf{a}\cap\mathbf{g}=(0,a,1)=A$, $\mathbf{b}\cap\mathbf{g}=(0,b,1)=B$, $\mathbf{c}\cap\mathbf{g}=(0,c,1)=C$, $\mathbf{d}\cap\mathbf{g}=(0,d,1)=D$, $\mathbf{k}^{-1}\cap\mathbf{g}=(0,1,k)=K^{-1}$. We immediately have:

$$\begin{aligned} \lfloor\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d}\rfloor &= (a,b;c,d) = (A,B;C,D)\\ \big\lfloor\mathbf{k}^{-1},\mathbf{b};\mathbf{c},\mathbf{d}\big\rfloor &= \big(k^{-1},b;c,d\big) = \big(K^{-1},B;C,D\big)\\ \big\lfloor\mathbf{a},\mathbf{k}^{-1};\mathbf{c},\mathbf{d}\big\rfloor &= \big(a,k^{-1};c,d\big) = \big(A,K^{-1};C,D\big)\\ \big\lfloor\mathbf{a},\mathbf{b};\mathbf{k}^{-1},\mathbf{d}\big\rfloor &= \big(a,b;k^{-1},d\big) = \big(A,B;K^{-1},D\big)\\ \big\lfloor\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{k}^{-1}\big\rfloor &= \big(a,b;c,k^{-1}\big) = \big(A,B;C,K^{-1}\big)\,. \end{aligned}$$ □

As a result of this theorem, by adapting the needed results from those obtained for the cross-ratio of the points on the line $\mathbf{g}$ in [6] to the elements of $\mathcal{L}_U$, we can give consecutively the following results without proof.

The first result, analogous to [6, Lemma 7], gives another statement of the definition of the cross-ratio.

3.2. Corollary. *Let* $\mathbf{a}=[0,1,a]$, $\mathbf{b}=[0,1,b]$, $\mathbf{c}=[0,1,c]$, $\mathbf{d}=[0,1,d]\in\mathcal{L}_U$ *be pairwise non-neighbour lines. Then*

$$\lfloor\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d}\rfloor = \langle\big((a-b)^{-1}-(a-d)^{-1}\big)\big((a-b)^{-1}-(a-c)^{-1}\big)^{-1}\rangle.$$

The second result, a fact given as a statement in [6, p. 251], gives an important result about the cross-ratio of the elements of $\mathcal{L}_U$.

3.3. Corollary. *Every cross-ratio consists only of elements of* $\mathcal{A}\setminus(\{0,1\}+\mathbf{I})$. *Conversely, the conjugacy class of any such element appears as a cross-ratio; given three pairwise non-neighbour lines* $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathcal{L}_U$ *and an element* $r\in\mathcal{A}\setminus(\{0,1\}+\mathbf{I})$, *then there is a (unique if* $r\in\mathbf{Z}(\varepsilon)$*) line* $\mathbf{d}\in\mathcal{L}_U$, $\mathbf{d}\nsim\mathbf{a},\mathbf{b},\mathbf{c}$, *with* $\lfloor\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d}\rfloor=\langle r\rangle$. □

The third result, analogous to [6, Theorem 2], is the following

3.4. Corollary. *The transformations*

$$t_u(\mathbf{x})=x+u \text{ where } u\in\mathcal{A},\ r_u(\mathbf{x})=xu \text{ where } u\in\mathcal{A}\setminus\mathbf{I},$$
$$i(\mathbf{x})=x^{-1},\ l_u(\mathbf{x})=ux=\left(ir_u^{-1}i\right)(x) \text{ where } u\in\mathcal{A}\setminus\mathbf{I},$$

which are defined on L_U, *preserve cross-ratios.* □

Moreover, we can state another result analogous to [5, Corollary (iii)].

3.5. Corollary. *The group* Λ *generated by the above transformations coincides with the group of projectivities of a point in* $\mathbf{M}(\mathcal{A})$. □

As a result of Lemma 2.4 and Theorem 3.1 we can also give the following result.

3.6. Corollary. *Let* $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{t}$ *be four pairwise non-neighbour lines of* L_U. *Then*

$$\lfloor\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}\rfloor=(\mathbf{x}\cap\mathbf{l},\mathbf{y}\cap\mathbf{l},\mathbf{z}\cap\mathbf{l},\mathbf{t}\cap\mathbf{l}),$$

where $\mathbf{l}$ *is not near to* U. □

By the last corollary, it is possible to extend the definition of the cross-ratio of the lines of $\mathcal{L}_U$ to the whole plane $\mathbf{M}(\mathcal{A})$. By Lemma 2.4 we can state this extension as the following definition.

3.7. Definition. Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{t}$ be four pairwise non-neighbour lines passing through any point P of $\mathbf{M}(\mathcal{A})$. Take any line $\mathbf{l}$, where $\mathbf{l}$ is not near to P and U. Then

$$\lfloor\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}\rfloor:=\lfloor(\mathbf{x}\cap\mathbf{l})\,U,(\mathbf{y}\cap\mathbf{l})\,U,(\mathbf{z}\cap\mathbf{l})\,U,(\mathbf{t}\cap\mathbf{l})\,U\rfloor.$$

Consequently, we have constructed the relation between cross-ratios of points and lines. For the relation in Moufang planes, see [8, 10].

Although we have constructed the relation between cross-ratios of points and lines above, we would like to find a simpler way to calculate the cross-ratio of lines passing through any point P. To obtain this we need the following two lemmas.

3.8. Lemma. *Let* $\{O,U,V,E\}$ *be a basis of* $\mathbf{M}(\mathcal{A})$, *where* $O=(0,0,1)$, $U=(1,0,0)$, $V=(0,1,0)$, $E=(1,1,1)$ (*see* [2, Section 4]). *Then*

(a) *If* $P=(x,y,1)$ *and* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ *are pairwise non-neighbour lines passing through* P, *then*
  (i) *If* P *is not near to* $\mathbf{g}$, *then* $x\notin\mathbf{I}$. *In this case, the cross-ratio is* $\lfloor\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d}\rfloor=\lfloor\sigma(\mathbf{a}),\sigma(\mathbf{b});\sigma(\mathbf{c}),\sigma(\mathbf{d})\rfloor$, *where* $\sigma=\sigma_{\mathbf{g}}(U,P)$.
  (ii) *If* P *is near to* $\mathbf{g}$, *then* $x\in\mathbf{I}$. *In this case, the cross-ratio is* $\lfloor\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d}\rfloor=\lfloor\sigma(\mathbf{a}),\sigma(\mathbf{b});\sigma(\mathbf{c}),\sigma(\mathbf{d})\rfloor$, *where* $\sigma=\sigma_{[1,0,1]}(U,P)$.

(b) *If* $P=(1,y,z)$, *then the cross-ratio is* $\lfloor\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d}\rfloor=\lfloor\sigma(\mathbf{a}),\sigma(\mathbf{b});\sigma(\mathbf{c}),\sigma(\mathbf{d})\rfloor$, *where* $\sigma=\sigma_{\mathbf{g}}(U,P)$.

(c) *If* $P=(w,1,z)$, *then the cross-ratio is* $\lfloor\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d}\rfloor=\lfloor\sigma(\mathbf{a}),\sigma(\mathbf{b});\sigma(\mathbf{c}),\sigma(\mathbf{d})\rfloor$, *where* $\sigma=\sigma_{[1,1,0]}(U,P)$. □

Thus we have the following lemma.

3.9. Lemma. *Let* σ *be the perspectivity in Lemma 3.8. Then,*

(a) *Let* $P=(x,y,1)$ *and* $\mathbf{a}=[a,1,y-xa]$, $\mathbf{k}^{-1}=[1,k,x-yk]$ *be the lines passing through* P.

(i) *If* $x \notin \mathbf{I}$, *then* $\sigma(\mathbf{a}) = [0, 1, y - xa]$, $\sigma\left(\mathbf{k}^{-1}\right) = \left[0, k(yk - x)^{-1}, 1\right]$,

(ii) *If* $x \in \mathbf{I}$ *then* $\sigma(\mathbf{a}) = [0, 1, (1 - x)a + y]$.

Also, $\sigma\left(\mathbf{k}^{-1}\right) = \left[0, k(1 - x + yk)^{-1}, 1\right]$.

(b) *Let* $P = (1, y, z)$ *and* $\mathbf{a} = [y - za, 1, a]$, $\mathbf{k}^{-1} = [z - yk, k, 1]$ *be the lines passing through* P. *Then* $\sigma(\mathbf{a}) = [0, 1, a]$, $\sigma\left(\mathbf{k}^{-1}\right) = [0, k, 1]$.

(c) *Let* $P = (w, 1, z)$ *and* $\mathbf{a} = [1, w - za, a]$, $\mathbf{k}^{-1} = [k, z - wk, 1]$ *be the lines passing through* P. *Then* $\sigma(\mathbf{a}) = \left[0, 1, a(1 - w + za)^{-1}\right]$.

Also, $\sigma\left(\mathbf{k}^{-1}\right) = [0, (1 - w)k + z, 1]$.

Proof. We give a detailed proof for the case (i) only, since the method for the others is the same. Immediately,

$$\begin{aligned}\sigma(\mathbf{a}) &= (\mathbf{a} \cap \mathbf{g}) \cup U \\ &= ([a, 1, y - xa] \cap [1, 0, 0]) \cup (1, 0, 0) \\ &= (0, y - xa, 1) \cup (1, 0, 0) \\ &= [0, 1, y - xa]\end{aligned}$$

and

$$\begin{aligned}\sigma\left(\mathbf{k}^{-1}\right) &= \left(\mathbf{k}^{-1} \cap \mathbf{g}\right) \cup U \\ &= ([1, k, x - yk] \cap [1, 0, 0]) \cup (1, 0, 0) \\ &= \left(0, 1, k(yk - x)^{-1}\right) \cup (1, 0, 0) \\ &= \left[0, k(yk - x)^{-1}, 1\right].\end{aligned}$$

□

This lemma enables us to make easily some calculations in the proof of the next theorem.

Now we are ready to give the simple way for calculating the cross-ratio of concurrent lines in $\mathbf{M}(\mathcal{A})$.

3.10. Theorem. *According to type of common point, the cross-ratio of the concurrent lines can be calculated as follows:*

If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ *and* $\mathbf{k}^{-1}$ *are pairwise non-neighbour lines*

(a) *Passing through* $P = (x, y, 1)$, *where* $\mathbf{a} = [a, 1, y - xa]$, $\mathbf{b} = [b, 1, y - xb]$, $\mathbf{c} = [c, 1, y - xc]$, $\mathbf{d} = [d, 1, y - xd]$ *are not near to* V *and* $\mathbf{k}^{-1} = [1, k, x - yk]$ *is near to* V*;*

(b) *Passing through* $P = (1, y, z)$, *where* $\mathbf{a} = [y - za, 1, a]$, $\mathbf{b} = [y - zb, 1, b]$, $\mathbf{c} = [y - zc, 1, c]$, $\mathbf{d} = [y - zd, 1, d]$ *are not near to* V *and* $\mathbf{k}^{-1} = [z - yk, k, 1] \sim UV = [0, 0, 1]$*;*

(c) *Passing through* $P = (w, 1, z)$, *where* $\mathbf{a} = [1, w - za, a]$, $\mathbf{b} = [1, w - zb, b]$, $\mathbf{c} = [1, w - zc, c]$, $\mathbf{d} = [1, w - zd, d]$ *are near to* V *and* $\mathbf{k}^{-1} = [k, z - wk, 1] \sim UV$*;*

then

$$\begin{aligned}\lfloor \mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d} \rfloor &= (a, b; c, d) \\ \left\lfloor \mathbf{k}^{-1}, \mathbf{b}; \mathbf{c}, \mathbf{d} \right\rfloor &= \left(k^{-1}, b; c, d\right) \\ \left\lfloor \mathbf{a}, \mathbf{k}^{-1}; \mathbf{c}, \mathbf{d} \right\rfloor &= \left(a, k^{-1}; c, d\right) \\ \left\lfloor \mathbf{a}, \mathbf{b}; \mathbf{k}^{-1}, \mathbf{d} \right\rfloor &= \left(a, b; k^{-1}, d\right) \\ \left\lfloor \mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{k}^{-1} \right\rfloor &= \left(a, b; c, k^{-1}\right)\end{aligned}$$

Proof. We separate the proof into three cases, as given in the statement of the theorem.

Case (a). There are two cases where $x \notin \mathbf{I}$ and $x \in \mathbf{I}$.

(a.1). If $x \notin \mathbf{I}$, then under the perspectivity $\sigma = \sigma_{\mathbf{g}}(U, P)$, the lines $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{k}^{-1}$ transform to $\mathbf{a}_1 = [0, 1, y - xa]$, $\mathbf{b}_1 = [0, 1, y - xb]$, $\mathbf{c}_1 = [0, 1, y - xc]$, $\mathbf{d}_1 = [0, 1, y - xd]$ and $\mathbf{k}_1^{-1} = \left[0, k(yk - x)^{-1}, 1\right]$, respectively. Therefore, with $\gamma = l_{(-x)^{-1}} \circ t_{-y} \in \Lambda$, we get

$$\begin{aligned}
\lfloor \mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d} \rfloor &= \lfloor \mathbf{a}_1, \mathbf{b}_1; c_1, \mathbf{d}_1 \rfloor = (y - xa, y - xb; y - xc, y - xd) \\
&= (\gamma(y - xa), \gamma(y - xb); \gamma(y - xc), \gamma(y - xd)) \\
&= (a, b; c, d),
\end{aligned}$$

and

$$\begin{aligned}
\left\lfloor \mathbf{k}^{-1}, \mathbf{b}; \mathbf{c}, \mathbf{d} \right\rfloor &= \left\lfloor \mathbf{k}_1^{-1}, \mathbf{b}_1; \mathbf{c}_1, \mathbf{d}_1 \right\rfloor = ((yk - x)k^{-1}, y - xb; y - xc, y - xd) \\
&= (\left(y - xk^{-1}\right), (y - xb); (y - xc), (y - xd)) \\
&= (\gamma\left(y - xk^{-1}\right), \gamma(y - xb); \gamma(y - xc), \gamma(y - xd)) \\
&= (k^{-1}, b; c, d).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left\lfloor \mathbf{a}, \mathbf{k}^{-1}; \mathbf{c}, \mathbf{d} \right\rfloor &= \left(a, k^{-1}; c, d\right), \\
\left\lfloor \mathbf{a}, \mathbf{b}; \mathbf{k}^{-1}, \mathbf{d} \right\rfloor &= \left(a, b; k^{-1}, d\right), \\
\left\lfloor \mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{k}^{-1} \right\rfloor &= \left(a, b; c, k^{-1}\right).
\end{aligned}$$

(a.2). If $x \in \mathbf{I}$, then under the perspectivity $\sigma = \sigma_{[1,0,1]}(U, P)$, the lines $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{k}^{-1}$ transform to $\mathbf{a}_1 = [0, 1, (1 - x)a + y]$, $\mathbf{b}_1 = [0, 1, (1 - x)b + y]$, $\mathbf{c}_1 = [0, 1, (1 - x)c + y]$, $\mathbf{d}_1 = [0, 1, (1 - x)d + y]$ and $\mathbf{k}_1^{-1} = \left[0, k(1 - x + yk)^{-1}, 1\right]$, respectively. Therefore, with $\gamma = l_{(1-x)^{-1}} \circ t_{-y} \in \Lambda$, we have

$$\begin{aligned}
\lfloor \mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d} \rfloor &= \lfloor \mathbf{a}_1, \mathbf{b}_1; \mathbf{c}_1, \mathbf{d}_1 \rfloor \\
&= ((1 - x)a + y, (1 - x)b + y; (1 - x)c + y, (1 - x)d + y) \\
&= \big(\gamma((1 - x)a + y), \gamma((1 - x)b + y); \gamma((1 - x)c + y), \\
&\qquad \gamma((1 - x)d + y)\big) \\
&= (a, b; c, d),
\end{aligned}$$

and

$$\begin{aligned}
\left\lfloor \mathbf{k}^{-1}, \mathbf{b}; \mathbf{c}, \mathbf{d} \right\rfloor &= \left\lfloor \mathbf{k}_1^{-1}, \mathbf{b}_1; \mathbf{c}_1, \mathbf{d}_1 \right\rfloor \\
&= \left((1 - x + yk)k^{-1}, (1 - x)b + y; (1 - x)c + y, (1 - x)d + y\right) \\
&= \left((1 - x)k^{-1} + y, (1 - x)b + y; (1 - x)c + y, (1 - x)d + y\right) \\
&= \big(\gamma\left((1 - x)k^{-1} + y\right), \gamma((1 - x)b + y); \gamma((1 - x)c + y), \\
&\qquad \gamma((1 - x)d + y)\big) \\
&= (k^{-1}, b; c, d).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left\lfloor \mathbf{a}, \mathbf{k}^{-1}; \mathbf{c}, \mathbf{d} \right\rfloor &= \left(a, k^{-1}; c, d\right), \\
\left\lfloor \mathbf{a}, \mathbf{b}; \mathbf{k}^{-1}, \mathbf{d} \right\rfloor &= \left(a, b; k^{-1}, d\right), \\
\left\lfloor \mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{k}^{-1} \right\rfloor &= \left(a, b; c, k^{-1}\right).
\end{aligned}$$

Case (b). Let $\mathbf{a} = [y - za, 1, a]$, $\mathbf{b} = [y - zb, 1, b]$, $\mathbf{c} = [y - zc, 1, c]$, $\mathbf{d} = [y - zd, 1, d]$ and $\mathbf{k}^{-1} = [z - yk, k, 1]$ be pairwise non-neighbour lines passing through $P = (1, y, z)$. Then the perspectivity $\sigma = \sigma_{\mathbf{g}}(U, P)$ transforms the lines $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{k}^{-1}$ to $\mathbf{a}_1 = [0, 1, a]$, $\mathbf{b}_1 = [0, 1, b]$, $\mathbf{c}_1 = [0, 1, c]$, $\mathbf{d}_1 = [0, 1, d]$ and $\mathbf{k}_1^{-1} = [0, k, 1]$, respectively. Therefore,

$$\lfloor \mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d} \rfloor = \lfloor \mathbf{a}_1, \mathbf{b}_1; \mathbf{c}_1, \mathbf{d}_1 \rfloor = (a, b; c, d),$$
$$\lfloor \mathbf{k}^{-1}, \mathbf{b}; \mathbf{c}, \mathbf{d} \rfloor = \lfloor \mathbf{k}_1^{-1}, \mathbf{b}_1; \mathbf{c}_1, \mathbf{d}_1 \rfloor = (k^{-1}, b; c, d),$$

and similarly,

$$\lfloor \mathbf{a}, \mathbf{k}^{-1}; \mathbf{c}, \mathbf{d} \rfloor = \left(a, k^{-1}; c, d\right),$$
$$\lfloor \mathbf{a}, \mathbf{b}; \mathbf{k}^{-1}, \mathbf{d} \rfloor = \left(a, b; k^{-1}, d\right),$$
$$\lfloor \mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{k}^{-1} \rfloor = \left(a, b; c, k^{-1}\right).$$

Case (c). Let $\mathbf{a} = [1, w - za, a]$, $\mathbf{b} = [1, w - zb, b]$, $\mathbf{c} = [1, w - zc, c]$, $\mathbf{d} = [1, w - zd, d]$ and $\mathbf{k}^{-1} = [k, z - wk, 1]$ be pairwise non-neighbour lines passing through $P = (w, 1, z)$. Then the perspectivity $\sigma = \sigma_{[1,1,0]}(U, P)$ transforms the lines $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{k}^{-1}$ to $\mathbf{a}_1 = \left[0, 1, a(1 - w + za)^{-1}\right]$, $\mathbf{b}_1 = \left[0, 1, b(1 - w + zb)^{-1}\right]$, $\mathbf{c}_1 = \left[0, 1, c(1 - w + zc)^{-1}\right]$, $\mathbf{d}_1 = \left[0, 1, d(1 - w + zd)^{-1}\right]$ and $\mathbf{k}_1^{-1} = [0, (1 - w)k + z, 1]$, respectively. Therefore, with $\gamma = i \circ l_{(1-w)^{-1}} \circ t_{-z} \circ i \in \Lambda$, we have

$$\begin{aligned}
\lfloor \mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d} \rfloor &= \lfloor \mathbf{a}_1, \mathbf{b}_1; \mathbf{c}_1, \mathbf{d}_1 \rfloor \\
&= \Big(a(1 - w + za)^{-1}, b(1 - w + zb)^{-1}; c(1 - w + zc)^{-1}, \\
&\qquad d(1 - w + zd)^{-1}\Big) \\
&= \Big(\gamma\left(a(1 - w + za)^{-1}\right), \gamma\left(b(1 - w + zb)^{-1}\right); \gamma\left(c(1 - w + zc)^{-1}\right) \\
&\qquad , \gamma\left(d(1 - w + zd)^{-1}\right)\Big) \\
&= (a, b; c, d),
\end{aligned}$$

and

$$\begin{aligned}
\lfloor \mathbf{k}^{-1}, \mathbf{b}; \mathbf{c}, \mathbf{d} \rfloor &= \lfloor \mathbf{k}_1^{-1}, \mathbf{b}_1; \mathbf{c}_1, \mathbf{d}_1 \rfloor \\
&= \Big(((1 - w)k + z)^{-1}, b(1 - w + zb)^{-1}; c(1 - w + zc)^{-1}, \\
&\qquad d(1 - w + zd)^{-1}\Big) \\
&= \Big(\gamma\left(((1 - w)k + z)^{-1}\right), \gamma\left(b(1 - w + zb)^{-1}\right); \\
&\qquad \gamma\left(c(1 - w + zc)^{-1}\right), \gamma\left(d(1 - w + zd)^{-1}\right)\Big) \\
&= (k^{-1}, b; c, d),
\end{aligned}$$

and similarly

$$\lfloor \mathbf{a}, \mathbf{k}^{-1}; \mathbf{c}, \mathbf{d} \rfloor = \left(a, k^{-1}; c, d\right)$$
$$\lfloor \mathbf{a}, \mathbf{b}; \mathbf{k}^{-1}, \mathbf{d} \rfloor = \left(a, b; k^{-1}, d\right)$$
$$\lfloor \mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{k}^{-1} \rfloor = \left(a, b; c, k^{-1}\right). \qquad \square$$

As a result of this theorem, one can easily compute the cross-ratio of any four pairwise non-neighbour concurrent lines because of the following facts:

(i) By the results of Case (a), the cross-ratio of the lines passing through the point $P = (x, y, 1)$ can be calculated by using the first coordinates of the lines not near to V and the inverse of the second coordinate of the line near to V.

(ii) By the results of Case (b), the cross-ratio the lines passing through the point $P = (1, y, z)$ can be calculated by using the last coordinates of the lines not near to V and the inverse of the second coordinate of the line neighbour to UV.

(iii) By the results of Case (c), the cross-ratio of the lines passing through the point $P = (w, 1, z)$ can be calculated by using the last coordinates of the lines near to V and the inverse of the first coordinate of the line neighbour to UV.

We now give an important theorem, the analogue of Lemma 2.4.

3.11. Theorem. *In $\mathbf{M}(\mathcal{A})$, perspectivities preserve cross-ratios.*

Proof. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be pairwise non-neighbour lines passing through a point P in $\mathbf{M}(\mathcal{A})$, $\sigma_{\mathbf{e}}(U, P)$ be the perspectivity given in Lemma 3.8 i.e.

$$\lfloor \mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d} \rfloor = \lfloor \sigma_{\mathbf{e}}(\mathbf{a}), \sigma_{\mathbf{e}}(\mathbf{b}); \sigma_{\mathbf{e}}(\mathbf{c}), \sigma_{\mathbf{e}}(\mathbf{d}) \rfloor,$$

and $\sigma_{\mathbf{h}}(U, P)$ a perspectivity such that $\mathbf{h} \not\sim \mathbf{e}$ and $\mathbf{h}$ are not near to P, U. It is sufficient to show that $\sigma_{\mathbf{h}}(U, P)$ preserves the cross-ratio. Since $\sigma = \sigma_{\mathbf{e}}\sigma_{\mathbf{h}}^{-1}$ is a projectivity of U, by Corollaries 3.5 and 3.4, it preserves the cross-ratio. Thus:

$$\begin{aligned} \lfloor \sigma_{\mathbf{h}}(\mathbf{a}), \sigma_{\mathbf{h}}(\mathbf{b}); \sigma_{\mathbf{h}}(\mathbf{c}), \sigma_{\mathbf{h}}(\mathbf{d}) \rfloor &= \lfloor \sigma_{\mathbf{e}}(\mathbf{a}), \sigma_{\mathbf{e}}(\mathbf{b}); \sigma_{\mathbf{e}}(\mathbf{c}), \sigma_{\mathbf{e}}(\mathbf{d}) \rfloor \\ &= \lfloor \mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d} \rfloor. \end{aligned}$$ □

We can state the following as a direct result of this theorem.

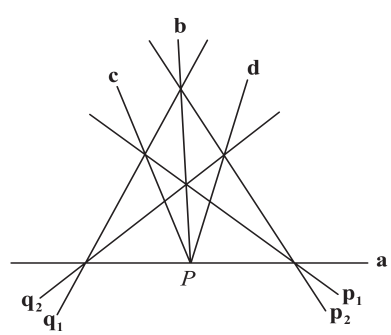
3.12. Corollary. *Cross-ratios are preserved by projectivities.* □

Now we can give the following definition in $\mathbf{M}(\mathcal{A})$, dual to the definitions of harmonicity and harmonic position given in [1, Definition 11] for points.

3.13. Definition. In $\mathbf{M}(\mathcal{A})$, any four pairwise non-neighbour lines $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ passing through P are called *harmonic* if $\lfloor \mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d} \rfloor = \langle -1 \rangle$, and we let $h(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ represent the statement: $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are harmonic.

Let P be any point in $\mathbf{M}(\mathcal{A})$. Then the pairwise non-neighbour lines $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ passing through P are said to be in *harmonic position* if there exists a quadrilateral ($\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}_1$, $\mathbf{q}_2$) such that $\mathbf{p}_1\mathbf{p}_2 \cup \mathbf{q}_1\mathbf{q}_2 = \mathbf{a}$, $\mathbf{p}_1\mathbf{q}_2 \cup \mathbf{p}_2\mathbf{q}_1 = \mathbf{b}$, $\mathbf{p}_1\mathbf{q}_1 \cup P = \mathbf{c}$, and $\mathbf{p}_2\mathbf{q}_2 \cup P = \mathbf{d}$, (see Figure 1). We let $H(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ represent the statement: $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are in harmonic position.

Figure 1



Now we can state consecutively two lemmas and a theorem which are necessary for the proof of Theorem 3.18. Their proofs can be easily obtained from the proofs of theorems given in [1], by using the principle of duality.

3.14. Lemma. [1, Lemma 12] *In* $\mathbf{M}(\mathcal{A})$*, if* f_1, f_2 *are elations with axis* $\mathbf{p}$ *and* $f_1(\mathbf{q}) = f_2(\mathbf{q}) \neq \mathbf{q}$ *for some lines* $\mathbf{q} \nsim \mathbf{p}$*, then* $f_1|_{\mathbf{pq}} = f_2|_{\mathbf{pq}}$. □

3.15. Lemma. [1, Lemma 13] *In* $\mathbf{M}(\mathcal{A})$*, if* $H(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ *then there exist a point* P *and an elation* f *with axis* $\mathbf{a}$*, center* P*, such that* $f : \mathbf{c}, \mathbf{b} \to \mathbf{b}, \mathbf{d}$. □

3.16. Theorem. [1, Theorem 14] *In* $\mathbf{M}(\mathcal{A})$*, if* $H(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ *and* $H(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}')$*, then* $\mathbf{d} = \mathbf{d}'$. □

As a result of Theorem 3.16, we have the following:

3.17. Corollary. *If* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{c} \in \mathcal{L}_U$ *are pairwise non-neighbour lines and* $\mathbf{d}$ *is constructed from* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{c}$*,* $\mathbf{p}_1$*,* $\mathbf{p}_2$ *where* $\mathbf{p}_1$*,* $\mathbf{p}_2$ *are not near to* U *and* $\mathbf{p}_1 \nsim \mathbf{p}_2$ *via the configuration in Figure 1, then the line* $\mathbf{d}$ *is uniquely determined by* $\mathbf{a}$*,* $\mathbf{b}$ *and* $\mathbf{c}$*. That is, the line* $\mathbf{d}$ *is independent of the choice of* $\mathbf{p}_1$ *and* $\mathbf{p}_2$. □

For any $x \in \mathcal{A}$, $\langle x\rangle^{-1}$ and $1 - \langle x\rangle$ are defined in the obvious way as $\langle x^{-1}\rangle$ and $\langle 1-x\rangle$, respectively. In this situation we can give the following results about cross-ratios of lines, analogous to the results obtained in [1] for cross-ratios of points (which were discovered by Möbius for the real projective plane [13, p. 152]).

$$
\begin{aligned}
\lfloor \mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d}\rfloor &= \lfloor \mathbf{b}, \mathbf{a}; \mathbf{d}, \mathbf{c}\rfloor = \lfloor \mathbf{c}, \mathbf{d}; \mathbf{a}, \mathbf{b}\rfloor = \lfloor \mathbf{d}, \mathbf{c}; \mathbf{b}, \mathbf{a}\rfloor = \langle w\rangle \\
\lfloor \mathbf{b}, \mathbf{a}; \mathbf{c}, \mathbf{d}\rfloor &= \lfloor \mathbf{a}, \mathbf{b}; \mathbf{d}, \mathbf{c}\rfloor = \lfloor \mathbf{d}, \mathbf{c}; \mathbf{a}, \mathbf{b}\rfloor = \lfloor \mathbf{c}, \mathbf{d}; \mathbf{b}, \mathbf{a}\rfloor = \langle w\rangle^{-1} \\
\lfloor \mathbf{a}, \mathbf{c}; \mathbf{b}, \mathbf{d}\rfloor &= \lfloor \mathbf{b}, \mathbf{d}; \mathbf{a}, \mathbf{c}\rfloor = \lfloor \mathbf{c}, \mathbf{a}; \mathbf{d}, \mathbf{b}\rfloor = \lfloor \mathbf{d}, \mathbf{b}; \mathbf{c}, \mathbf{a}\rfloor = 1 - \langle w\rangle \\
\lfloor \mathbf{b}, \mathbf{c}; \mathbf{a}, \mathbf{d}\rfloor &= \lfloor \mathbf{a}, \mathbf{d}; \mathbf{b}, \mathbf{c}\rfloor = \lfloor \mathbf{d}, \mathbf{a}; \mathbf{c}, \mathbf{b}\rfloor = \lfloor \mathbf{c}, \mathbf{b}; \mathbf{d}, \mathbf{a}\rfloor = 1 - \langle w\rangle^{-1} \\
\lfloor \mathbf{c}, \mathbf{a}; \mathbf{b}, \mathbf{d}\rfloor &= \lfloor \mathbf{d}, \mathbf{b}; \mathbf{a}, \mathbf{c}\rfloor = \lfloor \mathbf{a}, \mathbf{c}; \mathbf{d}, \mathbf{b}\rfloor = \lfloor \mathbf{b}, \mathbf{d}; \mathbf{c}, \mathbf{a}\rfloor = \langle 1 - w\rangle^{-1} \\
\lfloor \mathbf{c}, \mathbf{b}; \mathbf{a}, \mathbf{d}\rfloor &= \lfloor \mathbf{d}, \mathbf{a}; \mathbf{b}, \mathbf{c}\rfloor = \lfloor \mathbf{a}, \mathbf{d}; \mathbf{c}, \mathbf{b}\rfloor = \lfloor \mathbf{b}, \mathbf{c}; \mathbf{d}, \mathbf{a}\rfloor = \langle 1 - w^{-1}\rangle^{-1}
\end{aligned} \tag{3.1}
$$

where $w \in \lfloor \mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d}\rfloor$. Hence, there exist at most six different values of the cross-ratio, depending on the order of the lines. We will need the results (3.1) in the proof of the following theorem which is the analogue of [1, Theorem 16].

3.18. Theorem. *In* $\mathbf{M}(\mathcal{A})$*,* $H(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ *if and only if* $h(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$*, where* $\mathbf{a}$*,* $\mathbf{b}$*,* $\mathbf{c}$*,* $\mathbf{d} \in \mathcal{L}_U$.

Proof. Suppose that the lines $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in \mathcal{L}_U$ are in harmonic position and also firstly that none of them is neighbour to UV. Then we will show that $\lfloor \mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d}\rfloor = \langle -1\rangle$.

Let $\mathbf{a} = [0, 1, a]$, $\mathbf{b} = [0, 1, b]$, $\mathbf{c} = [0, 1, c]$, $\mathbf{d} = [0, 1, d]$. Without loss of generality, by Corollary 3.17, we may assume that $\mathbf{p}_1 = \mathbf{g}$ and $\mathbf{p}_2 = [1, 1, a]$. Then $\mathbf{bp}_2 = (b - a, b, 1)$, $\mathbf{bp}_2 \cup \mathbf{cp}_1 = \mathbf{q}_1 = \left[(a - b)^{-1}(c - b), 1, c\right]$, and

$$\mathbf{bp}_1 \cup \mathbf{aq}_1 = \mathbf{q}_2 = \left[\left(\left((a - b)^{-1}(c - b)\right)(a - c)^{-1}\right)(a - b), 1, b\right].$$

Since $\mathbf{dp}_2 = (d - a, d, 1)$ and $\mathbf{dp}_2 \in \mathbf{q}_2$, we have

$$
\begin{aligned}
&(d - a)\left(\left(\left((a - b)^{-1}(c - b)\right)(a - c)^{-1}\right)(a - b)\right) + b = d \\
&\quad\Longrightarrow \left(\left((a - b)^{-1}(c - b)\right)(a - c)^{-1}\right)(a - b) = (d - a)^{-1}(d - b) \\
&\quad\Longrightarrow \left((a - b)^{-1}(c - b)\right)(a - c)^{-1} = \left((d - a)^{-1}(d - b)\right)(a - b)^{-1} \\
&\quad\Longrightarrow \left((a - b)^{-1}(c - a + a - b)\right)(a - c)^{-1} = \left((d - a)^{-1}(d - a + a - b)\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times (a - b)^{-1} \\
&\quad\Longrightarrow \left((a - b)^{-1}(c - a) + 1\right)(a - c)^{-1} = \left(1 + (d - a)^{-1}(a - b)\right)(a - b)^{-1} \\
&\quad\Longrightarrow (b - a)^{-1} + (a - c)^{-1} = (a - b)^{-1} + (d - a)^{-1}
\end{aligned}
$$

$$\begin{aligned}
&\Longrightarrow\ (b-a)^{-1}-(d-a)^{-1}=(a-b)^{-1}-(a-c)^{-1}\\
&\Longrightarrow\ -\left((a-b)^{-1}-(a-d)^{-1}\right)=(a-b)^{-1}-(a-c)^{-1}\\
&\Longrightarrow\ \left((a-b)^{-1}-(a-d)^{-1}\right)\left((a-b)^{-1}-(a-c)^{-1}\right)^{-1}=-1,
\end{aligned}$$

and the last equality (by Corollary 3.2) means that $h(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$.

Secondly, if $\mathbf{p}_2\mathbf{q}_2\cup U\sim UV$, then $\mathbf{d}=[0,n,1]$ and

$$\mathbf{p}_2\mathbf{q}_2=(1,1+na,n),$$

where $n=\left(-2(a-b)^{-1}-(c-a)^{-1}\right)\in\mathbf{I}$. In this case, by the definition of the cross-ratio, we get

$$\begin{aligned}
\lfloor\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d}\rfloor&=(a,b;c,n^{-1})\\
&=\left\langle\left((1-na)^{-1}(1-nb)\right)\left((b-c)^{-1}(a-c)\right)\right\rangle\\
&=\left\langle\left((1+na)(1-nb)\right)\left((b-c)^{-1}(a-c)\right)\right\rangle\\
&=\left\langle\left(1+n(a-b)\right)\left((b-c)^{-1}(a-c)\right)\right\rangle\\
&=\left\langle\left(1+n(a-b)\right)\left((b-c)^{-1}(a-c)\right)\right\rangle,
\end{aligned}$$

and then by substituting n in the last equality we obtain

$$\begin{aligned}
\lfloor\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d}\rfloor&=\langle\left(1+\left(-2(a-b)^{-1}-(c-a)^{-1}\right)(a-b)\right)\left((b-c)^{-1}(a-c)\right)\rangle\\
&=\langle-\left(1+(c-a)^{-1}(a-b)\right)\left((b-c)^{-1}(a-c)\right)\rangle\\
&=\langle-\left(1+(c-a)^{-1}((a-c)+(c-b))\right)\left((b-c)^{-1}(a-c)\right)\rangle\\
&=\langle-\left((c-a)^{-1}(c-b)\right)\left((b-c)^{-1}(a-c)\right)\rangle\\
&=\langle-\left((c-a)^{-1}(b-c)\right)\left((b-c)^{-1}(c-a)\right)\rangle\\
&=\langle-1\rangle.
\end{aligned}$$

This means that $h(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$, even if $\mathbf{d}$ is neighbour to UV. If any one of the lines $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is neighbour to UV then the proof of this part follows from (3.1).

Conversely, let $h(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$. Existence of the point $\mathbf{d}'$ such that $H(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}')$ is obvious from Definition 3.13. Then $H(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}')$ implies $h(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}')$ (from the first part of the theorem). So, we have $h(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}')$ and $h(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$. Finally, by Corollary 3.3, we have $\mathbf{d}=\mathbf{d}'$, which gives $H(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$. □

We can also give a proof of the last theorem in an alternative way. All of the following results are concerned with this.

3.19. Theorem. *$H(A,B,C,D)$ iff $H(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$, where A,B,C,D are four pairwise non-neighbour collinear points and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are such that $A\in\mathbf{a}$, $B\in\mathbf{b}$, $C\in\mathbf{c}$, $D\in\mathbf{d}$ are four pairwise non-neighbour concurrent lines.*

Proof. Let $H(A,B,C,D)$, where $A,B,C,D\in\mathbf{q}$. Let $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$ be concurrent lines on R, where R is not near to $\mathbf{q}$. Then we can choose a point $Q\in\mathbf{b}$ where Q is not neighbour to B and R. In this case, if $P:=\mathbf{c}\cap AQ$, then it is clear that P is not neighbour to Q and R. In this case if $S:=\mathbf{a}\cap BP$, then it is clear that S is not neighbour to P, Q and R. So, we have obtained a 4-gon (P,Q,R,S). Then it must be that $D=\mathbf{q}\cap SQ$ since $H(A,B,C,D)$. If $\mathbf{p}:=AQ$, $\mathbf{r}:=BS$, $\mathbf{s}:=DS$, then we have found a quadrilateral $(\mathbf{p},\mathbf{q},\mathbf{r},\mathbf{s})$ embedded in a configuration like the one of Figure 1. This implies $H(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$. The remainder of the proof follows from the principle of duality. □

Combining the results of the last Theorem, [1, Theorem 16] and Theorem 3.1, we immediately obtain the following:

3.20. Corollary.

$$H(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) \iff H(A,B,C,D) \iff h(A,B,C,D) \iff h(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}),$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in \mathcal{L}_U$ *and* A, B, C, $D \in \mathbf{g}$. □

3.21. Remark. Corollary 3.20 is also valid for concurrent lines passing through any point of $\mathbf{M}(\mathcal{A})$ if we can show that $H(A,B,C,D)$ iff $h(A,B,C,D)$, where A,B,C,D are points on any line of $\mathbf{M}(\mathcal{A})$. A paper related to the derivation of this result is under review. So, we have completely given the relation between harmonicity and harmonic position, both for concurrent lines and collinear points.

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