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Computers and Mathematics with Applications 56 (2008) 898–908

An International Journal
computers & mathematics with applications

www.elsevier.com/locate/camwa

Interpolation function of the (h, q)-extension of twisted Euler numbers

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Received 24 May 2007; received in revised form 31 July 2007; accepted 7 January 2008

Abstract

In [H. Ozden, Y. Simsek, I.N. Cangul, Generating functions of the (h, q)-extension of Euler polynomials and numbers, Acta Math. Hungarica, in press (doi:10.1007/510474-008-7139-1)], by using p-adic q-invariant integral on $\mathbb{Z}_p$ in the fermionic sense, Ozden et al. constructed generating functions of the (h, q)-extension of Euler polynomials and numbers. They defined (h, q)-Euler zeta functions and (h, q)-Euler l-functions. They also raised the following problem: "*Find a p-adic twisted interpolation function of the generalized twisted (h, q)-Euler numbers, $E^{(h)}_{n,\chi,w}(q)$*". The aim of this paper is to give a partial answer to this problem. Therefore, we constructed twisted (h, q)-partial zeta function and twisted p-adic (h, q)-Euler l-functions:

$$l^{(h)}_{E,p,\xi,q}(s,\chi) = 2 \sum_{\substack{m=1\\(m,p)=1}}^{\infty} \frac{\chi(m)(-1)^m \xi^m q^{hm}}{m^s},$$

which interpolate (h, q)-extension of Euler numbers, at negative integers:

$$l^{(h)}_{E,p,\xi,q}(-n,\chi) = E^{(h)}_{n,\xi,\chi_n}(q) - p^n \chi_n(p) E^{(h)}_{n,\xi^p,\chi_n}(q^p).$$

By using this interpolation function and twisted (h, q)-partial zeta function, we proved distribution relations of the (h, q)-extension of generalized Euler polynomials. Consequently we find a partial answer to the above question.

Keywords: p-adic q-deformed fermionic integral; Twisted q-Euler numbers and polynomials; Zeta and l-functions; Twisted p-adic (h, q)-l-functions

1. Introduction, definitions and notations

p-adic numbers were invented by Kurt Hensel around the end of the nineteenth century. In spite of their being already one hundred years old, these numbers are still today enveloped in an aura of mystery within the scientific community. Although they have penetrated several mathematical fields, Number Theory, Algebraic Geometry, Algebraic Topology, Analysis, Mathematical Physics, String Theory, Field Theory, Stochastic Differential Equations

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0898-1221/$ - see front matter
doi:10.1016/j.camwa.2008.01.020

on real Banach Spaces and Manifolds and other parts of the natural sciences, they have yet to reveal their full potential in (for example) physics. While solving mathematical and physical problems and while constructing and investigating measures on manifolds, the p-adic numbers are used. There is an unexpected connection of the p-adic Analysis with q-Analysis and Quantum Groups, Quantum Top, and thus with Noncommutative Geometry, and q-Analysis is a sort of q-deformation of the ordinary analysis. Spherical functions on Quantum Groups are q-special functions cf. [2–25]. Kubota and Leopoldt proved the existence of meromorphic functions, $L_p(s, \chi)$, which is defined over the p-adic number field. This function interpolates the nth generalized Bernoulli numbers associate with the primitive Dirichlet character χ, and $\chi_n = \chi w^{-n}$, with the Teichmüller character cf. [3–5,8,11,12,16,19,21,23,25–33]. Young [24] defined p-adic integral representation for the two-variable p-adic L-functions, introduced by Fox [4]. For powers of the Teichmüller character, he used the integral representation to extend the L-function to the large domain, in which it is a meromorphic function in the first variable and an analytical element in the second. These integral representations imply systems of congruences for the generalized Bernoulli polynomials. In [12], Kim constructed the two-variable p-adic q-L-function, which interpolates the generalized q-Bernoulli polynomials. This function is the q-extension of the two-variable p-adic L-function. He gave a p-adic integral representation for this two-variable p-adic q-L-function. He also derived q-extension of the generalized formula of Diamond and Ferrero and Greenberg formula for the two variable p-adic L-function in terms of the p-adic gamma and log gamma functions.

Twisted q-Bernoulli and Euler numbers and polynomials are very important not only in Mathematics and Statistics, but also Mathematical Physics. Recently, these numbers and polynomials have been studied by several authors cf. [9,19,21,30,31,34–38]. In [10,39], by using q-Volkenborn integration, Kim constructed the new (h, q)-extension of the Bernoulli numbers and polynomials. He defined (h, q)-extension of the zeta functions which interpolate new (h, q)-extension of the Bernoulli numbers and polynomials. In [21,40], the second author defined (h, q)-extension of Bernoulli numbers and polynomials. He also constructed their interpolation functions at negative integers. Ozden et al. [41,42] studied on (h, q)-extension of Euler numbers and polynomials. In [29], Kim and Rim constructed the q-twisted Euler numbers and polynomials, by using p-adic q-invariant integral on $\mathbb{Z}_p$ in the fermionic sense. They also defined interpolation functions of them at negative integers, explicitly.

In this paper, we define twisted (h, q)-partial zeta function, which interpolate (h, q)-extension of Euler polynomials at negative integers. We find the relation between twisted (h, q)-partial zeta function and the twisted (h, q)-zeta function. By using this function, we constructed twisted p-adic (h, q)-Euler l-functions, which interpolate (h, q)-extension of Euler numbers, at negative integers. This result gave us a partial answer of the problem Ozden et al. [1], which is given by: "*Find a p-adic twisted interpolation function of the generalized twisted (h, q)-Euler numbers, $E^{(h)}_{n,\chi,w}(q)$.*"

Ozden et al. [1], by using p-adic q-Volkenborn integration, they constructed generating functions of the new twisted (h, q)-Euler numbers and generalized twisted (h, q)-Euler numbers and polynomials. By applying the Mellin transformation and derivative operator to these generating functions, they defined integral representation of the new twisted (h, q)-zeta functions and twisted (h, q)-l_E-functions which interpolate twisted (h, q)-Euler numbers and generalized twisted (h, q)-Euler numbers at nonpositive integers.

Throughout this paper $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of rational integers, the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{Z}_+ = \mathbb{Z}^+ \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $| p |_p = p^{-v_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, then we normally assume $| q - 1 |_p < 1$, so that $q^x = \exp(x \log q)$ for $| x |_p \leq 1$. If $q \in \mathbb{C}$, then we assume that $| q | < 1$. Let

$$[x]_q = \begin{cases} \dfrac{1-q^x}{1-q}, & \text{if } q \neq 1 \\ x, & \text{if } q = 1. \end{cases}$$

For

$$f \in UD(\mathbb{Z}_p, \mathbb{C}_p) = \{f \mid f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the p-adic invariant q-integral on $\mathbb{Z}_p$ was defined by Kim [10,28] as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) \mathrm{d}\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} q^x f(x),$$

where and p is a odd prime number. Recently, many applications of this integral have been given. For detail see cf. [10,11,21–23,32,34,39–49].

The q-deformed p-adic invariant integral on $\mathbb{Z}_p$, in the fermionic sense is defined by [28]:

$$I_{-1}(f) = \lim_{q\to -1} I_q(f) = \int_{\mathbb{Z}_p} f(x)\mathrm{d}\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x f(x). \tag{1.1}$$

By using p-adic q-integral on $\mathbb{Z}_p$, Kim[44] proved the integral equations related to the p-adic q-integral.

Let p be a fixed prime. For a fixed positive integer f with $(p, f) = 1$, we set

$$\mathbb{X} = \mathbb{X}_f = \lim_{\overleftarrow{N}} \mathbb{Z}/fp^N\mathbb{Z},$$
$$\mathbb{X}_1 = \mathbb{Z}_p,$$
$$\mathbb{X}^* = \bigcup_{\substack{0<a<fp \\ (a,p)=1}} a + fp\mathbb{Z}_p$$

and

$$a + fp^N\mathbb{Z}_p = \left\{x \in \mathbb{X} \mid x \equiv a (\mathrm{mod}\, fp^N)\right\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < fp^N$. For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$,

$$\int_{\mathbb{Z}_p} f(x)\mathrm{d}\mu_{-1}(x) = \int_{\mathbb{X}} f(x)\mathrm{d}\mu_{-1}(x), \quad \text{cf. [14,27–29,39,41,49].}$$

From (1.1), Kim [28] gave the following interesting identity:

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{1.2}$$

where $f_1(x) = f(x + 1)$.

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2\sum_{x=0}^{n-1} (-1)^{n-1-x} f(x), \tag{1.3}$$

where $f_n(x) = f(x + n), n \in \mathbb{Z}^+$.

In [1], Ozden et al. constructed generating functions of the twisted (h, q)-Euler numbers and polynomials. By using these generating functions and the q-deformed p-adic invariant integral on $\mathbb{Z}_p$ in the fermionic sense, they defined twisted (h, q)-Euler numbers, polynomials and twisted generalized (h, q)-Euler numbers and polynomials with attached to Dirichlet character. Moreover, they proved twisted version of Witt's formula for the (h, q)-Euler numbers and polynomials.

Let

$$T_p = \bigcup_{n\ge 1} C_{p^n} = \lim_{n\to\infty} C_{p^n},$$

where $T_{p^n} = \left\{\xi \in \mathbb{C}_p \mid \xi^{p^n} = 1\right\}$ is the cyclic group of order p^n. For $\xi \in T_p$, we denote by $\phi_\xi : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \to \xi^x$ (see [9,21,29,36]).

We give application of the q-deformed p-adic invariant integral on $\mathbb{Z}_p$ in the fermionic sense related to [1]. By substituting $f_{\xi,q}(x, t) = \xi^x q^{hx} \mathrm{e}^{tx}$ into (1.2), we define twisted (h, q)-extension of Euler numbers, $E_{n,\xi}^{(h)}(q)$ by means of the following generating function:

$$F_{\xi,q}^{(h)}(t) = I_{-1}(\xi^x q^{hx} \mathrm{e}^{tx}) = \frac{2}{\xi q^h \mathrm{e}^t + 1} = \sum_{n=0}^{\infty} E_{n,\xi}^{(h)}(q) \frac{t^n}{n!} \quad \text{cf. [1].} \tag{1.4}$$

We note that if $\xi = 1$, then $E_{n,\xi}^{(h)}(q) = E_n^{(h)}(q)$ and $F_{\xi,q}^{(h)}(t) = F_q^{(h)}(t) = \frac{2}{q^h \mathrm{e}^t+1}$ cf. [42]. If $q \to 1$, then $F_q^{(h)}(t) \to F(t) = \frac{2}{\mathrm{e}^t+1} = \sum_{n=1}^{\infty} E_n \frac{t^n}{n!}$, where E_n is usual Euler numbers, cf. [23,27,37].

Twisted (h, q)-extension of Euler polynomials $E^{(h)}_{n,\xi}(z, q)$ are defined by means of the generating function

$$F^{(h)}_{\xi,q}(t, z) = F^{(h)}_{\xi,q}(t)\mathrm{e}^{tz} = \sum_{n=0}^{\infty} E^{(h)}_{n,\xi}(z, q)\frac{t^n}{n!} \quad \text{cf. [1]}.$$

We note that $E^{(h)}_{n,\xi}(0, q) = E^{(h)}_{n,\xi}(q)$. If $\xi = 1$, then $E^{(h)}_{n,\xi}(z, q) \to E^{(h)}_{n}(z, q)$ and $F^{(h)}_{\xi,q}(t, z) \to F^{(h)}_{q}(t, z) = F^{(h)}_{q}(t)\mathrm{e}^{tz}$ cf. [42].

Twisted versions of Witt's formula for $E^{(h)}_{n,\xi}(z, q)$ and $E^{(h)}_{n,\xi}(q)$ are given by the following theorem:

Theorem 1 (*[1]*). *For $h \in \mathbb{Z}$, $\xi \in T_p$ and $q \in \mathbb{C}_p$ with $| q - 1 |_p < 1$, we obtain*

$$E^{(h)}_{n,\xi}(q) = \int_{\mathbb{Z}_p} \xi^x q^{hx} x^n \mathrm{d}\mu_{-1}(x),$$

and

$$E^{(h)}_{n,\xi}(z, q) = \int_{\mathbb{Z}_p} \xi^x q^{hx} (x + z)^n \mathrm{d}\mu_{-1}(x). \tag{1.5}$$

Remark 1. If $q \to 1$, $\xi = 1$, then $F_{\xi,q}(t, z) \to F(t, z) = \frac{2}{\mathrm{e}^t+1}\mathrm{e}^{tz}$ and $E^{(h)}_{n,\xi}(z, q) \to E_n(z)$, the usual Euler polynomials, cf. [23,42].

Theorem 2 (*[1]*). *Let $n \in \mathbb{Z}_+$. Then we have*

$$E^{(h)}_{n,\xi}(z, q) = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} E^{(h)}_{k,\xi}(q).$$

The generalized twisted (h, q)-extension of Euler polynomials, $E^{(h)}_{n,\chi,\xi}(z, q)$ are defined by means of the following generating function: let χ be a Dirichlet character with conductor f (=odd)

$$F^{(h)}_{\chi,\xi,q}(t, z) = \sum_{a=0}^{f-1} \frac{2(-1)^a \chi(a)\xi^a q^{ha}\mathrm{e}^{(z+a)t}}{\xi^f q^{fh}\mathrm{e}^{ft} + 1} = \sum_{n=0}^{\infty} E^{(h)}_{n,\chi,\xi}(z, q)\frac{t^n}{n!} \quad \text{cf. [1]}.$$

Note that, by substituting $z = 0$ in the above, $E^{(h)}_{n,\chi,\xi}(0, q) = E^{(h)}_{n,\chi,\xi}(q)$ is the nth twisted (h, q)-extension of Euler number. If $\xi = 1$, then $E^{(h)}_{n,\chi,\xi}(q) \to E^{(h)}_{n,\chi}(q)$ cf. [42].

$E^{(h)}_{n,\chi,\xi}(z, q)$ is given explicitly in the following theorem [1]:

Theorem 3. *Let χ be a Dirichlet character. We have*

$$E^{(h)}_{n,\chi,\xi}(z, q) = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} E^{(h)}_{k,\chi,\xi}(q).$$

Observe that if $\xi = 1$, then $F^{(h)}_{\chi,\xi,q}(t, z)$ reduces to $F^{(h)}_{\chi,q}(t, z)$:

$$F^{(h)}_{\chi,q}(t, z) = \sum_{a=0}^{f-1} \frac{2(-1)^a \chi(a) q^{ha}\mathrm{e}^{(z+a)t}}{q^h\mathrm{e}^{ft} + 1}, \quad \text{cf. [42]}.$$

If $q \to 1$ in the above, then $F_{\chi,q}(t, z) \to F_{\chi}(t, z) = F_{\chi}(t)\mathrm{e}^{zt} = \sum_{a=0}^{f-1} \frac{2(-1)^a\chi(a)\mathrm{e}^{(a+z)t}}{\mathrm{e}^{ft}+1}$ and $E_{n,\chi}(z, q) \to E_{n,\chi}(z)$ are the usual generalized Euler polynomials cf. [23,42].

Here, we assume that $q \in \mathbb{C}$ with $| q |< 1$ and $s \in \mathbb{C}$.

Ozden et al. [1] defined twisted (h, q)-extensions of Hurwitz type Euler zeta functions as follows:

Definition 1. Let $s \in \mathbb{C}$, $z \in \mathbb{R}^+$. We define

$$\zeta_{E,\xi,q}^{(h)}(s, z) = 2\sum_{n=0}^{\infty} \frac{(-1)^n \xi^n q^{nh}}{(n+z)^s}. \tag{1.6}$$

$\zeta_{E,\xi,q}^{(h)}(s, z)$ is an analytical function in the whole complex s-plane.

Remark 2. Observe that if $z = 1$, then twisted (h, q)-extensions of Euler zeta function is defined by

$$\zeta_{E,\xi,q}^{(h)}(s) = 2\sum_{n=1}^{\infty} \frac{(-1)^n \xi^n q^{hn}}{n^s}, \quad \text{cf. [42]}.$$

By (1.6), for $\xi = 1$, we easily see that

$$\zeta_{E,1,q}^{(h)}(s, z) = \zeta_{E,q}^{(h)}(s, z) = 2\sum_{n=0}^{\infty} \frac{(-1)^n q^{nh}}{(n+z)^s}, \quad \text{cf. [42]}.$$

If $\xi = 1$ and $q \to 1$ in (1.6), then we obtain

$$\zeta_E(s, z) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+z)^s},$$

cf. [14,28,29,42]. For q-zeta and q-Hurwitz zeta functions similar results were obtained cf. (see for detail [23,36–39, 49]).

The value of $\zeta_{E,\xi,q}^{(h)}(s, z)$ at negative integers is given explicitly as follows:

Theorem 4 (*[1]*). *Let $n \in \mathbb{Z}_+$. Then we have*

$$\zeta_{E,\xi,q}^{(h)}(-n, z) = E_{n,\xi}^{(h)}(z, q).$$

Twisted (h, q)-extensions of l-function is defined by

Definition 2 (*[1]*). Let χ be a Dirichlet character. For $s \in \mathbb{C}$, we have

$$l_{E,\xi,q}^{(h)}(s, \chi) = \sum_{n=1}^{\infty} \frac{2(-1)^n q^{nh} \chi(n) \xi^n}{n^s}.$$

$l_{E,\xi,q}^{(h)}(s, \chi)$ is an analytical function in the whole complex s-plane.
Note that if $\xi = 1$, then, we have in the above

$$l_{E,q}^{(h)}(s, \chi) = \sum_{n=1}^{\infty} \frac{2(-1)^n q^{nh} \chi(n)}{n^s}, \quad \text{cf. [42]},$$

and

$$\lim_{q\to 1} l_{E,q}^{(h)}(s, \chi) = l_E(s, \chi) = \sum_{n=1}^{\infty} \frac{2(-1)^n \chi(n)}{n^s} \quad \text{cf. [14,28,29,42]}.$$

Relation between $\zeta_{E,\xi,q}^{(h)}(s, z)$ and $l_{E,\xi,q}^{(h)}(s, \chi)$ is given by the following theorem:

Theorem 5 (*[1]*). *Let $s \in \mathbb{C}$. Let χ be a Dirichlet character with conductor f (=odd). We have*

$$l^{(h)}_{E,\xi,q}(s,\chi) = \frac{1}{f^s}\sum_{a=1}^{f}(-1)^a q^{ha}\xi^a\chi(a)\zeta^{(h)}_{E,\xi^f,q^f}\left(s,\frac{a}{f}\right). \tag{1.7}$$

The value of twisted (h,q)-l_E-function at negative integers is given explicitly by the following theorem:

Theorem 6 (*[1]*). *Let $n \in \mathbb{Z}_+$. Then we have*

$$l^{(h)}_{E,\xi,q}(-n,\chi) = E^{(h)}_{n,\chi,\xi}(q).$$

2. Partial twisted (h,q)-zeta function

In this section, we construct a twisted partial (h,q)-Euler zeta function. We assume that $q \in \mathbb{C}$ with $|\, q \,| < 1$. Let $\mathbb{R}$, be the field of real numbers and let ξ be the r-th root of unity.

Let $s \in \mathbb{C}$ and $a, F \in \mathbb{Z}$ with F is an odd integer and $0 < a < F$. Then twisted partial (h,q)-Euler zeta function is as follows:

$$H^{(h)}_{E,\xi,q}(s,a|F) = 2\sum_{\substack{m\equiv a(\mathrm{mod}F)\\ m>0}}\frac{(-1)^m\xi^m q^{hm}}{m^s}. \tag{2.1}$$

We give relationship between $H^{(h)}_{E,\xi,q}(s,a|F)$ and $\zeta^{(h)}_{E,\xi,q}(s,z)$ as follows:

Substituting $m = a + nF$ with F is odd into (2.1), we have

$$\begin{aligned} H^{(h)}_{E,\xi,q}(s,a|F) &= 2\sum_{\substack{m\equiv a(\mathrm{mod}F)\\ m>0}}\frac{(-1)^m\xi^m q^{hm}}{m^s} \\ &= 2\sum_{n=0}^{\infty}\frac{(-1)^{a+nF}\xi^{a+nF}q^{h(a+nF)}}{(a+nF)^s} \\ &= \frac{(-1)^a\xi^a q^{ha}}{F^s}2\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{nF}q^{nF}}{\left(n+\frac{a}{F}\right)^s} \\ &= \frac{(-1)^a\xi^a q^{ha}}{F^s}\zeta^{(h)}_{E,\xi^F,q^F}\left(s,\frac{a}{F}\right). \end{aligned} \tag{2.2}$$

By substituting $s = -n$, $n \in \mathbb{Z}_+$ in the above and using Theorem 4, after some elementary calculations, we arrive at the following theorem:

Theorem 7. *Let F be odd and $s \in \mathbb{C}$. Then we have*

$$H^{(h)}_{E,\xi,q}(s,a|F) = \frac{(-1)^a\xi^a q^{ha}}{F^s}\zeta^{(h)}_{E,\xi^F,q^F}\left(s,\frac{a}{F}\right).$$

Let $n \in \mathbb{Z}_+$, then we have

$$H^{(h)}_{E,\xi,q}(-n,a|F) = (-1)^a q^{ha}\xi^a F^n E^{(h)}_{n,\xi^F}\left(\frac{a}{F},q^F\right).$$

Let χ be a Dirichlet character with conductor f_χ and $f_\chi|F$. By using Theorem 2 in Theorem 7, we get

$$H^{(h)}_{E,\xi,q}(-n,a|F) = (-1)^a\xi^a q^{ha}F^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{a}{F}\right)^{n-k}E^{(h)}_{n,\xi^F}(q^F). \tag{2.3}$$

From Theorems 5 and 7, we have

$$l_{E,\xi,q}^{(h)}(s,\chi)=\sum_{a=1}^{F}\chi(a)H_{E,\xi,q}^{(h)}(s,a|F). \tag{2.4}$$

For $s=-n$ with $n\in\mathbb{Z}_+$

$$l_{E,\xi,q}^{(h)}(-n,\chi)=\sum_{a=1}^{F}\chi(a)H_{E,\xi,q}^{(h)}(-n,a|F). \tag{2.5}$$

By using Theorem 7 in the above, we have

$$l_{E,\xi,q}^{(h)}(-n,\chi)=\sum_{a=1}^{F}(-1)^a\,\chi(a)q^{ha}\xi^aF^nE_{n,\xi^F}^{(h)}\left(\frac{a}{F},q^F\right). \tag{2.6}$$

By using Theorem 6 in the above, we arrive at the following theorem:

Theorem 8. *Let χ be a Dirichlet character with conductor F. Then we have*

$$E_{n,\xi,\chi}^{(h)}(q)=\sum_{a=1}^{F}(-1)^a\,\chi(a)q^{ha}\xi^aF^nE_{n,\xi^F}^{(h)}\left(\frac{a}{F},q^F\right).$$

Remark 3. By applying the q-deformed p-adic invariant integral on $\mathbb{Z}_p$, in the fermionic sense, Ozden et al. [1] proved Theorem 8 by the different method. This theorem gives us the distribution relation of the generalized (h,q) twisted Euler numbers. Distribution relations of Bernoulli and Euler numbers are very important in Number Theory, Statistics and Measure Theory.

By using (2.3) and (2.6), we have

$$l_{E,\xi,q}^{(h)}(-n,\chi)=\sum_{a=1}^{F}(-1)^a\chi(a)\xi^aq^{ha}F^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{a}{F}\right)^{n-k}E_{k,\xi^F}^{(h)}(q^F).$$

From above, we have

$$l_{E,\xi,q}^{(h)}(-n,\chi)=\sum_{a=1}^{F}(-1)^a\chi(a)\xi^aq^{ha}a^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{F}{a}\right)^{k}E_{k,\xi^F}^{(h)}(q^F). \tag{2.7}$$

By using (2.3), we have

$$H_{E,\xi,q}^{(h)}(-n,a|F)=(-1)^a\xi^aq^{ha}a^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{F}{a}\right)^{k}E_{k,\xi^F}^{(h)}(q^F).$$

By using the above equation, we modify twisted partial (h,q)-zeta function as follows:

Definition 3. Let $s\in\mathbb{C}$. Then we define

$$H_{E,\xi,q}^{(h)}(s,a|F)=(-1)^a\xi^aq^{ha}a^{-s}\sum_{k=0}^{\infty}\binom{-s}{k}\left(\frac{F}{a}\right)^{k}E_{k,\xi^F}^{(h)}(q^F). \tag{2.8}$$

By using (2.5) and (2.8), we obtain

$$l_{E,\xi,q}^{(h)}(s,\chi)=\sum_{a=1}^{F}(-1)^a\chi(a)\xi^aq^{ha}a^{-s}\sum_{k=0}^{\infty}\binom{-s}{k}\left(\frac{F}{a}\right)^{k}E_{k,\xi^F}^{(h)}(q^F), \tag{2.9}$$

where $s\in\mathbb{C}$.

3. Twisted p-adic (h, q)-Euler l_E-function

Here, we use some notations which are due to Washington [8] and Kim [13]. The integer p^* is defined by $p^* = p$ if $p > 2$ and $p^* = 4$ if $p = 2$ cf. [8, p. 51]. Let w denote the Teichmüller character, having conductor $f_w = p^*$. For an arbitrary character χ with conductor $f = f_\chi$. We define $\chi_n = \chi w^{-n}$, where $n \in \mathbb{Z}$, in the sense of the product of characters. In this section, if $q \in \mathbb{C}_p$, then we assume $\mid 1 - q \mid_p < 1$. Let

$$\langle a \rangle = w^{-1}(a)a = \frac{a}{w(a)}.$$

Note that $\langle a \rangle \equiv 1 (\operatorname{mod} f_w)$ cf. [8, p. 51].

We are now ready to construct twisted p-adic (h, q)-l_E-function, which interpolate (h, q)-extension of Euler numbers, at negative integers. So we obtain a partial answer to the question of Ozden et al. [1]: "*Find a p-adic twisted interpolation function of the generalized twisted (h, q)-Euler numbers, $E^{(h)}_{n,\chi,w}(q)$*".

In Section 2, we define analytical functions on whole complex s-plane. In this section, we construct p-adic analogues of these functions, our method is similar to that of [8,13]. Let χ be Dirichlet character with conductor d (=odd) and F be a positive integer multiple of p and d. We define p-adic analogues of (2.9) as follows:

$$l^{(h)}_{E,p,\xi,q}(s, \chi) = \sum_{a=1}^{F} (-1)^a \chi(a) \xi^a q^{ha} \langle a \rangle^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} \left(\frac{F}{a}\right)^k E^{(h)}_{k,\xi^F}(q^F),$$

and we also define p-adic analogues of (2.8) as follows:

$$H^{(h)}_{E,p,\xi,q}(s, a|F) = (-1)^a \xi^a q^{ha} \langle a \rangle^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} \left(\frac{F}{a}\right)^k E^{(h)}_{k,\xi^F}(q^F), \tag{3.1}$$

where $p \nmid a$ and $p^* | F$.

Remark 4. $w^{-s}(a)\langle a \rangle^s$ and

$$\sum_{k=0}^{\infty} \binom{-s}{k} \left(\frac{F}{a}\right)^k E^{(h)}_{k,\xi^F}(q^F),$$

where F is multiple of p and f_χ, are analytical in $D = \{s \in \mathbb{C}_p \mid\mid s \mid_p < p^* p^{-\frac{1}{p-1}}\}$ cf. [7,20,8]. Therefore, $l^{(h)}_{E,p,\xi,q}(s, \chi)$ and $H^{(h)}_{E,p,\xi,q}(s, a|F)$ are analytical on D.

By (2.8), we have

$$\begin{aligned} H^{(h)}_{E,p,\xi,q}(-n, a|F) &= (-1)^a \xi^a q^{ha} \langle a \rangle^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{F}{a}\right)^k E^{(h)}_{k,\xi^F}(q^F) \\ &= (-1)^a \xi^a q^{ha} a^n w^{-n}(a) \sum_{k=0}^{n} \binom{n}{k} \left(\frac{a}{F}\right)^{-k} E^{(h)}_{k,\xi^F}(q^F) \\ &= w^{-n}(a) H^{(h)}_{E,\xi,q}(-n, a|F). \end{aligned} \tag{3.2}$$

Thus we get

$$H^{(h)}_{E,p,\xi,q}(-n, a|F) = w^{-n}(a) H^{(h)}_{E,\xi,q}(-n, a|F).$$

Definition 4. Let $s \in D$. Let χ be Dirichlet character with conductor d (=odd) and F be a positive integer multiple of p and d. Then we define

$$l^{(h)}_{E,p,\xi,q}(s, \chi) = \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi(a) H^{(h)}_{E,p,\xi,q}(s, a|F). \tag{3.3}$$

By substituting (3.2) into (3.3), Definition 4 is modified to the following definition:

Definition 5. Let $s \in D$. Then we define

$$l^{(h)}_{E,p,\xi,q}(s,\chi) = 2 \sum_{\substack{m=1 \\ (m,p)=1}}^{\infty} \frac{\chi(m)(-1)^m \xi^m q^{hm}}{m^s}.$$

By using Definition 5, we easily obtain

$$l^{(h)}_{E,p,\xi,q}(s,\chi) = 2 \sum_{m=1}^{\infty} \frac{\chi(m)(-1)^m \xi^m q^{hm}}{m^s} - 2 \sum_{m=1}^{\infty} \frac{\chi(mp)(-1)^m \xi^{mp} q^{hpm}}{p^s m^s}. \tag{3.4}$$

By using Definition 2, we obtain the following corollary:

Corollary 1. *Let $s \in D$. Then we have*

$$l^{(h)}_{E,p,\xi,q}(s,\chi) = l^{(h)}_{E,\xi,q}(s,\chi) - p^{-s}\chi(p) l^{(h)}_{E,\xi^p,q^p}(s,\chi).$$

The function, $l^{(h)}_{E,p,\xi,q}(s,\chi)$ is analytical on D. This function interpolates twisted generalized (h,q)-Euler numbers at negative integers, which is given as follows:

By substituting $s = -n$, $n \in \mathbb{Z}_+$, into (3.3), we have

$$l^{(h)}_{E,p,\xi,q}(-n,\chi) = \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi(a) H^{(h)}_{E,p,\xi,q}(-n,a|F).$$

By (3.2) and (3.4), we get

$$\begin{aligned} l^{(h)}_{E,p,\xi,q}(-n,\chi) &= \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi_n(a) H^{(h)}_{E,\xi,q}(-n,a|F) \\ &= \sum_{a=1}^{F} \chi_n(a) H^{(h)}_{E,\xi,q}(-n,a|F) - \sum_{a=1}^{\frac{F}{p}} \chi_n(ap) H^{(h)}_{E,\xi,q}(-n,ap|F) \\ &= \sum_{a=1}^{F} \chi_n(a) H^{(h)}_{E,\xi,q}(-n,a|F) - \chi_n(p) \sum_{a=1}^{\frac{F}{p}} \chi_n(a) H^{(h)}_{E,\xi,q}(-n,ap|F) \end{aligned}$$

By using Theorem 7 and (3.1) in the above, we have

$$\begin{aligned} l^{(h)}_{E,p,\xi,q}(-n,\chi) &= \sum_{a=1}^{F} \chi_n(a)(-1)^a \xi^a q^{ha} F^n E^{(h)}_{n,\xi^F}\left(\frac{a}{F}, q^F\right) \\ &\quad - p^n \chi_n(p) \sum_{a=1}^{\frac{F}{p}} \chi_n(a)(-1)^{pa} q^{hpa} \xi^{pa} \left(\frac{F}{p}\right)^n E^{(h)}_{n,(\xi^p)^{\frac{F}{p}}}\left(\frac{a}{\frac{F}{p}}, (q^p)^{\frac{F}{p}}\right) \\ &= E^{(h)}_{n,\xi,\chi_n}(q) - p^n \chi_n(p) E^{(h)}_{n,\xi^p,\chi_n}(q^p). \end{aligned}$$

Thus we arrive at the following theorem:

Theorem 9. *Let $n \in \mathbb{Z}_+$. Then we have*

$$l^{(h)}_{E,p,\xi,q}(-n,\chi) = E^{(h)}_{n,\xi,\chi_n}(q) - p^n \chi_n(p) E^{(h)}_{n,\xi^p,\chi_n}(q^p).$$

Observe that

$$\lim_{q\to 1} l^{(h)}_{E,p,\xi,q}(s,\chi) = l_{E,p,\xi}(s,\chi) = 2 \sum_{\substack{m=1\\(m,p)=1}}^{\infty} \frac{\chi(m)(-1)^m \xi^m}{m^s},$$

which is called twisted p-adic l_E-function. This function interpolates twisted generalized Euler numbers at negative integers.

In [42], Ozden and Simsek gave the following Witt's formula for the numbers $E^{(h)}_{n,\chi,q}(x)$

$$\int_{\mathbb{X}} y^n \chi(y) q^{hy} \mathrm{d}\mu_{-1}(y) = E^{(h)}_{n,\chi}(q).$$

By using the above equation, we obtain

$$l^{(h)}_{E,p,\xi,q}(s,\chi) = \int_{\mathbb{X}^*} \langle y\rangle^{-s} \chi(y) \xi^y q^{hy} \mathrm{d}\mu_{-1}(y).$$

Conclusion 1. *Thus we derive p-adic (h,q)-twisted interpolation function which interpolates $E^{(h)}_{n,\xi,\chi}$ numbers. This gives a part of answer to the question in* [1]. *For $q \to 1$, Definition 5 reduces to the ordinary twisted p-adic l_E-function, which interpolates twisted generalized Euler numbers at negative integers by using Theorem 9. p adic (h,q) twisted interpolation function, $l^{(h)}_{E,p,\xi,q}(s,\chi)$, for $q \to 1$, coincides with that of* [18]. *These functions are used not only in Mathematics and Mathematical analysis but also used in Physics and its applications.*

Acknowledgements

We thank the referees for their valuable comments on this present form. The second author is supported by the research fund of Akdeniz University.

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