Available online at www.sciencedirect.com

SCIENCE DIRECT®

Applied Mathematics Letters 18 (2005) 191–198

Applied Mathematics Letters

www.elsevier.com/locate/aml

A new class of Salagean-type harmonic univalent functions

Sibel Yalçin

Uludağ Üniversitesi, Fen Ed. Fak. Matematik, Bölümü 16059, Bursa, Turkey

Received 1 October 2003; received in revised form 1 January 2004; accepted 1 May 2004

Abstract

We define and investigate a new class of Salagean-type harmonic univalent functions. We obtain coefficient conditions, extreme points, distortion bounds, convex combination and radii of convex for the above class of harmonic univalent functions.

MSC: 30C45; 30C50; 31A05

Keywords: Harmonic univalent functions; Salagean derivative

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain $\mathfrak{D}$ is said to be harmonic in $\mathfrak{D}$ if both u and v are real harmonic in $\mathfrak{D}$. In any simply connected domain we can write $f = h + \bar{g}$, where h and g are analytic in $\mathfrak{D}$. A necessary and sufficient condition for f to be locally univalent and sense preserving in $\mathfrak{D}$ is that $|h'(z)| > |g'(z)|$, $z \in \mathfrak{D}$.

Denote by S_H the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1}$$

E-mail address: skarpuz@uludag.edu.tr.

0893-9659/$ - see front matter
doi:10.1016/j.aml.2004.05.003

In 1984 Clunie and Sheil-Small [2] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses.

The differential operator D^m was introduced by Salagean [5]. For $f = h + \bar{g}$ given by (1), Jahangiri et al. [4] defined the modified Salagean operator of f as

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)} \tag{2}$$

where

$$D^m h(z) = z + \sum_{k=2}^{\infty} k^m a_k z^k \qquad \text{and} \qquad D^m g(z) = \sum_{k=1}^{\infty} k^m b_k z^k.$$

For $0 \le \alpha < 1$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$ and $z \in U$, we let $S_H(m, n; \alpha)$ denote the family of harmonic functions f of the form (1) such that

$$\operatorname{Re}\left\{\frac{D^m f(z)}{D^n f(z)}\right\} > \alpha \tag{3}$$

where $D^m f$ is defined by (2).

We let the subclass $\bar{S}_H(m, n; \alpha)$ consist of harmonic functions $f_m = h + \bar{g}_m$ in $\bar{S}_H(m, n; \alpha)$ so that h and g_m are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad g_m(z) = (-1)^{m-1} \sum_{k=1}^{\infty} b_k z^k; \quad a_k, b_k \ge 0. \tag{4}$$

The class $\bar{S}_H(m, n; \alpha)$ includes a variety of well-known subclasses of S_H. For example, $\bar{S}_H(1, 0; \alpha) \equiv \mathcal{F}(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are starlike of order α in U, $\bar{S}_H(2, 1; \alpha)$ is the class of sense-preserving, harmonic univalent functions f which are convex of order α in U, and $\bar{S}_H(n + 1, n; \alpha) \equiv \bar{H}(n, \alpha)$ is the class of Salagean-type harmonic univalent functions.

For the harmonic functions f of the form (1) with $b_1 = 0$, Avcı and Zlotkiewicz [1] showed that if $\sum_{k=2}^{\infty} k^2(|a_k| + |b_k|) \le 1$ then $f \in HK$, and Silverman [6] proved that the above coefficient condition is also necessary if $f = h + \bar{g}$ has negative coefficients. Later, Silverman and Silvia [7] improved the results of [1,6] to the case b_1 not necessarily zero.

For the harmonic functions f of the form (4) with $m = 1$, Jahangiri [3] showed that $f \in \mathcal{F}(\alpha)$ if and only if $\sum_{k=2}^{\infty}(k - \alpha)|a_k| + \sum_{k=1}^{\infty}(k + \alpha)|b_k| \le 1 - \alpha$ and $f \in \bar{S}_H(2, 1; \alpha)$ if and only if $\sum_{k=2}^{\infty} k(k - \alpha)|a_k| + \sum_{k=1}^{\infty} k(k + \alpha)|b_k| \le 1 - \alpha$. In this note, we extend the above results to the families $S_H(m, n; \alpha)$ and $\bar{S}_H(m, n; \alpha)$. We also obtain extreme points, distortion bounds, convolution conditions, and convex combinations for $\bar{S}_H(m, n; \alpha)$.

2. Main results

We begin with a sufficient coefficient condition for functions in $S_H(m, n; \alpha)$.

Theorem 1. *Let $f = h + \bar{g}$ be so that h and g are given by (1). Furthermore, let*

$$\sum_{k=1}^{\infty}\left(\frac{k^m - \alpha k^n}{1 - \alpha}|a_k| + \frac{k^m - (-1)^{m-n}\alpha k^n}{1 - \alpha}|b_k|\right) \le 2 \tag{5}$$

where $a_1 = 1$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$ and $0 \leq \alpha < 1$; then f is sense-preserving, harmonic univalent in U and $f \in S_H(m, n; \alpha)$.

Proof. If $z_1 \neq z_2$, then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1 - \left|\frac{\sum_{k=1}^{\infty} b_k(z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k(z_1^k - z_2^k)}\right|$$
$$> 1 - \frac{\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=2}^{\infty} k|a_k|} \geq 1 - \frac{\sum_{k=1}^{\infty} \frac{k^m-(-1)^{m-n}\alpha k^n}{1-\alpha}|b_k|}{1 - \sum_{k=2}^{\infty} \frac{k^m-\alpha k^n}{1-\alpha}|a_k|} \geq 0$$

which proves univalence. Note that f is sense preserving in U. This is because

$$|h'(z)| \geq 1 - \sum_{k=2}^{\infty} k|a_k||z|^{k-1} > 1 - \sum_{k=2}^{\infty} \frac{k^m - \alpha k^n}{1-\alpha}|a_k| \geq \sum_{k=1}^{\infty} \frac{k^m - (-1)^{m-n}\alpha k^n}{1-\alpha}|b_k|$$
$$> \sum_{k=1}^{\infty} \frac{k^m - (-1)^{m-n}\alpha k^n}{1-\alpha}|b_k||z|^{k-1} \geq \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \geq |g'(z)|.$$

Using the fact that $\operatorname{Re} w > \alpha$ if and only if $|1 - \alpha + w| > |1 + \alpha - w|$, it suffices to show that

$$|(1-\alpha)D^n f(z) + D^m f(z)| - |(1+\alpha)D^n f(z) - D^m f(z)| > 0. \tag{6}$$

Substituting for $D^n f(z)$ and $D^m f(z)$ in (6) yields, by (5) we obtain

$$|(1-\alpha)D^n f(z) + D^m f(z)| - |(1+\alpha)D^n f(z) - D^m f(z)| > 0$$
$$= \left|(2-\alpha)z + \sum_{k=2}^{\infty}[k^n(1-\alpha) + k^m]a_k z^k + (-1)^n \sum_{k=1}^{\infty}[k^n(1-\alpha) + (-1)^{m-n}k^m]\overline{b_k z^k}\right|$$
$$- \left|\alpha z + \sum_{k=2}^{\infty}[k^m - (1+\alpha)k^n]a_k z^k - (-1)^n \sum_{k=1}^{\infty}[(-1)^{m-n}k^m - (1+\alpha)k^n]\overline{b_k z^k}\right|$$
$$\geq 2(1-\alpha)|z| - \sum_{k=2}^{\infty} 2[k^m - \alpha k^n]|a_k||z|^k - \sum_{k=1}^{\infty} |(1+\alpha)k^n + (-1)^{m-n}k^m||b_k||z|^k$$
$$- \sum_{k=1}^{\infty} |(-1)^{m-n}k^m - (1+\alpha)k^n||b_k||z|^k$$
$$= \begin{cases} 2(1-\alpha)|z| - 2\sum_{k=2}^{\infty}[k^m - \alpha k^n]|a_k||z|^k - 2\sum_{k=1}^{\infty}[k^m + \alpha k^n]|b_k||z|^k, & m-n \text{ is odd} \\ 2(1-\alpha)|z| - 2\sum_{k=2}^{\infty}[k^m - \alpha k^n]|a_k||z|^k - 2\sum_{k=1}^{\infty}[k^m - \alpha k^n]|b_k||z|^k, & m-n \text{ is even} \end{cases}$$

$$= 2(1-\alpha)|z|\left\{1-\sum_{k=2}^{\infty}\frac{k^m-\alpha k^n}{1-\alpha}|a_k||z|^{k-1}-\sum_{k=1}^{\infty}\frac{k^m-(-1)^{m-n}\alpha k^n}{1-\alpha}|b_k||z|^{k-1}\right\}$$

$$> 2(1-\alpha)\left\{1-\left(\sum_{k=2}^{\infty}\frac{k^m-\alpha k^n}{1-\alpha}|a_k|+\sum_{k=1}^{\infty}\frac{k^m-(-1)^{m-n}\alpha k^n}{1-\alpha}|b_k|\right)\right\}.$$

This last expression is non-negative by (5), and so the proof is complete. □

The harmonic univalent functions

$$f(z) = z+\sum_{k=2}^{\infty}\frac{1-\alpha}{k^m-\alpha k^n}x_k z^k+\sum_{k=1}^{\infty}\frac{1-\alpha}{k^m-(-1)^{m-n}\alpha k^n}\overline{y_k z^k}, \tag{7}$$

where $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k| = 1$, show that the coefficient bound given by (5) is sharp. The functions of the form (7) are in $S_H(m, n; \alpha)$ because

$$\sum_{k=1}^{\infty}\left(\frac{k^m-\alpha k^n}{1-\alpha}|a_k|+\frac{k^m-(-1)^{m-n}\alpha k^n}{1-\alpha}|b_k|\right) = 1+\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k| = 2.$$

In the following theorem, it is shown that the condition (5) is also necessary for functions $f_m = h+\bar{g}_m$ where h and g_m are of the form (4).

Theorem 2. *Let $f_m = h+\bar{g}_m$ be given by (4). Then $f_m \in \bar{S}_H(m, n; \alpha)$ if and only if*

$$\sum_{k=1}^{\infty}\left[(k^m-\alpha k^n)a_k+(k^m-(-1)^{m-n}\alpha k^n)b_k\right] \le 2(1-\alpha). \tag{8}$$

Proof. Since $\bar{S}_H(m, n; \alpha) \subset S_H(m, n; \alpha)$, we only need to prove the "only if" part of the theorem. To this end, for functions f_m of the form (4), we notice that the condition $\mathrm{Re}\{D^m f_m(z)/D^n f_m(z)\} > \alpha$ is equivalent to

$$\mathrm{Re}\left\{\frac{(1-\alpha)z-\sum_{k=2}^{\infty}(k^m-\alpha k^n)a_k z^k+(-1)^{2m-1}\sum_{k=1}^{\infty}(k^m-(-1)^{m-n}\alpha k^n)b_k\bar{z}^k}{z-\sum_{k=2}^{\infty}k^n a_k z^k+(-1)^{m+n-1}\sum_{k=1}^{\infty}k^n b_k\bar{z}^k}\right\} \ge 0. \tag{9}$$

The above required condition (9) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0 \le z = r < 1$, we must have

$$\frac{1-\alpha-\sum_{k=2}^{\infty}(k^m-\alpha k^n)a_k r^{k-1}-\sum_{k=1}^{\infty}(k^m-(-1)^{m-n}\alpha k^n)b_k r^{k-1}}{1-\sum_{k=2}^{\infty}k^n a_k r^{k-1}-(-1)^{m-n}\sum_{k=1}^{\infty}k^n b_k r^{k-1}} \ge 0. \tag{10}$$

If the condition (8) does not hold, then the numerator in (10) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in (0, 1) for which the quotient in (10) is negative. This contradicts the required condition for $f_m \in \bar{S}_H(m, n; \alpha)$ and so the proof is complete. □

Next we determine the extreme points of closed convex hulls of $\bar{S}_H(m, n; \alpha)$ denoted by $clco\bar{S}_H(m, n; \alpha)$.

Theorem 3. *Let f_m be given by (4). Then $f_m \in \bar{S}_H(m,n;\alpha)$ if and only if $f_m(z) = \sum_{k=1}^{\infty}(x_k h_k(z) + y_k g_{m_k}(z))$, where $h_1(z) = z$, $h_k(z) = z - \frac{1-\alpha}{k^m-\alpha k^n} z^k$, $(k = 2,3,\ldots)$, and $g_{m_k}(z) = z + (-1)^{m-1}\frac{1-\alpha}{k^m-(-1)^{m-n}\alpha k^n}\bar{z}^k$, $(k = 1,2,\ldots)$, $x_k \ge 0$, $y_k \ge 0$, $x_1 = 1 - \sum_{k=2}^{\infty}(x_k + y_k) \ge 0$. In particular, the extreme points of $\bar{S}_H(m,n;\alpha)$ are $\{h_k\}$ and $\{g_{m_k}\}$.*

Proof. Suppose

$$\begin{aligned} f_m(z) &= \sum_{k=1}^{\infty}(x_k h_k(z) + y_k g_{m_k}(z)) \\ &= \sum_{k=1}^{\infty}(x_k + y_k)z - \sum_{k=2}^{\infty}\frac{1-\alpha}{k^m-\alpha k^n}x_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty}\frac{1-\alpha}{k^m-(-1)^{m-n}\alpha k^n}y_k\bar{z}^k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty}\frac{k^m-\alpha k^n}{1-\alpha}\left(\frac{1-\alpha}{k^m-\alpha k^n}x_k\right) + \sum_{k=1}^{\infty}\frac{k^m-(-1)^{m-n}\alpha k^n}{1-\alpha}\left(\frac{1-\alpha}{k^m-(-1)^{m-n}\alpha k^n}y_k\right) \\ &\quad = \sum_{k=2}^{\infty}x_k + \sum_{k=1}^{\infty}y_k = 1 - x_1 \le 1 \end{aligned}$$

and so $f_m \in \bar{S}_H(m,n;\alpha)$.

Conversely, if $f_m \in clco\bar{S}_H(m,n;\alpha)$, then $a_k \le \frac{1-\alpha}{k^m-\alpha k^n}$ and $b_k \le \frac{1-\alpha}{k^m-(-1)^{m-n}\alpha k^n}$. Set

$$x_k = \frac{k^m-\alpha k^n}{1-\alpha}a_k, \quad (k = 2,3,\ldots), \qquad \text{and} \qquad y_k = \frac{k^m-(-1)^{m-n}\alpha k^n}{1-\alpha}b_k, \quad (k = 1,2,\ldots).$$

Then note that by Theorem 2, $0 \le x_k \le 1$, $(k = 2,3,\ldots)$ and $0 \le y_k \le 1$, $(k = 1,2,\ldots)$. We define $x_1 = 1 - \sum_{k=2}^{\infty}x_k - \sum_{k=1}^{\infty}y_k$ and note that, by Theorem 2, $x_1 \ge 0$. Consequently, we obtain $f_m(z) = \sum_{k=1}^{\infty}(x_k h_k(z) + y_k g_k(z))$ as required. □

The following theorem gives the distortion bounds for functions in $\bar{S}_H(m,n;\alpha)$ which yields a covering result for this class.

Theorem 4. *Let $f_m \in \bar{S}_H(m,n;\alpha)$. Then for $|z| = r < 1$ we have*

$$|f_m(z)| \le (1+b_1)r + \frac{1}{2^n}\left(\frac{1-\alpha}{2^{m-n}-\alpha} - \frac{1-(-1)^{m-n}\alpha}{2^{m-n}-\alpha}b_1\right)r^2, \quad |z| = r < 1,$$

and

$$|f_m(z)| \ge (1-b_1)r - \frac{1}{2^n}\left(\frac{1-\alpha}{2^{m-n}-\alpha} - \frac{1-(-1)^{m-n}\alpha}{2^{m-n}-\alpha}b_1\right)r^2, \quad |z| = r < 1.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f_m \in \bar{S}_H(m,n;\alpha)$. Taking the absolute value of f_m we have

$$\begin{aligned} |f_m(z)| &\le (1+b_1)r + \sum_{k=2}^{\infty}(a_k+b_k)r^k \le (1+b_1)r + \sum_{k=2}^{\infty}(a_k+b_k)r^2 \\ &= (1+b_1)r + \frac{1-\alpha}{2^n(2^{m-n}-\alpha)}\sum_{k=2}^{\infty}\frac{2^n(2^{m-n}-\alpha)}{1-\alpha}(a_k+b_k)r^2 \end{aligned}$$

$$\le (1+b_1)r + \frac{(1-\alpha)r^2}{2^n(2^{m-n}-\alpha)} \sum_{k=2}^{\infty} \left(\frac{k^m - \alpha k^n}{1-\alpha} a_k + \frac{k^m - (-1)^{m-n}\alpha k^n}{1-\alpha} b_k \right)$$

$$\le (1+b_1)r + \frac{1}{2^n} \left(\frac{1-\alpha}{2^{m-n}-\alpha} - \frac{1-(-1)^{m-n}\alpha}{2^{m-n}-\alpha} b_1 \right) r^2. \quad \square$$

The following covering result follows from the left hand inequality in Theorem 4.

Corollary 5. *Let f_m of the form (4) be so that $f_m \in \bar{S}_H(m,n;\alpha)$. Then*

$$\left\{ w : |w| < \frac{2^m - 1 - (2^n-1)\alpha}{2^m - \alpha 2^n} - \frac{2^m - 1 - (2^n - (-1)^{m-n})\alpha}{2^m - \alpha 2^n} b_1 \right\} \subset f_m(U).$$

Remark 1. For $m = 1$, $n = 0$; $m = 2$, $n = 1$ and $m = n + 1$ the above covering result given in [3,4], respectively.

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form $f_m(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{m-1} \sum_{k=1}^{\infty} b_k \bar{z}^k$ and $F_m(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^{m-1} \sum_{k=1}^{\infty} B_k \bar{z}^k$ we define the convolution of two harmonic functions f_m and F_m as

$$(f_m * F_m)(z) = f_m(z) * F_m(z) = z - \sum_{k=2}^{\infty} a_k A_k z^k + (-1)^{m-1} \sum_{k=1}^{\infty} b_k B_k \bar{z}^k. \tag{11}$$

Using this definition, we show that the class $\bar{S}_H(m,n;\alpha)$ is closed under convolution.

Theorem 6. *For $0 \le \beta \le \alpha < 1$ let $f_m \in \bar{S}_H(m,n;\alpha)$, and $F_m \in \bar{S}_H(m,n;\beta)$. Then $f_m * F_m \in \bar{S}_H(m,n;\alpha) \subset \bar{S}_H(m,n;\beta)$.*

Proof. Let $f_m(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{m-1} \sum_{k=1}^{\infty} b_k \bar{z}^k$ be in $\bar{S}_H(m,n;\alpha)$ and $F_m(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^{m-1} \sum_{k=1}^{\infty} B_k \bar{z}^k$, be in $\bar{S}_H(m,n;\beta)$. Then the convolution $f_m * F_m$ is given by (11). We wish to show that the coefficients of $f_m * F_m$ satisfy the required condition given in Theorem 2. For $F_m \in \bar{S}_H(m,n;\beta)$ we note that $A_k < 1$ and $B_k < 1$. Now, for the convolution function $f_m * F_m$ we obtain

$$\sum_{k=2}^{\infty} \frac{k^m - \beta k^n}{1-\beta} a_k A_k + \sum_{k=1}^{\infty} \frac{k^m - (-1)^{m-n}\beta k^n}{1-\beta} b_k B_k$$

$$\le \sum_{k=2}^{\infty} \frac{k^m - \beta k^n}{1-\beta} a_k + \sum_{k=1}^{\infty} \frac{k^m - (-1)^{m-n}\beta k^n}{1-\beta} b_k$$

$$\le \sum_{k=2}^{\infty} \frac{k^m - \alpha k^n}{1-\alpha} a_k + \sum_{k=1}^{\infty} \frac{k^m - (-1)^{m-n}\alpha k^n}{1-\alpha} b_k \le 1$$

since $0 \le \beta \le \alpha < 1$ and $f_m \in \bar{S}_H(m,n;\alpha)$. Therefore $f_m * F_m \in \bar{S}_H(m,n;\alpha) \subset \bar{S}_H(m,n;\beta)$. $\square$

Now we show that $\bar{S}_H(m,n;\alpha)$ is closed under convex combinations of its members.

Theorem 7. *The class $\bar{S}_H(m,n;\alpha)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, \ldots$ let $f_{m_i} \in \bar{S}_H(m,n;\alpha)$, where f_{m_i} is given by

$$f_{m_i}(z) = z - \sum_{k=2}^{\infty} a_{k_i} z^k + (-1)^{m-1} \sum_{k=1}^{\infty} |b_{k_i}| \bar{z}^k.$$

Then by (8),

$$\sum_{k=1}^{\infty}\left(\frac{k^m-\alpha k^n}{1-\alpha}a_{k_i}+\frac{k^m-(-1)^{m-n}\alpha k^n}{1-\alpha}b_{k_i}\right)\le 2. \tag{12}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \le t_i \le 1$, the convex combination of f_{m_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{m_i}(z) = z - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{k_i}\right) z^k + (-1)^{m-1}\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i b_{k_i}\right)\bar{z}^k. \tag{13}$$

Then by (12),

$$\begin{aligned}&\sum_{k=1}^{\infty}\left[\frac{k^m-\alpha k^n}{1-\alpha}\sum_{i=1}^{\infty} t_i a_{k_i}+\frac{k^m-(-1)^{m-n}\alpha k^n}{1-\alpha}\sum_{i=1}^{\infty} t_i b_{k_i}\right]\\ &\quad=\sum_{i=1}^{\infty} t_i\left\{\sum_{k=1}^{\infty}\left[\frac{k^m-\alpha k^n}{1-\alpha}a_{k_i}+\frac{k^m-(-1)^{m-n}\alpha k^n}{1-\alpha}b_{k_i}\right]\right\}\\ &\quad\le 2\sum_{i=1}^{\infty} t_i = 2.\end{aligned}$$

This is the condition required by (8) and so $\sum_{i=1}^{\infty} t_i f_{m_i}(z) \in \bar{S}_H(m, n; \alpha)$. □

Theorem 8. *If $f_m \in \bar{S}_H(m, n; \alpha)$ then f_m is convex in the disc*

$$|z| \le \min_k\left\{\frac{(1-\alpha)(1-b_1)}{k[1-\alpha-(1-(-1)^{m-n}\alpha)b_1]}\right\}^{1/(k-1)}, \quad k = 2, 3, \ldots.$$

Proof. Let $f_m \in \bar{S}_H(m, n; \alpha)$, and let r, $0 < r < 1$, be fixed. Then $r^{-1} f_m(rz) \in \bar{S}_H(m, n; \alpha)$ and we have

$$\begin{aligned}\sum_{k=2}^{\infty} k^2(a_k+b_k) &= \sum_{k=2}^{\infty} k(a_k+b_k)(kr^{k-1})\\ &\le \sum_{k=2}^{\infty}\left(\frac{k^m-\alpha k^n}{1-\alpha}a_k+\frac{k^m-(-1)^{m-n}\alpha k^n}{1-\alpha}b_k\right)kr^{k-1}\\ &\le 1-b_1\end{aligned}$$

provided

$$kr^{k-1} \le \frac{1-b_1}{1-\frac{1-(-1)^{m-n}\alpha}{1-\alpha}b_1}$$

which is true if

$$r \le \min_k\left\{\frac{(1-\alpha)(1-b_1)}{k[1-\alpha-(1-(-1)^{m-n}\alpha)b_1]}\right\}^{1/(k-1)}, \quad k = 2, 3, \ldots. \quad □$$

References

[1] Y. Avcı, E. Zlotkiewicz, On harmonic univalent mappings, Ann. Univ. Mariae Curie Sklodowska Sect. A 44 (1990) 1–7.
[2] J. Clunie, T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A I Math. 9 (1984) 3–25.
[3] J.M. Jahangiri, Harmonic functions starlike in the unit disk, J. Math. Anal. Appl. 235 (1999) 470–477.
[4] J.M. Jahangiri, G. Murugusundaramoorthy, K. Vijaya, Salagean-type harmonic univalent functions, South. J. Pure Appl. Math. 2 (2002) 77–82.
[5] G.S. Salagean, Subclass of univalent functions, Complex Analysis-Fifth Romanian Finish Seminar, Bucharest, 1, 1983, pp. 362–372.
[6] H. Silverman, Harmonic univalent function with negative coefficients, J. Math. Anal. Appl. 220 (1998) 283–289.
[7] H. Silverman, E.M. Silvia, Subclasses of harmonic univalent functions, New Zealand J. Math. 28 (1999) 275–284.