

## ON SLANT SUBMANIFOLDS OF NEUTRAL KAEHLER MANIFOLDS

K. Arslan, A. Carriazo, B.-Y. Chen and C. Murathan

**Abstract.** An indefinite Riemannian manifold is called *neutral* if its index is equal to one half of its dimension and an indefinite Kaehler manifold is called neutral Kaehler if its complex index is equal to the half of its complex dimension. In the first part of this article, we extend the notion of slant surfaces in Lorentzian Kaehler surfaces to slant submanifolds in neutral Kaehler manifolds; moreover, we characterize slant submanifolds with parallel canonical structures. By applying the results obtained in the first part we completely classify slant surfaces with parallel mean curvature vector and minimal slant surfaces in the Lorentzian complex plane in the second part of this article.

### 1. INTRODUCTION

Let  $\tilde{M}_i^m$  be a complex  $m$ -dimensional indefinite Kaehler manifold with complex index  $i$ . Thus,  $\tilde{M}_i^m$  is endowed with an almost complex structure  $J$  and with an indefinite Riemannian metric  $\tilde{g}$ , which is  $J$ -Hermitian, i.e., for all  $p \in \tilde{M}_i^m$ , we have

$$(1.1) \quad \tilde{g}(JX, JY) = \tilde{g}(X, Y), \quad \forall X, Y \in T_p\tilde{M}_i^m,$$

$$(1.2) \quad \tilde{\nabla}J = 0,$$

where  $\tilde{\nabla}$  is the Levi-Civita connection of  $\tilde{g}$ . It follows that  $J$  is integrable.

The complex index of  $\tilde{M}_i^m$  is defined as the complex dimension of the largest complex negative definite subspace of the tangent space. When  $m = 2n$  and the

---

Received and accepted November 4, 2008.

Communicated by J. C. Yao.

2000 *Mathematics Subject Classification*: Primary 53C40; Secondary 53C42, 53B25.

*Key words and phrases*: Slant submanifold, Neutral Kaehler manifold, Neutral complex space form, Minimal surface, Lorentzian complex plane.

A part of this article was done while the second and third authors visited Uludağ University, Turkey, supported by TÜBİTAK research fellowship. Both authors would like to express their many thanks for the hospitality they received during their visits.

complex index is  $n$ , the indefinite Kaehler manifold  $\tilde{M}_n^{2n}$  is called a *neutral Kaehler manifold*. A neutral Kaehler surface is nothing but a Lorentzian Kaehler surface.

The simplest examples of neutral Kaehler manifolds are the neutral complex space forms defined as follows: Let  $\mathbf{C}^{2n}$  denote the complex  $2n$ -plane with complex coordinates  $z_1, \dots, z_{2n}$ . Then the  $\mathbf{C}^{2n}$  endowed with  $g_{n,2n}$ , i.e., the real part of the Hermitian form

$$b_{n,2n}(z, w) = - \sum_{k=1}^n \bar{z}_k w_k + \sum_{j=n+1}^{2n} \bar{z}_j w_j, \quad z, w \in \mathbf{C}^{2n},$$

defines a flat indefinite complex space form with complex index  $n$ . We simply denote this flat neutral Kaehler manifold  $(\mathbf{C}^{2n}, g_{n,2n})$  by  $\mathbf{C}_n^{2n}$ .

Consider  $S_{2n}^{4n+1} = \{z \in \mathbf{C}_n^{2n+1}; b_{n,2n+1}(z, z) = 1\}$ , which is an indefinite real space form of constant sectional curvature one. The Hopf fibration

$$\pi : S_{2n}^{4n+1} \rightarrow CP_n^{2n} : z \mapsto z \cdot \mathbf{C}^*$$

is a submersion and there is a unique neutral metric of complex index  $n$  on  $CP_n^{2n}$  such that  $\pi$  is a Riemannian submersion. The pseudo-Riemannian manifold  $CP_n^{2n}$  is a neutral complex space form of positive holomorphic sectional curvature 4.

Analogously, consider  $H_{2n}^{4n+1} = \{z \in \mathbf{C}_{n+1}^{2n+1}; b_{n+1,2n+1}(z, z) = -1\}$ , which is an indefinite real space form of constant sectional curvature  $-1$ . The Hopf fibration

$$\pi : H_{2n}^{4n+1} \rightarrow CH_n^{2n} : z \mapsto z \cdot \mathbf{C}^*$$

is a submersion and there exists a unique neutral metric on  $CH_n^{2n}$  such that  $\pi$  is a Riemannian submersion. The pseudo-Riemannian manifold  $CH_n^{2n}$  is a Lorentzian complex space form of constant holomorphic sectional curvature  $-4$ .

We denote by  $\langle \cdot, \cdot \rangle$  the inner product induced from the neutral metrics on neutral manifolds. A tangent vector  $v$  of a neutral manifold  $M_n^{2n}$  is called *space-like* (respectively, *time-like*) if  $\langle v, v \rangle > 0$  (respectively,  $\langle v, v \rangle < 0$ ). A vector  $v$  is called *null* or *light-like* if it is a nonzero vector and it satisfies  $\langle v, v \rangle = 0$ .

A distribution  $\mathcal{D}$  on a neutral manifold  $M_n^{2n}$  is called *space-like* (respectively, *time-like*) if each nonzero vector  $v \in \mathcal{D}$  is *space-like* (respectively, *time-like*).

The notion of slant submanifolds in Kaehler manifolds (or more generally, in almost Hermitian manifolds) was introduced and studied in 1990 by the third author in [6, 7]. Since then such submanifolds have been investigated extensively by many geometers and many interesting results were obtained (see [7] and [8, Chapter 18] for more details). Moreover, contact and Sasakian versions of slant submanifolds have been studied in [2, 3, 4, 14, 18] among others.

In this article, we define the notion of slant submanifolds in neutral Kaehler manifolds as a natural extension of the notion of slant surfaces in Lorentzian Kaehler

surfaces studied in [9, 11, 13]. Some fundamental and classification results for slant submanifolds in neutral Kaehler manifolds are obtained. In particular, we characterize slant submanifolds with parallel canonical structures in section 4 and section 5. In section 6, slant surfaces with parallel mean curvature vector are completely classified. In the last section, we classify slant minimal surfaces in the Lorentzian complex plane.

2. BASIC FORMULAS AND FUNDAMENTAL EQUATIONS

Let  $\tilde{M}$  be an indefinite Kaehler manifold. Denote by  $\tilde{R}$  the Riemann-Christoffel curvature tensor of  $\tilde{M}$ . Assume that  $M$  is a pseudo-Riemannian submanifold of  $\tilde{M}$ .

Denote by  $\nabla$  and  $\tilde{\nabla}$  the Levi Civita connections on  $M$  and  $\tilde{M}$ , respectively. The formulas of Gauss and Weingarten are given by (cf. [5, 7, 19])

$$(2.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$(2.2) \quad \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$$

for tangent vector fields  $X, Y$  and a normal vector field  $\xi$ , where  $h$ ,  $A$  and  $D$  are the second fundamental form, the shape operator and the normal connection. For each  $\xi \in T_p^\perp M$ , the shape operator  $A_\xi$  is a symmetric endomorphism of the tangent space  $T_p M$  at  $p \in M$ .

The shape operator and the second fundamental form are related by

$$(2.3) \quad \langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle$$

for  $X, Y$  tangent to  $M$  and  $\xi$  normal to  $M$ .

For a vector  $\tilde{X} \in T_p \tilde{M}$ ,  $p \in M$ , we denote by  $\tilde{X}^\top$  and  $\tilde{X}^\perp$  the tangential and the normal components of  $\tilde{X}$ , respectively. The equations of Gauss, Codazzi and Ricci are given respectively by

$$(2.4) \quad (\tilde{R}(X, Y)Z)^\top = R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X,$$

$$(2.5) \quad (\tilde{R}(X, Y)Z)^\perp = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z),$$

$$(2.6) \quad (\tilde{R}(X, Y)\xi)^\perp = h(A_\xi X, Y) - h(X, A_\xi Y) + R^D(X, Y)\xi$$

for vector fields  $X, Y$  and  $Z$  tangent to  $M$ ,  $\xi$  normal to  $M$ , where  $\bar{\nabla}h$  and  $R^D$  are defined respectively by

$$(2.7) \quad (\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

$$(2.8) \quad R^D(X, Y) = [D_X, D_Y] - D_{[X, Y]}.$$

The  $R^D$  is known as the curvature tensor of the normal bundle.

The mean curvature vector  $H$  is defined by

$$(2.9) \quad H = \frac{1}{n} \operatorname{trace} h, \quad n = \dim_{\mathbf{R}} M.$$

The mean curvature vector is said to be parallel in the normal bundle if  $DH = 0$  holds identically.

A submanifold  $M$  in an indefinite Kaehler manifold is called *minimal* if its mean curvature vector vanishes identically; and  $M$  is called *quasi-minimal* if its mean curvature vector is nonzero and light-like at each point on  $M$ .

### 3. BASICS OF SLANT SUBMANIFOLDS

An isometric immersion  $\Psi: M \rightarrow \tilde{M}$  of a manifold  $M$  into a neutral Kaehler manifold  $\tilde{M}$  is called *purely real* if the almost complex structure  $J$  on  $\tilde{M}$  carries the tangent bundle of  $M$  into a transversal bundle, that is  $J(TM) \cap TM = \{0\}$ . Obviously, every purely real immersion contains no complex points.

Let  $\Psi: M \rightarrow \tilde{M}$  be a purely real immersion. For each tangent vector  $X$ , we put

$$(3.1) \quad JX = PX + FX,$$

where  $PX$  and  $FX$  are the tangential and the normal components of  $JX$ . Clearly,  $P$  is an endomorphism of the tangent bundle  $TM$  of  $M$  and  $F$  is a normal-bundle-valued 1-form on  $TM$ .

Similarly, for each normal vector  $\xi$  of  $M$ , we put

$$(3.2) \quad J\xi = t\xi + f\xi,$$

where  $t\xi$  and  $f\xi$  are the tangential and the normal components of  $J\xi$ . Then  $f$  is an endomorphism of the normal bundle and  $t$  is a tangent-bundle-valued 1-form on the normal bundle.

For vectors  $X, Y$  tangent to  $M$ , it follows from (1.1) and (3.1) that

$$(3.3) \quad \langle PX, Y \rangle = -\langle X, PY \rangle.$$

Thus, we have

$$(3.4) \quad \langle P^2 X, Y \rangle = \langle X, P^2 Y \rangle.$$

**Example 3.1.** (Slant surfaces in Lorentzian Kaehler surfaces). Let  $\Psi : M_1^2 \rightarrow \tilde{M}_1^2$  be an isometric immersion of a Lorentz surface into a Lorentzian Kaehler surface. Then,  $\Psi$  is always purely real (see [9, Proposition 3.1]).

Let  $e_1, e_{1^*}$  be an orthonormal frame on  $M_1^2$  such that

$$(3.5) \quad \langle e_1, e_1 \rangle = 1, \quad \langle e_1, e_{1^*} \rangle = 0, \quad \langle e_{1^*}, e_{1^*} \rangle = -1.$$

Then, it follows from (3.3) and (3.5) that  $Pe_1 = \gamma e_{1^*}$  for some nonzero real-valued function  $\gamma$ . So, we find from (3.1) and (3.5) that  $\langle Fe_1, Fe_1 \rangle = 1 + \gamma^2 > 1$ . Hence,  $Fe_1$  is a space-like normal vector field. Therefore, there exists a nonzero real number  $\alpha$  such that

$$(3.6) \quad Je_1 = \sinh \alpha e_{1^*} + \cosh \alpha \xi_1$$

for some unit space-like normal vector field  $\xi_1$ .

It follows from (3.3) and (3.6) that

$$(3.7) \quad Je_{1^*} = \sinh \alpha e_1 + \cosh \alpha \xi_{1^*}$$

for some unit time-like normal vector field  $\xi_{1^*}$ . By applying (1.1), (3.6) and (3.7), we get

$$(3.8) \quad \langle \xi_1, \xi_1 \rangle = 1, \quad \langle \xi_1, \xi_{1^*} \rangle = 0, \quad \langle \xi_{1^*}, \xi_{1^*} \rangle = -1.$$

Also, from (3.6) and (3.7) we find  $P^2 = (\sinh^2 \alpha)I$ . The immersion  $\Psi : M_1^2 \rightarrow \tilde{M}_1^2$  is called  $\theta$ -slant if the function  $\alpha$  is constant  $\theta$  (see [13]).

If we denote the distributions on  $M_1^2$  spanned by  $e_1$  and  $e_2$  by  $\mathcal{D}_s^1$  and  $\mathcal{D}_t^1$ , respectively, then we have the orthogonal decomposition:  $TM_1^2 = \mathcal{D}_s^1 \oplus \mathcal{D}_t^1$  such that  $P(\mathcal{D}_s^1) = \mathcal{D}_t^1$  and  $P(\mathcal{D}_t^1) = \mathcal{D}_s^1$ .

Now, we extend the above notion of slant surfaces in a Lorentzian Kaehler surface to slant submanifolds in a neutral Kaehler manifold.

**Definition 1.** An isometric immersion  $\Psi : M_n^{2n} \rightarrow \tilde{M}_n^{2n}$  of a neutral  $2n$ -manifold into a neutral Kaehler manifold of complex dimension  $2n$  is called  $\theta$ -slant if there exist a real number  $\theta$  and an orthogonal decomposition:

$$(3.9) \quad TM_n^{2n} = \mathcal{D}_s^n \oplus \mathcal{D}_t^n$$

of the tangent bundle  $TM_n^{2n}$  such that

- (a)  $\mathcal{D}_s^n$  is a space-like distribution and  $\mathcal{D}_t^n$  a time-like distribution;
- (b)  $P(\mathcal{D}_s^n) = \mathcal{D}_t^n$  and  $P(\mathcal{D}_t^n) = \mathcal{D}_s^n$ ;

$$(c) P^2 = (\sinh^2 \theta)I,$$

where  $P$  is defined by (3.1).

The real number  $\theta$  in the definition is called the *slant angle*. A slant submanifold is called *Lagrangian* if its slant angle is equal to zero. A slant submanifold is called *proper slant* if it is not Lagrangian.

For a Lagrangian submanifold  $M_n^{2n}$  in a neutral Kaehler manifold  $\tilde{M}_n^{2n}$ , the almost complex structure  $J$  on  $\tilde{M}_n^{2n}$  interchanges the tangent bundle of  $M_n^{2n}$  with its normal bundle of  $M_n^{2n}$ .

Assume that  $\Psi : M_n^{2n} \rightarrow \tilde{M}_n^{2n}$  is a  $\theta$ -slant immersion with distributions  $\mathcal{D}_s^n$  and  $\mathcal{D}_t^n$  given above. Let  $e_1, \dots, e_n$  be an orthonormal frame of the space-like distribution  $\mathcal{D}_s^n$ . Then it follows from (3.3) and condition (c) in Definition 1 that

$$(3.10) \quad \langle Pe_i, Pe_j \rangle = -\langle e_i, P^2 e_j \rangle = -\delta_{ij} \sinh^2 \theta.$$

Hence, if we put

$$(3.11) \quad Pe_i = \sinh \theta e_{i^*}, \quad i = 1, \dots, n,$$

then  $e_{1^*}, \dots, e_{n^*}$  form an orthonormal frame of the time-like distribution  $\mathcal{D}_t^n$ . Also, it follows from (3.11) and  $P^2 = (\sinh^2 \theta)I$  that

$$(3.12) \quad Pe_{i^*} = \sinh \theta e_i, \quad i = 1, \dots, n.$$

Next, let us put  $Fe_i = \cosh \theta \xi_i, i = 1, \dots, n$ . Then we have

$$(3.13) \quad Je_i = \sinh \theta e_{i^*} + \cosh \theta \xi_i, \quad i = 1, \dots, n.$$

From  $\langle Je_i, Je_j \rangle = \delta_{ij}$  and (3.13) we know that  $\xi_1, \dots, \xi_n$  are orthonormal space-like normal vector fields of  $M_n^{2n}$ .

Similarly, if we put  $Fe_{i^*} = \cosh \theta \xi_{i^*}, i = 1, \dots, n$ . Then we obtain

$$(3.14) \quad Je_{i^*} = \sinh \theta e_i + \cosh \theta \xi_{i^*}, \quad i = 1, \dots, n,$$

where  $\xi_{1^*}, \dots, \xi_{n^*}$  are orthonormal time-like normal vectors. Moreover, it is easy to verify that  $\xi_1, \dots, \xi_n, \xi_{1^*}, \dots, \xi_{n^*}$  form an orthonormal frame of the normal bundle of the slant immersion  $\Psi : M_n^{2n} \rightarrow \tilde{M}_n^{2n}$ .

From (1.1), (3.13) and (3.14) we also have

$$(3.15) \quad J\xi_i = -\cosh \theta e_i - \sinh \theta \xi_{i^*}, \quad J\xi_{i^*} = -\cosh \theta e_{i^*} - \sinh \theta \xi_i$$

for  $i = 1, \dots, n$ .

The frame

$$\{e_1, \dots, e_n, e_{1^*}, \dots, e_{n^*}, \xi_1, \dots, \xi_n, \xi_{1^*}, \dots, \xi_{n^*}\}$$

chosen above is called an *adapted slant frame* of the  $\theta$ -slant immersion.

**Remark 3.1.** Let  $\{e_1, e_{1^*}, \xi_1, \xi_{1^*}\}$  be an adapted slant frame of a  $\theta$ -slant surface  $M_1^2$  in a Lorentzian Kaehler surface  $\tilde{M}_1^2$ . If we put

$$(3.16) \quad \hat{e}_1 = \frac{e_{1^*} + e_1}{\sqrt{2}}, \quad \hat{e}_2 = \frac{e_{1^*} - e_1}{\sqrt{2}}, \quad \hat{\xi}_1 = \frac{\xi_{1^*} + \xi_1}{\sqrt{2}}, \quad \hat{\xi}_{1^*} = \frac{\xi_{1^*} - \xi_1}{\sqrt{2}},$$

then we have

$$(3.17) \quad \begin{aligned} \langle \hat{e}_1, \hat{e}_1 \rangle &= \langle \hat{e}_{1^*}, \hat{e}_{1^*} \rangle = 0, \quad \langle \hat{e}_1, \hat{e}_{1^*} \rangle = -1, \\ J\hat{e}_1 &= \sinh \theta \hat{e}_1 + \sinh \theta \hat{\xi}_1, \quad J\hat{e}_{1^*} = \sinh \theta \hat{e}_{1^*} + \sinh \theta \hat{\xi}_{1^*}, \\ J\hat{\xi}_1 &= -\cosh \theta \hat{e}_1 - \sinh \theta \hat{\xi}_1, \quad J\hat{\xi}_{1^*} = -\cosh \theta \hat{e}_{1^*} - \sinh \theta \hat{\xi}_{1^*}. \end{aligned}$$

Such a pseudo-orthonormal frame  $\{\hat{e}_1, \hat{e}_{1^*}, \hat{\xi}_1, \hat{\xi}_{1^*}\}$  on the  $\theta$ -slant surface is called an *adapted pseudo-orthonormal slant frame*.

Obviously, adapted pseudo-orthonormal slant frames can also be defined for slant immersions from neutral manifolds  $M_n^{2n}$  into neutral Kaehler manifolds  $\tilde{M}_n^{2n}$  in a similar way.

#### 4. SLANT SUBMANIFOLDS WITH $\nabla P = 0$

Let  $\Psi : M_n^{2n} \rightarrow \tilde{M}_n^{2n}$  be a  $\theta$ -slant immersion of a neutral manifold into a neutral Kaehler manifold. Let us choose an adapted slant frame  $e_1, \dots, e_n, e_{1^*}, \dots, e_{n^*}, \xi_1, \dots, \xi_n, \xi_{1^*}, \dots, \xi_{n^*}$  of  $\Psi$ . Put

$$(4.1) \quad \begin{aligned} \nabla_X e_i &= \sum_{j=1}^n \omega_i^j(X) e_j + \sum_{j=1}^n \omega_i^{j^*}(X) e_{j^*}, \\ \nabla_X e_{i^*} &= \sum_{j=1}^n \omega_{i^*}^j(X) e_j + \sum_{j=1}^n \omega_{i^*}^{j^*}(X) e_{j^*}, \quad i = 1, \dots, n, \end{aligned}$$

for  $X$  tangent to  $M_n^{2n}$ , where  $\nabla$  is the Levi-Civita connections of  $M_n^{2n}$ .

From  $\langle e_i, e_{j^*} \rangle = 0$  and (4.1) we obtain

$$(4.2) \quad \omega_i^{j^*} = \omega_{i^*}^j, \quad i, j = 1, \dots, n.$$

As usual, we define  $\nabla P$  by

$$(4.3) \quad (\nabla_X P)Y = \nabla_X(PY) - P(\nabla_X Y)$$

for  $X, Y$  tangent to  $M_n^{2n}$ . The endomorphism  $P$  is called *parallel* if  $\nabla P = 0$  holds identically. It follows from (4.1) and (4.2) that

$$\begin{aligned} (\nabla_X P)e_i &= \sinh \theta \sum_{j=1}^n \{\omega_{i^*}^{j^*}(X) - \omega_i^j(X)\} e_{j^*}, \\ (\nabla_X P)e_{i^*} &= \sinh \theta \sum_{j=1}^n \{\omega_i^j(X) - \omega_{i^*}^{j^*}(X)\} e_j, \quad i = 1, \dots, n, \end{aligned}$$

which imply the following.

**Proposition 4.1.** *Let  $\Psi : M_n^{2n} \rightarrow \tilde{M}_n^{2n}$  be a slant immersion of a neutral manifold into a neutral Kaehler manifold. Then we have  $\nabla P = 0$  if and only if with respect to an adapted slant frame of  $\Psi$  we have  $\omega_{i^*}^{j^*} = \omega_i^j, i, j = 1, \dots, n$ .*

An important application of Proposition 4.1 is the following.

**Corollary 4.1.** *Let  $\Psi : M_1^2 \rightarrow \tilde{M}_1^2$  be a purely real surface in a neutral Kaehler surface  $\tilde{M}_1^2$ . Then  $M_1^2$  is a slant surface in  $\tilde{M}_1^2$  if and only  $\nabla P = 0$  holds identically.*

*Proof.* Under the hypothesis, if  $M_1^2$  is slant in  $\tilde{M}_1^2$ , we have  $\omega_{1^*}^{1^*} = \omega_1^1 = 0$  with respect to an adapted slant frame  $e_1, e_{1^*}, \xi_1, \xi_{1^*}$ . Hence, by applying Proposition 4.1, we know that  $\nabla P = 0$  holds identically.

Conversely, assume that  $M_1^2$  is a purely real surface in  $\tilde{M}_1^2$  satisfying  $\nabla P = 0$ . Let  $e_1, e_{1^*}$  be an orthonormal frame satisfying (3.5) on  $M_1^2$ . Then there exists a function  $\alpha$  such that  $Pe_1 = \sinh \alpha e_{1^*}$  and  $P^2 = (\sinh^2 \alpha)I$ . Hence, we have

$$(4.4) \quad \begin{aligned} 0 &= (\nabla_X P)e_1 = \nabla_X(\sinh \alpha e_{1^*}) - P(\omega_1^{1^*}(X)e_{1^*}) \\ &= (X\alpha) \cosh \alpha e_{1^*} + \sinh \alpha \nabla_X e_{1^*} - \omega_1^{1^*}(X)Pe_{1^*}. \end{aligned}$$

Since  $\nabla_X e_{1^*}$  and  $Pe_{1^*}$  are parallel to  $e_1$ , (4.4) implies that  $\alpha$  is constant. Therefore, the surface is slant. ■

The next result characterizes slant submanifolds with  $\nabla P = 0$  in term of the shape operator.

**Proposition 4.2.** *Let  $\Psi : M_n^{2n} \rightarrow \tilde{M}_n^{2n}$  be a purely real immersion of a neutral manifold into a neutral Kaehler manifold. Then  $\nabla P = 0$  holds identically if and only if the shape operator satisfies*

$$(4.5) \quad A_{FY}Z = A_{FZ}Y$$

for vectors  $Y, Z$  tangent to  $M_n^{2n}$ .

*Proof.* Let  $\Psi : M_n^{2n} \rightarrow \tilde{M}_n^{2n}$  be a purely real immersion. Then it follows from (3.1), (3.3) and  $\tilde{\nabla}J = 0$  that

$$(4.6) \quad \begin{aligned} 0 &= \tilde{\nabla}_X(JY) - J\tilde{\nabla}_X Y \\ &= \tilde{\nabla}_X(PY) + \tilde{\nabla}_X(FY) - J\nabla_X Y - Jh(X, Y) \\ &= \nabla_X(PY) + h(X, PY) - A_{FY}X + D_X(FY) - P(\nabla_X Y) \\ &\quad - F(\nabla_X Y) - th(X, Y) - fh(X, Y) \end{aligned}$$



for  $X, Y$  tangent to  $M_n^{2n}$ . Thus, the tangential components of (4.6) yields

$$(\nabla_X P)Y = A_{FY}X + th(X, Y),$$

which implies the Proposition. ■

### 5. SLANT SUBMANIFOLDS WITH $\nabla F = 0$

Let  $\Psi : M_n^{2n} \rightarrow \tilde{M}_n^{2n}$  be a  $\theta$ -slant immersion of a neutral manifold into a neutral Kaehler manifold. For the normal-bundle-valued 1-form  $F$ , we define as usual that

$$(5.1) \quad (\nabla_X F)Y = D_X(FY) - F(\nabla_X Y)$$

for vectors  $Y, Z$  tangent to  $M_n^{2n}$ . With respect to an adapted slant frame

$$(5.2) \quad e_1, \dots, e_n, e_{1^*}, \dots, e_{n^*}, \xi_1, \dots, \xi_n, \xi_{1^*}, \dots, \xi_{n^*},$$

we put

$$(5.3) \quad \begin{aligned} D_X \xi_i &= \sum_{j=1}^n \Phi_i^j(X) \xi_j + \sum_{j=2}^n \Phi_i^{j^*}(X) \xi_{j^*}, \\ D_X \xi_{i^*} &= \sum_{j=1}^n \Phi_{i^*}^j(X) \xi_j + \sum_{j=2}^n \Phi_{i^*}^{j^*}(X) \xi_{j^*}, \quad i = 1, \dots, n. \end{aligned}$$

The next result characterizes slant submanifolds with  $\nabla F = 0$  in term of connection forms.

**Proposition 5.1.** *Let  $\Psi : M_n^{2n} \rightarrow \tilde{M}_n^{2n}$  be a slant immersion of a neutral manifold into a neutral Kaehler manifold. Then  $\nabla F = 0$  holds if and only if we have*

$$(5.4) \quad F\nabla_X Y = D_X FY$$

for  $X, Y$  tangent to  $M_n^{2n}$ ; or equivalently, with respect to an adapted slant frame, we have

$$(5.5) \quad \Phi_r^s = \omega_r^s, \quad r, s = 1, \dots, n, 1^*, \dots, n^*.$$

*Proof.* Let  $\Psi : M_n^{2n} \rightarrow \tilde{M}_n^{2n}$  be a  $\theta$ -slant immersion. We find from (3.13), (4.1) and (5.3) that

$$(D_X F)e_r = \cosh \theta \left\{ \sum_{j=1}^n (\Phi_r^j(X) - \omega_r^j(X)) \xi_j + \sum_{j=1}^n (\Phi_r^{j^*}(X) - \omega_r^{j^*}(X)) \xi_{j^*} \right\},$$

which implies the Proposition. ■

An immediate consequence of Proposition 5.1 is the following.

**Corollary 5.1.** *Let  $\Psi : M_n^{2n} \rightarrow \tilde{M}_n^{2n}$  be a slant immersion of a neutral manifold into a neutral Kaehler manifold. If  $\nabla F = 0$  holds, then the Riemannian curvature tensor  $R$  and the normal curvature tensors  $R^D$  satisfy*

$$(5.6) \quad F(R(X, Y)Z) = R^D(X, Y)FZ$$

for  $X, Y, Z$  tangent to  $M_n^{2n}$ . In particular, the slant submanifold  $M_n^{2n}$  is flat if and only if it has flat normal connection in  $\tilde{M}_n^{2n}$ .

The next result characterizes purely real submanifolds with  $\nabla F = 0$  in term of second fundamental form.

**Proposition 5.2.** *Let  $\Psi : M_n^{2n} \rightarrow \tilde{M}_n^{2n}$  be a purely real immersion of a neutral manifold into a neutral Kaehler manifold. Then  $\nabla F = 0$  holds if and only if the shape operator  $A$  satisfies*

$$(5.7) \quad fh(X, Y) = h(X, PY)$$

for vectors  $X, Y$  tangent to  $M_n^{2n}$ , or equivalently,

$$(5.8) \quad A_{f\xi}Y = -A_\xi(PY)$$

for  $Y$  tangent to  $M_n^{2n}$  and  $\xi$  normal to  $M_n^{2n}$ .

*Proof.* This is obtained by comparing the normal components of (4.6). ■

Proposition 5.2 implies the following.

**Corollary 5.2.** *Let  $\Psi : M_n^{2n} \rightarrow \tilde{M}_n^{2n}$  be a slant immersion of a neutral manifold into a neutral Kaehler manifold. If  $\nabla F = 0$  holds, then*

$$(5.9) \quad h(e_i, e_i) = h(e_{i^*}, e_{i^*}), \quad i = 1, \dots, n,$$

with respect to the adapted slant frame (5.2). In particular, a slant submanifold with  $\nabla F = 0$  in a neutral Kaehler manifold  $\tilde{M}_n^{2n}$  is a minimal submanifold.

*Proof.* Under the hypothesis, it follows from (3.11), (3.12) and (5.7) that

$$h(e_{i^*}, e_{i^*}) = \operatorname{csch} \theta fh(e_i, e_{i^*}) = h(e_i, e_i),$$

which implies the Corollary. ■

The following result characterizes minimal slant surfaces in a neutral Kaehler surface among purely real surfaces in term of  $\nabla F$ .

**Theorem 5.1.** *Let  $\Psi : M_1^2 \rightarrow \tilde{M}_1^2$  be a purely real surface in a neutral Kaehler surface. Then  $\nabla F = 0$  holds if and only if  $M_1^2$  is a minimal slant surface.*

*Proof.* Under the hypothesis, if  $\nabla F = 0$  holds, then the shape operator satisfies (5.8) by Proposition 5.2. Let  $e_1, e_{1^*}$  be an orthonormal frame on  $M_1^2$  satisfying (3.5). Then there is a function  $\alpha$  and normal vector fields  $\xi_1, \xi_{1^*}$  satisfying (3.6)-(3.8). Hence, we have

$$(5.10) \quad J\xi_1 = -\cosh \alpha e_1 - \sinh \alpha \xi_{1^*}, \quad J\xi_{1^*} = -\cosh \alpha e_{1^*} - \sinh \alpha \xi_1.$$

Thus, we obtain

$$(5.11) \quad \begin{aligned} A_{Fe_1} e_{1^*} &= \coth \alpha A_{\xi_1}(Pe_1) = -\operatorname{csch} \alpha \coth \alpha A_{f\xi_{1^*}}(Pe_1) \\ &= \cosh \alpha A_{\xi_{1^*}} e_1 = A_{Fe_{1^*}} e_1. \end{aligned}$$

Therefore, according to Proposition 4.2 and Corollary 4.1,  $M_1^2$  is a slant surface. Consequently,  $M_1^2$  is a minimal slant surface according to Corollary 5.2.

Conversely, if  $\Psi : M_1^2 \rightarrow \tilde{M}_1^2$  is a minimal slant surface, hence with respect to an adapted slant frame  $e_1, e_{1^*}, \xi_1, \xi_{1^*}$  we have

$$(5.12) \quad A_{\xi_1} e_{1^*} = A_{\xi_{1^*}} e_1.$$

Since  $M_1^2$  is minimal, we also have

$$(5.13) \quad h(e_{1^*}, e_{1^*}) = h(e_1, e_1).$$

So, it follows from (5.12) and (5.13) that the second fundamental form satisfies

$$(5.14) \quad h(e_1, e_1) = h(e_{1^*}, e_{1^*}) = \beta \xi_1 + \gamma \xi_{1^*}, \quad h(e_1, e_{1^*}) = -\gamma \xi_1 - \beta \xi_{1^*}$$

for some functions  $\beta, \gamma$ . Thus, after applying (3.11), (3.12), (3.15), and (5.14) we obtain (5.7). Consequently, the slant surface satisfies  $\nabla F = 0$ . ■

**Corollary 5.3.** *If  $\Psi : M_1^2 \rightarrow \tilde{M}_1^2$  is a minimal slant surface in a neutral Kaehler surface, then we have*

$$(5.15) \quad F\nabla_X Y = D_X FY$$

for  $X, Y$  tangent to  $M_1^2$ .

*Proof.* Follows from Theorem 5.1 and (5.1). ■

**Remark 5.1.** Let  $\Psi : M_n^{2n} \rightarrow \tilde{M}_n^{2n}$  be a slant immersion of a neutral manifold into a neutral Kaehler manifold and let  $t$  be the tangent-bundle-valued 1-form on the normal bundle defined by (3.2). Define  $\nabla t$  by

$$(\nabla_X t)\xi = \nabla_X(t\xi) - tD_X\xi$$

for any normal vector field  $\xi$  and tangent vector  $X$ . Then, we may prove that  $\nabla t = 0$  holds if and only if  $\nabla F = 0$  holds.

**Remark 5.2.** Let  $\Psi : M_n^{2n} \rightarrow \tilde{M}_n^{2n}$  be a slant immersion of a neutral manifold into a neutral Kaehler manifold and let  $f$  be the endomorphism of the normal bundle defined by (3.2). Define  $\nabla f$  by

$$(\nabla_X f)\xi = D_X(f\xi) - f(D_X\xi)$$

for any tangent vector  $X$  and normal vector field  $\xi$ . Then  $\nabla f = 0$  holds identically if and only if we have  $\Phi_{i*}^{j*} = \Phi_i^j$ ,  $i, j = 1, \dots, n$ , with respect to an adapted slant frame of  $\Psi$ .

## 6. CLASSIFICATION OF SLANT SURFACES WITH $DH = 0$ IN $\mathbf{C}_1^2$

The light cone  $\mathcal{LC}$  in  $\mathbf{C}_1^2$  is defined by  $\mathcal{LC} = \{v \in \mathbf{C}_1^2 : \langle v, v \rangle = 0\}$ .

In this section, we completely classify slant surfaces with parallel mean curvature vector in  $\mathbf{C}_1^2$ .

**Theorem 6.1.** *Let  $\Psi : M_1^2 \rightarrow \mathbf{C}_1^2$  be a slant surface in the Lorentzian complex plane  $\mathbf{C}_1^2$ . If  $M_1^2$  has parallel mean curvature vector, then either  $M_1^2$  is a minimal slant surface or, up to rigid motions,  $M_1^2$  is locally an open portion of one of the following nine types of flat slant surfaces in  $\mathbf{C}_1^2$ :*

- (a) *A Lagrangian surface defined by  $\Psi(x, y) = z(x)e^{iay}$ , where  $a$  is a nonzero real number and  $z$  is a null curve lying in the light cone  $\mathcal{LC}$  satisfying  $\langle iz', z \rangle = a^{-1}$ ;*
- (b) *A Lagrangian surface defined by*

$$\Psi(x, y) = \left( \frac{e^{icy}}{2c} \left( 2cx - i + 2 \int_0^y u(y)dy \right) - \frac{1}{c} \int_0^y e^{icy} u(y)dy, \right. \\ \left. \frac{e^{icy}}{2c} \left( 2cx + i + 2 \int_0^y u(y)dy \right) - \frac{1}{c} \int_0^y e^{icy} u(y)dy \right),$$

where  $c$  is a nonzero real number and  $u(y)$  is a nonzero real-valued function defined on an open interval  $I \ni 0$ ;

(c) *A Lagrangian surface defined by*

$$\Psi(x, y) = \left( \frac{x + ky}{\sqrt{2k}}, \frac{e^{2i(x-ky)}}{2\sqrt{2k}} \right),$$

where  $k$  is a positive real number;

(d) *A Lagrangian surface defined by*

$$\Psi(x, y) = \left( \frac{e^{2i(x+by)}}{2\sqrt{2b}}, \frac{x - by}{\sqrt{2b}} \right),$$

where  $b$  is a positive real number;

(e) *A Lagrangian surface defined by*

$$\Psi(x, y) = \frac{\sqrt{a}}{\sqrt{2b}} \left( \frac{e^{i(1+a^{-1})(ax+by)}}{a + 1}, \frac{e^{i(a^{-1}-1)(ax-by)}}{a - 1} \right),$$

where  $a$  and  $b$  are positive real numbers with  $a \neq 1$ ;

(f) *A Lagrangian surface defined by*

$$\Psi(x, y) = \frac{\sqrt{a}}{\sqrt{2k}} \left( \frac{e^{i(a^{-1}-1)(ax+ky)}}{a - 1}, \frac{e^{i(1+a^{-1})(ax-ky)}}{a + 1} \right),$$

where  $a$  and  $k$  are positive real numbers with  $a \neq 1$ ;

(g) *A Lagrangian surface defined by*

$$\Psi(x, y) = e^{(i-a)x+(i-a^{-1})by} \left( e^{2ax} - \frac{(a+i)^4 e^{2b^{-1}by}}{8b(1+a^2)^2}, e^{2ax} - \frac{e^{2a^{-1}by}}{8b} \right),$$

where  $a$  is a positive real number and  $b$  is a nonzero real number;

(h) *A proper slant surface with slant angle  $\theta$  defined by*

$$\Psi(x, y) = z(x) \frac{(2y \sinh \theta - a \cosh \theta)^{\frac{1}{2} - \frac{i}{2} \operatorname{csch} \theta}}{\sinh \theta - i},$$

where  $a$  is a real number and  $z(x)$  is a null curve lying in the light cone  $\mathcal{LC}$  which satisfies  $\langle z', iz \rangle = \cosh^2 \theta$ ;

(i) A proper slant surface with slant angle  $\theta$  defined by

$$\begin{aligned} \Psi(x, y) = & \left( \operatorname{sech}^2 \theta \int_0^y u(y)(2y \sinh \theta - a \cosh \theta)^{\frac{3}{2} - \frac{1}{2} \operatorname{csch} \theta} dy \right. \\ & + (2y \sinh \theta - a \cosh \theta)^{\frac{1}{2} - \frac{1}{2} \operatorname{csch} \theta} \left( x + \frac{i}{2} + \operatorname{sech}^2 \theta \int_0^y (2y \sinh \theta - a \cosh \theta) u(y) dy \right), \\ & \operatorname{sech}^2 \theta \int_0^y u(y)(2y \sinh \theta - a \cosh \theta)^{\frac{3}{2} - \frac{1}{2} \operatorname{csch} \theta} dy \\ & \left. + (2y \sinh \theta - a \cosh \theta)^{\frac{1}{2} - \frac{1}{2} \operatorname{csch} \theta} \left( x - \frac{i}{2} - \operatorname{sech}^2 \theta \int_0^y (2y \sinh \theta - a \cosh \theta) u(y) dy \right) \right), \end{aligned}$$

where  $a$  is a real number and  $u(y)$  is a nonzero real-valued function defined on an open interval  $I \ni 0$ .

*Proof.* Let  $\Psi : M_1^2 \rightarrow \mathbf{C}_1^2$  be a  $\theta$ -slant surface with  $DH = 0$  in  $\mathbf{C}_1^2$ . Then  $\langle H, H \rangle$  is constant. Thus, if  $\Psi$  is non-minimal, then either  $\Psi$  is quasi-minimal or  $\langle H, H \rangle$  is a nonzero constant. When  $\Psi$  is quasi-minimal, it follows from the main result of [12] that  $M_1^2$  is a flat surface given by cases (a), (b), (h) or (i) of the theorem. So, in the remaining part of the proof of this theorem, we assume that  $\langle H, H \rangle$  is a nonzero constant.

On the slant surface  $M_1^2$  we may choose an adapted pseudo-orthonormal slant frame  $\{e_1, e_{1^*}, \xi_1, \xi_{1^*}\}$  such that

$$(6.1) \quad \langle e_1, e_1 \rangle = \langle e_{1^*}, e_{1^*} \rangle = 0, \quad \langle e_1, e_{1^*} \rangle = -1,$$

$$(6.2) \quad \langle \xi_1, \xi_1 \rangle = \langle \xi_{1^*}, \xi_{1^*} \rangle = 0, \quad \langle \xi_1, \xi_{1^*} \rangle = -1,$$

$$(6.3) \quad J e_1 = \sinh \theta e_1 + \cosh \theta \xi_1, \quad J e_{1^*} = \sinh \theta e_{1^*} + \cosh \theta \xi_{1^*}$$

It follows from (5.3) and (6.2) that

$$(6.4) \quad D_X \xi_1 = \Phi(X) \xi_1, \quad D_X \xi_{1^*} = -\Phi(X) \xi_{1^*}, \quad \Phi = \Phi_1^1.$$

From Corollary 4.1 we have  $\nabla P = 0$ . Hence, after applying Proposition 4.2, we see that the second fundamental form satisfies

$$(6.5) \quad h(e_1, e_1) = \beta \xi_1 + \gamma \xi_{1^*}, \quad h(e_1, e_{1^*}) = \mu \xi_1 + \beta \xi_{1^*}, \quad h(e_{1^*}, e_{1^*}) = \lambda \xi_1 + \mu \xi_{1^*}$$

for some functions  $\beta, \gamma, \lambda, \mu$ .

From (2.9), (6.1) and (6.5), we know that the mean curvature vector is given by

$$(6.6) \quad H = -h(e_1, e_{1^*}) = -\mu \xi_1 - \beta \xi_{1^*}.$$

Since  $\langle H, H \rangle$  is a nonzero constant, (6.2) and (6.6) implies that  $\beta$  and  $\mu$  are nowhere zero. Moreover, since  $DH = 0$ , it follows from (6.4) and (6.6) that

$$(6.7) \quad \Phi = d(\ln \beta) = -d(\ln \mu).$$

Hence, we have  $\mu = b\beta^{-1}$  for some nonzero real number  $b$ .

From (4.1) and (6.1) we have

$$(6.8) \quad \nabla_X e_1 = \omega(X)e_1, \quad \nabla_X e_{1^*} = -\omega(X)e_{1^*}, \quad \omega = \omega_1^1.$$

Now, by applying Lemma 3.2 of [9], we also have

$$(6.9) \quad \omega(e_1) - \Phi(e_1) = 2\beta \tanh \theta, \quad \omega(e_{1^*}) - \Phi(e_{1^*}) = 2\mu \tanh \theta.$$

On the other hand, it follows from (6.4), (6.5), (6.7), (6.8) and the equation of Codazzi that

$$(6.10) \quad \omega = \Phi, \quad e_{1^*}\gamma = 3\gamma\omega(e_{1^*}), \quad e_1\lambda = -3\lambda\omega(e_1).$$

By combining (6.9) and the first equation in (6.10) we get  $\theta = 0$ . Hence,  $\Psi$  is a Lagrangian immersion. Therefore, (6.3) reduces to

$$(6.11) \quad Je_1 = \xi_1, \quad Je_{1^*} = \xi_{1^*}.$$

From (6.7) and  $\omega = \Phi$ , we have  $d\beta = \beta\omega$ . By applying this and (6.8) we derive that  $[\beta^{-1}e_1, \beta e_{1^*}] = 0$ . Thus, there exist coordinates  $\{x, y\}$  such that

$$(6.12) \quad \frac{\partial}{\partial x} = \beta^{-1}e_1, \quad \frac{\partial}{\partial y} = \beta e_{1^*}.$$

So, by (6.1) and (6.12) we know that the metric tensor  $g$  is

$$(6.13) \quad g = -(dx \otimes dy + dy \otimes dx),$$

which implies that the surface is flat and the Levi-Civita connection satisfies

$$(6.14) \quad \nabla_{\partial/\partial x} \frac{\partial}{\partial x} = \nabla_{\partial/\partial x} \frac{\partial}{\partial y} = \nabla_{\partial/\partial y} \frac{\partial}{\partial y} = 0.$$

From (6.12) and (6.14) we get

$$(6.15) \quad \omega(e_1) = \beta_x, \quad \omega(e_{1^*}) = \frac{\beta_y}{\beta^2}.$$

By applying (6.7), (6.10) and (6.15) we obtain

$$(6.16) \quad \gamma = p(x)\beta^3, \quad \lambda\beta^3 = q(y)$$

for some functions  $p(x), q(y)$ .

From (6.5), (6.11) and (6.12) we obtain

$$(6.17) \quad \begin{aligned} h \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) &= i\Psi_x + ip(x)\Psi_y, \\ h \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) &= ib\Psi_x + i\Psi_y, \\ h \left( \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right) &= iq(y)\Psi_x + ib\Psi_y, \end{aligned}$$

where  $b$  is a nonzero real number.

Since the surface is flat, (6.17) and the equation of Gauss yield  $p(x)q(y) = b$ . Thus, we must have  $p(x) = c$  and  $q(y) = b/c$  for some nonzero real number  $c$ . Consequently, we obtain from (6.14), (6.17) and the formula of Gauss that

$$(6.18) \quad \begin{aligned} \Psi_{xx} &= i\Psi_x + ic\Psi_y, \\ \Psi_{xy} &= ib\Psi_x + i\Psi_y, \\ \Psi_{yy} &= ibc^{-1}\Psi_x + ib\Psi_y. \end{aligned}$$

The first two equations in (6.18) imply

$$(6.19) \quad \Psi_{xxx} = 2i\Psi_{xx} + (1 - bc)\Psi_x.$$

**Case 1.**  $bc = 1$ . After solving (6.19) we obtain

$$(6.20) \quad \Psi(x, y) = e^{2ix}A(y) + xB(y) + C(y)$$

for some vector functions  $A(y), B(y)$  and  $C(y)$ . Substituting this into the first equation in (6.18) yields

$$(6.21) \quad A'(y) = 2ibA(y), \quad B'(y) = 0, \quad C'(y) = -bB(y).$$

By solving the three equations in (6.21) we have

$$A(y) = c_2 e^{2iby}, \quad B(y) = c_1, \quad C(y) = -c_1 by$$

for some vectors  $c_1, c_2$ . Combining these with (6.20) gives

$$(6.22) \quad \Psi(x, y) = c_1(x - by) + c_2 e^{2i(x+by)}.$$

Therefore, after choosing suitable initial conditions, we obtain cases (c) and (d) of the theorem.



**Case 2.**  $bc = a^2, a > 0$  and  $a \neq 1$ . In this case, after solving (6.19) we obtain

$$(6.23) \quad \Psi(x, y) = e^{i(1-a)x} (e^{2iax}A(y) + B(y)) + C(y).$$

Substituting this into the first equation in (6.18) we get

$$(6.24) \quad aA'(y) = i(1+a)bA(y), \quad aB'(y) = i(a-1)bB(y), \quad C'(y) = 0.$$

After solving the three equations in (6.24) we obtain from (6.23) that

$$(6.25) \quad \Psi(x, y) = e^{i(1-a)x} \left\{ c_1 e^{i(2ax+(1+a^{-1})by)} + c_2 e^{i(1-a^{-1})y} \right\} + c_0$$

for some vectors  $c_0, c_1, c_2$ . Thus, by choosing suitable initial conditions, we obtain cases (e) and (f).

**Case 3.**  $bc = -a^2, a > 0$ . In this case, after solving (6.19) in the similar way as case (2) we obtain

$$\Psi(x, y) = e^{(i-a)x} \left\{ c_1 e^{2ax+(i-a^{-1})by} + c_2 e^{(i+a^{-1})by} \right\} + c_0$$

for some vectors  $c_0, c_1, c_2$ . Hence, after choosing suitable initial conditions, we obtain case (g). ■

**Remark 6.1.** By direct computation, one can verify that the nine types of surfaces described in Theorem 6.1 are slant surfaces with nonzero parallel mean curvature vector.

**Remark 6.2.** In views of Theorem 6.1 and Theorem 1.1 of [7, page 50], we know that the situation of slant surfaces with parallel mean curvature vector in  $\mathbf{C}^2$  and in  $\mathbf{C}_1^2$  are quite different.

### 7. CLASSIFICATION OF MINIMAL SLANT SURFACES IN $\mathbf{C}_1^2$

In this section we classify minimal slant surfaces in  $\mathbf{C}_1^2$ .

**Theorem 7.1.** *Let  $\Psi : M_1^2 \rightarrow \mathbf{C}_1^2$  be a minimal slant surface in the Lorentzian complex plane  $\mathbf{C}_1^2$ . Then, up to rigid motions,  $M_1^2$  is locally an open portion of one of the following three types of surfaces:*

- (i) *A totally geodesic slant plane;*

(ii) A flat  $\theta$ -slant surface defined by

$$\begin{aligned}\Psi(x, y) &= \left( x - \frac{iy}{2} \cosh \theta \coth \theta + (i \operatorname{sech} \theta + \tanh \theta) K(y), \right. \\ &\quad \left. x - y + \frac{iy}{4} (\cosh 2\theta - 3) \operatorname{csch} \theta + (i \operatorname{sech} \theta + \tanh \theta) K(y) \right),\end{aligned}$$

where  $K(y)$  is a non-constant function;

(iii) A non-flat  $\theta$ -slant surface defined by

$$\begin{aligned}\Psi(x, y) &= \left( \int_0^y \frac{dy}{\sqrt{v'(y)}} - \frac{1 + i \sinh \theta}{2c} \int_0^x \frac{u(x) dx}{\sqrt{u'(x)}} + \frac{i \cosh \theta}{2b^{\frac{3}{2}}} \int_0^y \frac{v(y) dy}{\sqrt{v'(y)}} \right. \\ &\quad + b^{\frac{3}{2}} c (i \operatorname{sech} \theta - \tanh \theta) \int_0^x \frac{dx}{\sqrt{u'(x)}}, \int_0^y \frac{dy}{\sqrt{v'(y)}} + \frac{1 + i \sinh \theta}{2c} \int_0^x \frac{u(x) dx}{\sqrt{u'(x)}} \\ &\quad \left. - \frac{i \cosh \theta}{2b^{\frac{3}{2}}} \int_0^y \frac{v(y) dy}{\sqrt{v'(y)}} + b^{\frac{3}{2}} c (i \operatorname{sech} \theta - \tanh \theta) \int_0^x \frac{dx}{\sqrt{u'(x)}} \right),\end{aligned}$$

where  $b, c$  are nonzero real numbers, and  $u(x), v(y)$  are functions with  $u'(x) > 0$ ,  $v'(y) > 0$  defined respectively on open intervals  $I_1$  and  $I_2$  containing 0.

*Proof.* Let  $\Psi : M_1^2 \rightarrow \mathbf{C}_1^2$  be a minimal slant surface in the Lorentzian complex plane  $\mathbf{C}_1^2$ . Let  $\{e_1, e_{1^*}, \xi_1, \xi_{1^*}\}$  be an adapted pseudo-orthonormal slant frame of  $M_1^2$ . Then, as in the proof of Theorem 6.1, we have

$$(7.1) \quad \langle e_1, e_1 \rangle = \langle e_{1^*}, e_{1^*} \rangle = 0, \quad \langle e_1, e_{1^*} \rangle = -1,$$

$$(7.2) \quad \langle \xi_1, \xi_1 \rangle = \langle \xi_{1^*}, \xi_{1^*} \rangle = 0, \quad \langle \xi_1, \xi_{1^*} \rangle = -1,$$

$$(7.3) \quad J e_1 = \sinh \theta e_1 + \cosh \theta \xi_1, \quad J e_{1^*} = \sinh \theta e_{1^*} + \cosh \theta \xi_{1^*},$$

$$(7.4) \quad \nabla e_1 = \omega e_1, \quad \nabla e_{1^*} = -\omega e_{1^*}, \quad D \xi_1 = \Phi \xi_1, \quad D \xi_{1^*} = -\Phi \xi_{1^*}.$$

Since  $M_1^2$  is minimal slant, it follows from Proposition 5.1 and Theorem 5.1 that  $\Phi = \omega$ . Also, it follows from (2.9), (7.1), Corollary 4.1 and Proposition 4.2 that

$$(7.5) \quad h(e_1, e_1) = f \xi_{1^*}, \quad h(e_1, e_{1^*}) = 0, \quad h(e_{1^*}, e_{1^*}) = k \xi_1$$

for some real-valued functions  $f, k$ .

From (7.4), (7.5),  $\Phi = \omega$  and the equation of Codazzi we obtain

$$(7.6) \quad e_{1^*} f = 3f \omega(e_{1^*}), \quad e_1 k = -3k \omega(e_1).$$

We divide the proof into several cases:

**Case (a).**  $f = k = 0$ . In this case, the slant surface is totally geodesic. Hence, the surface is an open portion of a slant plane. This gives case (i) of the theorem.

**Case (b).**  $f = 0$  and  $k \neq 0$ . In this case, (7.5) and the equation of Gauss show that the slant surface is flat.

If we choose a local coordinate system  $\{x, y\}$  in  $M_1^2$  such that  $\partial/\partial x, \partial/\partial y$  are parallel to  $e_1, e_{1^*}$ , respectively, then the metric tensor of  $M_1^2$  takes the form:

$$(7.7) \quad g = -\psi^2(dx \otimes dy + dy \otimes dx)$$

for some nonzero real-valued function  $\psi = \psi(x, y)$ . We may put

$$(7.8) \quad \frac{\partial}{\partial x} = \psi e_1, \quad \frac{\partial}{\partial y} = \psi e_{1^*}.$$

The Levi-Civita connection satisfies

$$(7.9) \quad \nabla_{\partial/\partial x} \frac{\partial}{\partial x} = \frac{2\psi_x}{\psi} \frac{\partial}{\partial x}, \quad \nabla_{\partial/\partial x} \frac{\partial}{\partial y} = 0, \quad \nabla_{\partial/\partial y} \frac{\partial}{\partial y} = \frac{2\psi_y}{\psi} \frac{\partial}{\partial y}.$$

In view of (7.3) and (7.8) we have

$$(7.10) \quad h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = 0, \quad h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0, \quad h\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = k\psi^2\xi_1.$$

Thus, by applying (7.3), (7.8), (7.9) and (7.10), we know that the slant immersion satisfies

$$(7.11) \quad \begin{aligned} \Psi_{xx} &= \frac{2\psi_x}{\psi} \Psi_x, \quad \Psi_{xy} = 0, \\ \Psi_{yy} &= \frac{2\psi_y}{\psi} \Psi_y + (\operatorname{isech} \theta + \tanh \theta)k\psi \Psi_x. \end{aligned}$$

The compatibility conditions of this system are given by

$$(7.12) \quad \psi\psi_{xy} - \psi_x\psi_y = 0, \quad \psi k_x = -3k\psi_x.$$

The first condition in (7.12) implies  $\psi^2 = p(x)q(y)$  for some functions  $p(x)$  and  $q(y)$ . Thus, after replacing  $x$  and  $y$  by some anti-derivatives of  $p(x)$  and  $q(y)$ , respectively, we get

$$(7.13) \quad g = -(dx \otimes dy + dy \otimes dx).$$

Hence, the second condition in (7.12) becomes  $k_x = 0$  which implies that  $k = k(y)$ . Therefore, system (7.11) becomes

$$(7.14) \quad \begin{aligned} \Psi_{xx} &= \Psi_{xy} = 0, \\ \Psi_{yy} &= (i \operatorname{sech} \theta + \tanh \theta) k(y) \Psi_x. \end{aligned}$$

After solving this system and choosing suitable initial conditions, we obtain case (ii) of the theorem.

**Case (c).**  $k = 0$  and  $f \neq 0$ . After interchanging  $x$  and  $y$  this reduces to case (b).

**Case (d).**  $f, k \neq 0$ . We find from (7.4) and (7.6) that  $[f^{-\frac{1}{3}}e_1, k^{-\frac{1}{3}}e_2] = 0$ . Thus, there exist coordinates  $\{x, y\}$  such that

$$(7.15) \quad e_1 = f^{\frac{1}{3}} \frac{\partial}{\partial x}, \quad e_2 = k^{\frac{1}{3}} \frac{\partial}{\partial y}.$$

Hence, we find from (7.5) and (7.15) that

$$(7.16) \quad h \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) = f^{\frac{1}{3}} \xi_{1*}, \quad h \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = 0, \quad h \left( \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right) = k^{\frac{1}{3}} \xi_1.$$

Moreover, we know from (7.2) and (7.15) that

$$(7.17) \quad g = -\frac{1}{\delta} (dx \otimes dy + dy \otimes dx), \quad \delta = f^{\frac{1}{3}} k^{\frac{1}{3}}.$$

The Levi-Civita connection of (7.17) satisfies

$$(7.18) \quad \nabla_{\partial/\partial x} \frac{\partial}{\partial x} = -(\ln \delta)_x \frac{\partial}{\partial x}, \quad \nabla_{\partial/\partial x} \frac{\partial}{\partial y} = 0, \quad \nabla_{\partial/\partial y} \frac{\partial}{\partial y} = -(\ln \delta)_y \frac{\partial}{\partial y}.$$

Consequently, by (7.3), (7.16) and (7.18), we obtain

$$(7.19) \quad \begin{aligned} \Psi_{xx} &= -(\ln \delta)_x \Psi_x + f^{\frac{5}{6}} k^{-\frac{1}{6}} (i \operatorname{sech} \theta - \tanh \theta) \Psi_y, \\ \Psi_{xy} &= 0, \\ \Psi_{yy} &= -(\ln \delta)_y \Psi_y + f^{-\frac{1}{6}} k^{\frac{5}{6}} (i \operatorname{sech} \theta + \tanh \theta) \Psi_x. \end{aligned}$$

The compatibility conditions of system (7.19) are given by

$$(7.20) \quad (\ln \delta)_{xy} = -\delta^2,$$

$$(7.21) \quad (\ln f)_x = (\ln k)_x, \quad (\ln f)_y = (\ln k)_y.$$

After solving (7.20) we find the following solution:

$$(7.22) \quad f^{\frac{1}{3}}k^{\frac{1}{3}} = \frac{c\sqrt{u'(x)v'(y)}}{c^2v(y) - u(x)}$$

for some differentiable functions  $u(x), v(y)$  and nonzero real number  $c$  (see [10]). Also, it follows from (7.21) that  $f = b^3k$  for some nonzero constant  $b$ . Hence, by applying (7.22) and  $f = b^3k$ , we have

$$f = \frac{b^{\frac{3}{2}}c^{\frac{3}{2}}u'(x)^{\frac{3}{4}}v'(y)^{\frac{3}{4}}}{(c^2v(y) - u(x))^{\frac{3}{2}}}, \quad k = \frac{c^{\frac{3}{2}}u'(x)^{\frac{3}{4}}v'(y)^{\frac{3}{4}}}{b^{\frac{3}{2}}(c^2v(y) - u(x))^{\frac{3}{2}}}.$$

Without loss of generality, we may assume that  $u(x)$  and  $v(y)$  are defined on some open intervals  $I_1$  and  $I_2$  containing 0, respectively. Therefore, by combining these with (7.19), we obtain

$$(7.23) \quad \begin{aligned} \Psi_{xx} &= - \left( \ln \frac{c\sqrt{u'(x)v'(y)}}{c^2v(y) - u(x)} \right)_x \Psi_x + \frac{b^{\frac{3}{2}}c\sqrt{u'(x)v'(y)}}{c^2v(y) - u(x)} (i \operatorname{sech} \theta - \tanh \theta) \Psi_y, \\ \Psi_{xy} &= 0, \\ \Psi_{yy} &= - \left( \ln \frac{c\sqrt{u'(x)v'(y)}}{c^2v(y) - u(x)} \right)_y \Psi_y + \frac{c\sqrt{u'(x)v'(y)}(i \operatorname{sech} \theta + \tanh \theta)}{b^{\frac{3}{2}}(c^2v(y) - u(x))} \Psi_x. \end{aligned}$$

Solving the second equation in (7.23) gives

$$(7.24) \quad \Psi = z(x) + w(y)$$

for some  $C_1^2$ -valued functions  $z(x)$  and  $w(y)$ . By substituting this into the first equation in (7.23), we find

$$\left( \frac{z'(x)\sqrt{u'(x)}}{c^2v(y) - u(x)} \right)_x = \frac{b^{\frac{3}{2}}cu'(x)\sqrt{v'(y)}}{(c^2v(y) - u(x))^2} (i \operatorname{sech} \theta - \tanh \theta) w'(y).$$

From this we get

$$(7.25) \quad z'(x) = \frac{A(y)(c^2v(y) - u(x))}{c\sqrt{u'(x)v'(y)}} + \frac{b^{\frac{3}{2}}c(i \operatorname{sech} \theta - \tanh \theta)\sqrt{v'(y)}w'(y)}{\sqrt{u'(x)}},$$

which implies that

$$(7.26) \quad \begin{aligned} z(x) &= \frac{cv(y)A(y)}{\sqrt{v'(y)}} \int_0^x \frac{dx}{\sqrt{u'(x)}} - \frac{A(y)}{c\sqrt{v'(y)}} \int_0^x \frac{u(x)}{\sqrt{u'(x)}} dx \\ &\quad + b^{\frac{3}{2}}c(i \operatorname{sech} \theta - \tanh \theta)\sqrt{v'(y)}w'(y) \int_0^x \frac{dx}{\sqrt{u'(x)}} + B(y). \end{aligned}$$

On the other hand, substituting (7.24) into the last equation in (7.23) gives

$$(7.27) \quad \left( \frac{w'(y)\sqrt{v'(y)}}{c^2v(y) - u(x)} \right)_y = \frac{c\sqrt{u'(x)}v'(y)(\operatorname{isech}\theta + \tanh\theta)}{b^{\frac{3}{2}}(c^2v(y) - u(x))^2} z'(x).$$

By combining (7.25) and (7.27) we obtain

$$w''(y) + \frac{v''(y)}{2v'(y)}w'(y) = \frac{\operatorname{isech}\theta + \tanh\theta}{b^{\frac{3}{2}}}A(y).$$

After solving this second order differential equation we get

$$(7.28) \quad w(y) = \frac{\operatorname{isech}\theta + \tanh\theta}{b^{\frac{3}{2}}} \int_0^y \frac{\int_0^y \sqrt{v'(t)}A(t)dt}{\sqrt{v'(y)}} dy + c_1 \int_0^y \frac{dy}{\sqrt{v'(y)}} + c_0$$

for some vectors  $c_0, c_1 \in \mathbf{C}_1^2$ . From (7.24), (7.26) and (7.28) we obtain

$$(7.29) \quad \begin{aligned} \Psi(x, y) &= \frac{cv(y)A(y)}{\sqrt{v'(y)}} \int_0^x \frac{dx}{\sqrt{u'(x)}} - \frac{A(y)}{c\sqrt{v'(y)}} \int_0^x \frac{u(x)}{\sqrt{u'(x)}} dx + c_1 \int_0^y \frac{dy}{\sqrt{v'(y)}} \\ &+ c_1 b^{\frac{3}{2}} c (\operatorname{isech}\theta - \tanh\theta) \int_0^x \frac{dx}{\sqrt{u'(x)}} - c \int_0^x \frac{dx}{\sqrt{u'(x)}} \int_0^y \sqrt{v'(y)} A(y) dy \\ &+ \frac{\operatorname{isech}\theta + \tanh\theta}{b^{\frac{3}{2}}} \int_0^y \frac{\int_0^y \sqrt{v'(t)}A(t)dt}{\sqrt{v'(y)}} dy + B(y) + c_0. \end{aligned}$$

Substituting this into the second equation in (7.23) yields

$$2A'(y)v'(y) = A(y)v''(y),$$

which implies  $A(y) = c_2\sqrt{v'(y)}$  for some vector  $c_2$ . Therefore, (7.29) becomes

$$(7.30) \quad \begin{aligned} \Psi(x, y) &= c_1 \int_0^y \frac{dy}{\sqrt{v'(y)}} + c_1 b^{\frac{3}{2}} c (\operatorname{isech}\theta - \tanh\theta) \int_0^x \frac{dx}{\sqrt{u'(x)}} \\ &- \frac{c_2}{c} \int_0^x \frac{u(x)}{\sqrt{u'(x)}} dx + c_2 \frac{\operatorname{isech}\theta + \tanh\theta}{b^{\frac{3}{2}}} \int_0^y \frac{v(y)}{\sqrt{v'(y)}} dy + B(y) + c_0. \end{aligned}$$

Now, by substituting (7.30) into the first equation in (7.23), we get  $B'(y) = 0$ . Hence,  $B(y) = c_3$  for some vector  $c_3$ . Consequently, after applying a suitable translation on  $\mathbf{C}_1^2$ , we obtain

$$(7.31) \quad \begin{aligned} \Psi(x, y) &= c_1 \int_0^y \frac{dy}{\sqrt{v'(y)}} + c_1 b^{\frac{3}{2}} c (\operatorname{isech}\theta - \tanh\theta) \int_0^x \frac{dx}{\sqrt{u'(x)}} \\ &- \frac{c_2}{c} \int_0^x \frac{u(x)}{\sqrt{u'(x)}} dx + c_2 \frac{\operatorname{isech}\theta + \tanh\theta}{b^{\frac{3}{2}}} \int_0^y \frac{v(y)}{\sqrt{v'(y)}} dy. \end{aligned}$$

From (7.31) we have

$$(7.32) \quad \begin{aligned} \Psi_x &= \frac{b^{\frac{3}{2}}c^2c_1(\operatorname{isech} \theta - \tanh \theta) - c_2u(x)}{c\sqrt{u'(x)}}, \\ \Psi_y &= \frac{b^{\frac{3}{2}}c_1 + c_2(\operatorname{isech} \theta + \tanh \theta)v(y)}{b^{\frac{3}{2}}\sqrt{v'(y)}}. \end{aligned}$$

It follows from (7.3), (7.17), (7.22) and (7.32) that  $c_1$  and  $c_2$  are light-like vectors satisfying  $\langle c_1, c_2 \rangle = -1$  and  $\langle c_1, ic_2 \rangle = \sinh \theta$ . Therefore, up to rigid motions, we may choose  $c_1, c_2$  such that

$$c_1 = (1, 1), \quad c_2 = \frac{1}{2}(1 + i \sinh \theta, -1 - i \sinh \theta).$$

Consequently, we obtain case (iii) of the theorem from (7.31). ■

**Remark 7.1.** By direct computation we can verify that the maps given in case (ii) and case (iii) of Theorem 7.1 define minimal slant surfaces in  $\mathbf{C}_1^2$ .

#### REFERENCES

1. M. Barros and A. Romero, Indefinite Kaehler manifolds, *Math. Ann.*, **261** (1982), 44-62.
2. J. L. Cabrerizo, A. Carriazo, L. M. Fernández and M. Fernández, Slant submanifolds in Sasakian manifolds, *Glasgow. Math. J.*, **42** (2000), 125-138.
3. A. Carriazo, L. M. Fernández and M. B. Hans-Uber, Some slant submanifolds of  $S$ -manifolds, *Acta Math. Hungar.*, **107** (2005), 267-285.
4. A. Carriazo, L. M. Fernández and M. B. Hans-Uber, Minimal slant submanifolds of the smallest dimension in  $S$ -manifolds, *Rev. Mat. Iberoamericana*, **21** (2005), 47-66.
5. B.-Y. Chen, *Geometry of Submanifolds*, Mercer Dekker, New York, 1973.
6. B.-Y. Chen, Slant immersions, *Bull. Austral. Math. Soc.*, **41** (1990), 135-147.
7. B.-Y. Chen, *Geometry of Slant Submanifolds*, Katholieke Universiteit Leuven, Leuven, 1990.
8. B.-Y. Chen, Riemannian submanifolds, *Handbook of differential geometry*, Vol. I, 187-418, North-Holland, Amsterdam, 2000.
9. B.-Y. Chen, Minimal flat Lorentzian surfaces in Lorentzian complex space forms, *Publ. Math. Debrecen*, **73** (2008), 233-248.
10. B.-Y. Chen, Lagrangian minimal surfaces in Lorentzian complex plane, *Arch. Math. (Basel)*, **91** (2008), 366-371.
11. B.-Y. Chen, Nonlinear Klein-Gordon equations and Lorentzian minimal surfaces in Lorentzian complex space forms, *Taiwanese J. Math.*, **13** (2009), 1-24.

12. B.-Y. Chen and O. J. Garay, Quasi-minimal surfaces with parallel mean curvature vector in Lorentzian complex plane, *Results Math.*, (in press).
13. B.-Y. Chen and I. Mihai, Classification of quasi-minimal slant surfaces in Lorentzian complex space forms, *Acta Math. Hugar.*, **122** (2009), 307-328.
14. B.-Y. Chen and Y. Tazawa, Slant submanifolds of complex projective and complex hyperbolic spaces, *Glasgow Math. J.*, **42** (2000), 439-454.
15. B.-Y. Chen and L. Vrancken, Existence and uniqueness theorem for slant immersions and its applications, *Results Math.*, **31** (1997), 28-39; Addendum, *Results Math.*, **39** (2001), 18-22.
16. G. R. Jensen and M. Rigoli, Neutral surfaces in neutral four-spaces, *Matematiche (Catania)*, **45** (1990), 407-443.
17. G. Li and C. Wu, Slant immersions of complex space forms and Chen's inequality, *Acta Math. Sci. Ser. B Engl. Ed.*, **25** (2005), 223-232.
18. I. Mihai and Y. Tazawa, On 3-dimensional contact slant submanifolds in Sasakian space forms, *Bull. Austral. Math. Soc.*, **68** (2003), 275-283.
19. B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.

K. Arslan  
Department of Mathematics,  
Uludağ University,  
16059 Bursa, Turkey  
E-mail: arslan@uludag.edu.tr

A. Carriazo  
Department of Geometry and Topology,  
University of Seville,  
Apdo. de Correos 1160,  
41080-Sevilla, Spain  
E-mail: carriazo@us.es

B.-Y. Chen  
Department of Mathematics,  
Michigan State University,  
East Lansing, MI 48824-1027,  
U.S.A.  
E-mail: bychen@math.msu.edu

C. Murathan  
Department of Mathematics,  
Uludağ University,  
16059 Bursa, Turkey  
E-mail: cengiz@uludag.edu.tr