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CERTAIN CURVATURE CONDITIONS ON AN LP-SASAKIAN MANIFOLD WITH A COEFFICIENT α

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Abstract. The object of the present paper is to study certain curvature restriction on an LP-Sasakian manifold with a coefficient α. Among others it is shown that if an LP-Sasakian manifold with a coefficient α is a manifold of constant curvature, then the manifold is the product manifold. Also it is proved that a 3-dimensional Ricci semisymmetric LP-Sasakian manifold with a constant coefficient α is a spaceform.

1. Introduction

In 1989, Matsumoto [6] introduced the notion of LP-Sasakian manifolds. Then Mihai and Rosca [7] introduced the same notion independently and they obtained several results in this manifold. In a recent paper, De, Shaikh, and Sengupta [3] introduced the notion of LP-Sasakian manifolds with a coefficient α which generalizes the notion of LP-Sasakian manifolds. Recently, T. Ikawa and his coauthors [4], [5] studied Sasakian manifolds with Lorentzian metric and obtained several results in this manifold. The object of the present paper is to study certain curvature restriction on an LP-Sasakian manifold with a coefficient α. After preliminaries, in Section 3 it is shown that if an LP-Sasakian manifold M^n with a coefficient α is of constant curvature, then the vector field ξ is a concircular vector field and as an important consequence of this theorem we prove that such a manifold is the product manifold. In the last section we study a 3-dimensional LP-Sasakian manifold with a constant coefficient α.

2. Preliminaries

Let M^n be an n-dimensional differentiable manifold endowed with a (1, 1) tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g of type (0, 2) such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature

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$(-,+,+,\ldots,+)$, where T_pM denotes the tangent vector space of M at p and R is the real number space, which satisfies

$$\eta(\xi) = -1, \quad \phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$g(X,\ \xi) = \eta(X), \quad g(\phi\, X,\ \phi Y) = g(X,\ Y) + \eta(X)\eta(Y) \tag{2.2}$$

for all vectors fields X and Y. Then such a structure (ϕ, ξ, η, g) is termed as Lorentzian almost paracontact structure and the manifold M^n with the structure (ϕ, ξ, η, g) is called Lorentzian almost paracontact manifold [6]. In a Lorentzian almost paracontact manifold M^n, the following relations hold good [6]:

$$\phi\,\xi = 0, \quad \eta(\phi\, X) = 0, \tag{2.3}$$

$$\Omega(X,Y) = \Omega(Y,X), \text{ where } \Omega = g(X,\phi Y). \tag{2.4}$$

In the Lorentzian almost paracontact manifold M^n, if the relations

$$\begin{aligned}(\nabla_Z \Omega)(X,Y) = \alpha[&\{g(X,Z) + \eta(X)\eta(Z)\}\eta(Y) \\ &+ \{g(Y,Z) + \eta(Y)\eta(Z)\}\eta(X)], \quad (\alpha \neq 0)\end{aligned} \tag{2.5}$$

$$\Omega(X,Y) = \frac{1}{\alpha}(\nabla_X\, \eta)(Y), \tag{2.6}$$

hold where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g, then M^n is called an LP-Sasakian manifold with a coefficient α [3]. An LP-Sasakian manifold with a coefficient 1 is an LP-Sasakian manifold [6]. If a vector field V satisfies the equation of the following form:

$$\nabla_X V = \beta X + T(X)V,$$

where β is a non-zero scalar function and T is a non-zero 1-form, then V is called a torse-forming vector field [9]. In a Lorentzian manifold M^n, if we assume that ξ is a unit torse-forming vector field, then we have the equation:

$$(\nabla_X \eta)(Y) = \alpha[g(X,Y) + \eta(X)\eta(Y)], \tag{2.7}$$

where α is a non-zero scalar function. Hence the manifold admitting a unit torse-forming vector field satisfying (2.7) is an LP-Sasakian manifold with a coefficient α. Especially, if η satisfies

$$(\nabla_X \eta)(Y) = \epsilon[g(X,Y) + \eta(X)\eta(Y)], \quad \epsilon^2 = 1, \tag{2.8}$$

then M^n is called an LSP-Sasakian manifold [6]. In particular, if α satisfies (2.7) and the equation of the form:

$$\alpha(X) = p\eta(X), \quad \alpha(X) = \nabla_X \alpha, \tag{2.9}$$

where p is a scalar function. Then ξ is called a concircular vector field. A Riemannian manifold satisfying the condition $\nabla S = 0$, where S denotes the Ricci tensor is called Ricci-symmetric. A Riemannian manifold satisfying the condition $R(X,Y).S = 0$ is called Ricci-symmetric [8] where $R(X,Y)$ denotes the derivation of the tensor algebra at each point of the tangent space. Let us consider an LP-Sasakian manifold M^n $(\phi,\ \xi,\ \eta,\ g)$ with a coefficient α. Then we have the following relations [3]:

$$\eta(R(X,Y)Z) = -\alpha(X)\Omega(Y,Z) + \alpha(Y)\Omega(X,Z) + \alpha^2\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \tag{2.10}$$

$$S(X,\ \xi) = -\psi\alpha(X) + (n-1)\alpha^2\eta(X) + \alpha(\phi X), \tag{2.11}$$

where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold and $\psi = \mathrm{Trace}(\phi)$. We state the following results which will be needed in latter sections.

Lemma 2.1 ([3]). *In an LP-Sasakian manifold M^n with a non-constant coefficient α, one of the following cases occurs*:
(i) $\psi^2 = (n-1)^2$.
(ii) $\alpha(Y) = -p\eta(Y)$, *where* $p = \alpha(\xi)$.

Lemma 2.2 ([3]). *In a Lorentzian almost paracontact manifold M^n with structure $(\phi,\ \xi,\ \eta,\ g)$ satisfying $\Omega(X,Y) = \frac{1}{\alpha}(\nabla_X\eta)(Y)$, where α is a non-zero scalar function, the vector field ξ is torse-forming if and only if the relation $\psi^2 = (n-1)^2$ holds good.*

3. LP-Sasakian manifolds with a coefficient α which is of constant curvature

We consider an LP-Sasakian manifold which is of constant curvature. Then we have

$$R(X,Y,Z,W) = \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \tag{3.1}$$

From (3.1) we have

$$S(Y,Z) = \frac{r}{n}g(Y,Z) \tag{3.2}$$

which implies that the manifold is Einstein and hence the scalar curvature r of the manifold is given by [3]

$$r = n\{p\psi + (n-1)\alpha^2\}. \tag{3.3}$$

Putting $Z = \xi$ in (3.2) we have by virtue of (2.11)

$$\alpha(\phi Y) = \psi\alpha(Y) + \left\{\frac{r}{n} - (n-1)\alpha^2\right\}\eta(Y). \tag{3.4}$$

Again from (3.1) we have by virtue of (2.2)

$$\sum_{i=1}^{n}\epsilon_i R(e_i, Z, \phi Y, \phi e_i) = \frac{r}{n(n-1)}[\psi\Omega(Y,Z) - g(Y,Z) - \eta(Y)\eta(Z)], \tag{3.5}$$

where $\{e_i\}$ is an orthonormal basis of the tangent space at any point of the manifold and $\epsilon_i = g(e_i, e_i)$. Now in an LP-Sasakian manifold with a coefficient α we have the following relation [3]:

$$\begin{aligned} & S(Y,Z) - \sum_{i=1}^{n} \epsilon_i R(e_i, Z, \phi Y, \phi e_i) \\ &= \{\psi\alpha(Z) - \alpha(\phi Z) - (2n-3)\alpha^2\eta(Z)\}\eta(Y) \\ &\quad - (n-2)\alpha^2 g(Y,Z) + (p+\psi\alpha^2)\Omega(Y,Z). \end{aligned} \tag{3.6}$$

Using (3.2), (3.3), (3.4) and (3.5) in (3.6) we obtain

$$\begin{aligned} & \left\{2(n-1)\alpha^2 + \frac{np\psi}{n-1}\right\} g(Y,Z) - \left\{2\psi\alpha^2 + (1+\frac{\psi^2}{n-1})p\right\}\Omega(Y,Z) \\ &\qquad + \left\{2(n-1)\alpha^2 + \frac{np\psi}{n-1}\right\}\eta(Y)\eta(Z) = 0. \end{aligned} \tag{3.7}$$

We consider the case when α is not constant. In this case, taking a frame field and contracting over Y and Z we obtain from (3.7) that

$$[(n-1)^2 - \psi^2]\left\{2\alpha^2 + \frac{p\psi}{n-1}\right\} = 0.$$

From this equation we find either

$$\psi^2 = (n-1)^2, \tag{3.8}$$

or

$$p\psi = -2(n-1)\alpha^2. \tag{3.9}$$

If (3.9) holds, then from (3.3) we obtain

$$r = -n(n-1)\alpha^2,$$

from which we find that α is constant, since r is constant, which contradicts our assumption that α is non constant.

On the other hand, from (3.8) by virtue of Lemma 2.2 we conclude that ξ is torse-forming. We have that

$$(\nabla_X\eta)(Y) = \beta\{g(X,Y) + \eta(X)\eta(Y)\}.$$

Then from (2.6) we get

$$\begin{aligned} \Omega(X,Y) &= \frac{\beta}{\alpha}\{g(X,Y) + \eta(X)\eta(Y)\} \\ &= g\left(\frac{\beta}{\alpha}(X + \eta(X)\xi),\ Y\right) \\ \text{and} \quad \Omega(X,Y) &= g(\phi X, Y). \end{aligned}$$

Now from (3.4) and using $\phi(X) = X + \eta(X)\xi$ we obtain

$$\alpha(Y + \eta(Y)\xi) = \psi\alpha(Y) + \left\{\frac{n(p\psi + (n-1)\alpha^2)}{n} - (n-1)\alpha^2\right\}\eta(Y)$$

or, $\alpha(Y) + p\eta(Y) = \psi\alpha(Y) + p\psi\eta(Y)$

or, $\alpha(Y) - \psi\alpha(Y) = p\psi\eta(Y) - p\eta(Y)$

or, $(1-\psi)\alpha(Y) = p(-1+\psi)\eta(Y)$

or, $$\alpha(Y) = p\left(\frac{-1+\psi}{1-\psi}\right)\eta(Y) = -p\eta(Y).$$

In a similar way using $\phi(X) = -X+\eta(X)\xi$ in (3.4) we obtain $\alpha(X) = -p\eta(Y)$.

Since g is non-singular, we have

$$\phi(X) = \frac{\beta}{\alpha}(X + \eta(X)\xi)$$

and

$$\phi^2(X) = \left(\frac{\beta}{\alpha}\right)^2 (X + \eta(X)\xi).$$

It follows from (2.1) that $(\frac{\beta}{\alpha})^2 = 1$ and hence, $\alpha = \pm\beta$. Thus we have

(3.10) $$\phi(X) = \pm(X + \eta(X)\xi.$$

Thus in both the cases we obtain

$$\alpha(Y) = -p\eta(Y).$$

Hence we can state the following:

Theorem 3.1. *If an LP-Sasakian manifold M^n with a coefficient α is a manifold of constant curvature, then the vector field ξ is a concircular vector field.*

Again since ξ is a concircular vector field, we have

(3.11) $$\nabla_X\xi = \alpha[X + \eta(X)\xi],$$

where $\alpha(Y) = p\eta(Y)$, where p is a scalar function.

Let $\xi^\perp$ denote the $(n-1)$-dimensional distribution in an LP-Sasakian manifold with coefficient α orthogonal to ξ. If X and Y belong to $\xi^\perp$, where $Y \neq \lambda X$, then

(3.12) $$g(X,\xi) = 0$$

and

(3.13) $$g(Y,\xi) = 0.$$

Since $(\nabla_X g)(Y,\xi) = 0$, it follows from (3.11) and (3.13) that

$$g(\nabla_X Y,\xi) = g(\nabla_X\xi, Y) = \alpha g(X,Y).$$

Similarly, we get

$$g(\nabla_Y X,\xi) = g(\nabla_Y\xi, X) = \alpha g(X,Y).$$

Hence

(3.14) $$g(\nabla_X Y,\xi) = g(\nabla_Y X,\xi).$$

Now $[X,Y] = \nabla_X Y - \nabla_Y X$. Therefore

$$g([X,Y],\xi) = g(\nabla_X Y - \nabla_Y X, \xi) = 0 \text{ by } (3.14).$$

Hence $[X,Y]$ is orthogonal to ξ, i.e., $[X,Y]$ belong to $\xi^{\perp}$. Thus the distribution $\xi^{\perp}$ is involutive [2]. Hence from Frobenius' theorem [2] it follows that $\xi^{\perp}$ is integrable. This implies that if an LP-Sasakian manifold with a coefficient α is a manifold of constant curvature, then it is a product manifold. We can therefore state the following theorem.

Theorem 3.2. *If an LP-Sasakian manifold with a coefficient α is a manifold of constant curvature, then the manifold is the product manifold.*

4. 3-dimensional LP-Sasakian manifold with a constant coefficient α

Let us consider a 3-dimensional LP-Sasakian manifold with a constant coefficient α. In a 3-dimensional Riemannian manifold we have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{4.1}$$

where Q is the Ricci operator, i.e., $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold.

Since α is constant and dimension of the manifold is 3, equations (2.10) and (2.11) reduce to

$$\eta(R(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{4.2}$$

$$S(X,\xi) = 2\,\alpha^2\eta(X). \tag{4.3}$$

From (4.2) we get

$$R(X,Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y]. \tag{4.4}$$

Putting $Z = \xi$ in (4.1) and using (4.4) we have

$$\eta(Y)QX - \eta(X)QY = \left(\frac{r}{2} - \alpha^2\right)[\eta(Y)X - \eta(X)Y]. \tag{4.5}$$

Putting $Y = \xi$ in (4.5) and using (2.1) and (4.3), we get

$$QX = \frac{1}{2}\left\{(r - 2\alpha^2)X + (r - 6\alpha^2)\eta(X)\xi\right\} \tag{4.6}$$

i.e.,

$$S(X,Y) = \frac{1}{2}\left\{(r - 2\alpha^2)g(X,Y) + (r - 6\alpha^2)\eta(X)\eta(Y)\right\}.$$

An LP-Sasakian manifold is said to be a space form if the manifold is a space of constant curvature. We assume that $\psi =$ trace of $\phi \neq 0$, i.e., ξ is not harmonic [1].

Using (4.6) in (4.1), we get

$$\begin{aligned} R(X,Y)Z = \frac{r-4\alpha^2}{2}[g(Y,Z)X - g(X,Z)Y] + \frac{r-6\alpha^2}{2}[g(Y,Z)\eta(X)\xi \\ - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{4.7}$$

Hence we can state the following:

Theorem 4.1. *A 3-dimensional LP-Sasakian manifold with a constant coefficient α is a space form if and only if the scalar curvature $r = 6\alpha^2$.*

Next we consider a 3-dimensional LP-Sasakian manifold with constant coefficient α which satisfies the condition

$$R(X,Y).S = 0. \tag{4.8}$$

From (4.8) we have

$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0. \tag{4.9}$$

Again from (4.2) we get

$$R(X,\xi)Z = \alpha^2[\eta(Z)X - g(X,Z)\xi]. \tag{4.10}$$

Putting $Y = \xi$ in (4.9) and using (4.10) we get

$$\eta(U)S(X,V) - g(X,U)S(\xi,V) + \eta(V)S(U,X) - g(X,V)S(U,\xi) = 0. \tag{4.11}$$

Since $\alpha^2 \neq 0$ using (4.3) in (4.11) we have

$$\eta(U)S(X,V) - 2\alpha^2 g(X,U)\eta(V) + \eta(V)S(U,X) - 2\alpha^2 g(X,V)\eta(V) = 0. \tag{4.12}$$

Taking a frame field and contracting over X and U from (4.12) we obtain

$$S(\xi,V) - 8\alpha^2\eta(V) + r\eta(V) = 0. \tag{4.13}$$

Using (4.3) in (4.13) we obtain

$$(r - 6\alpha^2)\eta(V) = 0.$$

This gives $r = 6\alpha^2$ (since $\eta(V) \neq 0$), which implies by Theorem 4.1 that the manifold is a space form.

Hence we can state the following:

Theorem 4.2. *A 3-dimensional Ricci semi-symmetric LP-Sasakian manifold with a constant coefficient α is a space form.*

Since $\nabla S = 0$ implies $R(X,Y).S = 0$, we get the following:

Corollary. *A 3-dimensional Ricci symmetric LP-Sasakian manifold with a constant coefficient α is a space form.*

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