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MATHEMATICS

A characterization of ccr-curves in $\mathbb{R}^m$

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Abstract. We study the curve in $\mathbb{R}^m$ for which the ratios between two consecutive curvatures are constant (ccr-curves). We show that closed ccr-curves in Euclidean space $\mathbb{R}^m$ are of finite type. We also consider Frenet curves with constant harmonic curvatures and show that an immersed curve in $\mathbb{R}^{2n+1}$ with constant harmonic curvatures H_i at point $\gamma(s_0)$ has a Darboux vertex at that point.

Key words: differential geometry, Frenet curve, W-curve, curves of finite type, ccr-curve.

1. INTRODUCTION

Let $\gamma = \gamma(s) : I \to \mathbb{R}^m$ be a regular curve in $\mathbb{R}^m$ (i.e. $\|\gamma'\|$ is nowhere zero), where I is an interval in $\mathbb{R}$. The curve γ is called a *Frenet curve of rank r* ($r \in \mathbb{N}_0$) if $\gamma'(t)$, $\gamma''(t), \ldots, \gamma^{(r)}(t)$ are linearly independent and $\gamma'(t)$, $\gamma''(t), \ldots, \gamma^{(r+1)}(t)$ are no longer linearly independent for all t in I. In this case, $Im(\gamma)$ lies in an r-dimensional Euclidean subspace of $\mathbb{R}^m$. For each Frenet curve of rank r there occur an associated orthonormal r-frame $\{E_1, E_2, \ldots, E_r\}$ along γ, the Frenet r-frame, and $r-1$ functions $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}: I \longrightarrow \mathbb{R}$, and the Frenet curvatures, such that

$$\begin{bmatrix} E_1' \\ E_2' \\ E_3' \\ \ldots \\ E_r' \end{bmatrix} = v \begin{bmatrix} 0 & \kappa_1 & 0 & \ldots & 0 \\ -\kappa_1 & 0 & \kappa_2 & \ldots & 0 \\ 0 & -\kappa_2 & 0 & \ldots & 0 \\ \ldots & & & & \kappa_{r-1} \\ 0 & 0 & \ldots & -\kappa_{r-1} & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ \ldots \\ E_r \end{bmatrix},$$

where v is the speed of the curve.

In fact, to obtain $E_1, E_2, \ldots, E_r$ it is sufficient to apply the Gram–Schmidt orthonormalization process to $\gamma'(t)$, $\gamma''(t), \ldots, \gamma^{(r)}(t)$. Moreover, the functions $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}$ are easily obtained as by-product during this calculation. More precisely, $E_1, E_2, \ldots, E_r$ and $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}$ are determined by the following formulas [8]:

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$$
\begin{aligned}
v_1(t) \quad &: \quad = \gamma'(t); \quad E_1 := \frac{v_1}{\|v_1(t)\|}; \\
v_k(t) \quad &: \quad = \gamma^{(k)}(t) - \sum_{i=1}^{k-1} < \gamma^{(k)}(t), v_i(t) > \frac{v_i(t)}{\|v_i(t)\|^2}; \\
\kappa_{k-1}(t) \quad &: \quad = \frac{\|v_k(t)\|}{\|v_{k-1}(t)\|\,\|v_1(t)\|}; \\
E_k \quad &: \quad = \frac{v_k}{\|v_k(t)\|},
\end{aligned}
$$

where $k \in \{2,3,...,r\}$. It is natural and convenient to define Frenet curvatures $\kappa_r = \kappa_{r+1} = ... = \kappa_{m-1} = 0$. It is clear that $E_1, E_2, ..., E_r$ and $\kappa_1, \kappa_2, ..., \kappa_{r-1}$ can be defined for any regular curve (not necessarily a Frenet curve) in the neighbourhood of a point s_0 for which $\gamma'(s_0)$, $\gamma''(s_0), ..., \gamma^{(r)}(s_0)$ are linearly independent.

The notion of a generalized helix in $\mathbb{R}^3$, a curve making a constant angle with a fixed direction, can be generalized to higher dimensions in many ways. In [14], the same definition is proposed in $\mathbb{R}^m$. In [9], the definition is more restrictive: the fixed direction makes a constant angle with all the vectors of the Frenet frame. It is easy to check that the definition only works in the odd dimensional case. Moreover, in the same reference, it is proven that the definition is equivalent to the fact that the ratios $\frac{\kappa_2}{\kappa_1}, \frac{\kappa_4}{\kappa_3}, ..., \kappa_i$ being the curvatures, are constant.

In [15] Uribe-Vargas proved that the immersed curve in $\mathbb{R}^{2n+1}$, $n \geqslant 1$ has a Darboux vertex at point $\gamma(s_0)$ if and only if $(\frac{\kappa_1}{\kappa_2})' = 0, (\frac{\kappa_3}{\kappa_4})' = 0, ..., (\frac{\kappa_{2k-1}}{\kappa_{2k}})' = 0$.

Recently, Monterde [11] has considered the Frenet curves in $\mathbb{R}^m$ which have constant curvature ratios (i.e., $\frac{\kappa_2}{\kappa_1}, \frac{\kappa_3}{\kappa_2}, \frac{\kappa_4}{\kappa_3}...$ are constant). The Frenet curves with constant curvature ratios are called ccr-curves.

In the present study we prove that if the harmonic curvatures H_i of the immersed curve in $\mathbb{R}^{2n+1}$ are constant at point $\gamma(s_0)$, then γ has a Darboux vertex at that point.

We also prove that every closed ccr-curve is of finite type.

2. W-CURVES

Definition 1. *A Frenet curve of rank r for which $\kappa_1, \kappa_2, ..., \kappa_{r-1}$ are constant is called (generalized) screw line or helix* [6]. *Since these curves are trajectories of the* 1*-parameter group of the Euclidean transformations*, *Klein and Lie* [10] *called them W-curves.*

A unit speed *W-curve of rank* $2k$ in $\mathbb{R}^m$ has the parameterization of the form

$$\gamma(s) = a_0 + \sum_{i=1}^{k} (a_i \cos \mu_i s + b_i \sin \mu_i s)$$

and a unit speed *W-curve of rank* $(2k+1)$ has the parameterization of the form

$$\gamma(s) = a_0 + b_0 s + \sum_{i=1}^{k} (a_i \cos \mu_i s + b_i \sin \mu_i s),$$

where $a_0, b_0, a_1, ..., a_k, b_1, ..., b_k$ are constant vectors in $\mathbb{R}^m$ and $\mu_1 < \mu_2 < ... < \mu_k$ are positive real numbers.

So, a W-curve of rank 1 is a straight line, a W-curve of rank 2 is a circle, and a W-curve of rank 3 is a right circular helix.

The subset of $\mathbb{R}^{2n}$ parameterized by

$$\vec{x}(u_1, u_2, ..., u_n) = (r_1 \cos(u_1), r_1 \sin(u_1), r_2 \cos(u_2), r_2 \sin(u_2), ..., r_n \cos(u_n), r_n \sin(u_n)),$$

where $u_i \in \mathbb{R}$, is called a flat torus in $\mathbb{R}^{2n}$.

By analogy, the subset of $\mathbb{R}^{2n+1}$ parameterized by

$$\overrightarrow{x}(u_1,u_2,...,u_n)=(r_1\cos(u_1),r_1\sin(u_1),r_2\cos(u_2),r_2\sin(u_2),...,r_n\cos(u_n),r_n\sin(u_n),a),$$

where $u_i\in\mathbb{R}$ and a is a real constant, will be called a flat torus in $\mathbb{R}^{2n+1}$.

We give the following examples.

Example 1. Any curve in a flat torus of the kind

$$\alpha(t)=\overrightarrow{x}(m_1t,m_2t,...,m_nt)$$

has all its curvatures constant (i.e. W-curve).

These curves are the geodesics of the flat tori and it is proven in [13] that they are twisted curves if and only if the constants $m_i\neq m_j$ for all $i\neq j$. For closed twisted curves see also [13].

Example 2. (Helices in $\mathbb{S}^3$) Let $\mathbb{S}^3$ be the unit 3-sphere imbedded in the Euclidean 4-space $\mathbb{E}^4$. A model helix in $\mathbb{S}^3\subset\mathbb{E}^4$ is given by

$$\gamma(s)=(\cos\phi\cos(as),\cos\phi\sin(as),\sin\phi\cos(bs),\sin\phi\sin(bs)),$$

with

$$a^2\cos^2\phi+b^2\sin^2\phi=1.$$

Here s is the arclength parameter. It is easy to see that γ lies in the flat torus:

$$x_1^2+x_2^2=\cos^2\phi,\ x_3^2+x_4^2=\sin^2\phi.$$

Example 3. The Frenet curve $\alpha: I\to\mathbb{R}^4$ given by the parameterization

$$\alpha(s)=\frac{1}{\sqrt{r_1^2+r_2^2}}\left(\frac{r_1}{m_1}\sin(m_1s),-\frac{r_1}{m_1}\cos(m_1s),\frac{r_2}{m_2}\sin(m_2s),-\frac{r_2}{m_2}\cos(m_2s)\right)$$

is a spherical W-curve (with radius 1), (see [11]) where, $r_1^2m_2^2+r_2^2m_1^2=m_1^2m_2^2(r_1^2+r_2^2)$.

3. CURVES OF FINITE TYPE

Let $f(s)$ be a periodic continuous function with period $2\pi r$. Then it is well known that $f(s)$ has a Fourier series expansion given by

$$f(s)=\frac{a_0}{2}+a_1\cos\frac{s}{r}+a_2\cos\frac{2s}{r}+...+b_1\sin\frac{s}{r}+b_2\sin\frac{2s}{r}+...,$$

where a_k and b_k are the Fourier coefficients defined by

$$a_k=\frac{1}{\pi r}\int_{-\pi r}^{\pi r}f(s)\cos\frac{ks}{r}\mathrm{d}s,\qquad k=0,1,2,...,$$

$$b_k=\frac{1}{\pi r}\int_{-\pi r}^{\pi r}f(s)\sin\frac{ks}{r}\mathrm{d}s,\qquad k=1,2,... .$$

Let γ be a closed curve of length $2\pi r$. If $x:\gamma\to\mathbb{R}^m$ is an isometric immersion, then

$$x^{(j)}=\frac{\mathrm{d}^jx}{\mathrm{d}s^j}.$$

Because $\Delta = -\frac{d^2}{ds^2}$, we have

$$\Delta^j H = (-1)^j x^{(2j+2)}, \quad j = 0, 1, 2, \ldots .$$

If x is of finite type, each coordinate function x_i satisfies the following homogeneous ordinary differential equation with constant coefficients:

$$x_i^{(2k+2)} + c_1 x_i^{(2k)} + \ldots + c_{k-1} x_i^{(4)} + c_k x_i^{(2)} = 0, \quad i = 1, 2, \ldots, m$$

for some integer $k \geq 1$ and constant $c_1, \ldots, c_k$. Because our solutions x_i of the above differential equation are periodic solutions with period $2\pi r$, each x_i is a finite linear combination of the following particular solutions:

$$1, \quad \cos\left(\frac{n_i s}{r}\right), \quad \sin\left(\frac{m_i s}{r}\right), \quad n_i, m_i \in \mathbb{Z}.$$

Therefore, each x_i is of the form

$$x_i = c_i + \sum_{t=p_A}^{q_A} \left(a_A(t) \cos\frac{ts}{r} + b_A(t) \sin\frac{ts}{r} \right)$$

for some suitable constant $c_i, a_A(t), b_A(t)$ $(A = 1, \ldots, n)$ and integers p_A, q_A . Thus each x_i has a Fourier series expansion of finite sum. Similarly, if each x_i has a Fourier series expansion of finite sum, then x is of finite type (see [4,5,7]).

Theorem 1. [3] *Let γ be a closed curve of length $2\pi r$ in $\mathbb{R}^m$. Then isometric immersion $x : \gamma \to \mathbb{R}^m$ is of finite type if and only if the Fourier series expansion of each coordinate function of γ,*

$$\gamma(s) = a_0 + \sum_{t=1}^{\infty} \left(a_t \cos\frac{ts}{r} + b_t \sin\frac{ts}{r} \right)$$

has only finite nonzero terms.

Thus, using the above theorems, we have the following corollary.

Corollary 2. *Every closed k-type curve γ in $\mathbb{R}^m$ can be written in the form*

$$\gamma(s) = a_0 + \sum_{i=1}^{k} (a_i \cos\lambda_{t_i} s + b_i \sin\lambda_{t_i} s), \tag{1}$$

where $T(x) = \{t_1, t_2, \ldots t_k\}$ is the order of the curve and $a_0, a_1, \ldots, a_k, b_1, \ldots, b_k$ are vectors in $\mathbb{R}^m$ such that for any i in $\{1, 2, \ldots, k\}$, a_i and b_i are not simultaneously zero. Moreover, if $q = t_k$ is the upper order of γ, then $|a_q| = |b_q| \neq 0$.

Corollary 3. *Every null k-type curve γ in $\mathbb{R}^m$ can be written in the form*

$$\gamma(s) = a_0 + b_0 s + \sum_{i=1}^{k} (a_i \cos\lambda_{t_i} s + b_i \sin\lambda_{t_i} s), \tag{2}$$

where $a_0, b_0, a_1, \ldots, a_k, b_1, \ldots, b_k$ are vectors in $\mathbb{R}^m$ such that $b_0 \neq 0$ and for any i in $\{1, 2, \ldots, k\}$, a_i and b_i are not simultaneously zero. Moreover, $|a_q| = |b_q| \neq 0$, where q is the upper order of the curve γ.

From (1) and (2) we obtain the following corollary.

Corollary 4. [2]
1) *Every k-type curve of $\mathbb{R}^m$ lies in an affine $2k$-subspace $\mathbb{R}^{2k}$ of $\mathbb{R}^m$.*
2) *Every null k-type curve of $\mathbb{R}^m$ lies in an affine $(2k-1)$-subspace $\mathbb{R}^{2k-1}$ of $\mathbb{R}^m$.*

4. GENERALIZED HELICES

In the present section we give some well-known definitions of harmonic curvature and Darboux vertex of a curve in $\mathbb{R}^m$. We prove that the immersed curve in $\mathbb{R}^m$ with constant harmonic curvatures H_i at point $\gamma(s_0)$ has a Darboux vertex at that point.

Definition 2. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^m$ be a regular curve of rank r with unit speed. For $2 \leq j \leq r-2$, the functions $H_j : I \to \mathbb{R}$ defined by*

$$H_0 = 0, H_1 = \frac{\kappa_1}{\kappa_2},\ H_j = \left\{\nabla_{v_1} H_{j-1} + H_{j-2}\kappa_j\right\} \frac{1}{\kappa_{j+1}} \tag{3}$$

are called the harmonic curvatures of γ, where $\kappa_1, \kappa_2, ..., \kappa_{r-1}$ are Frenet curvatures of γ which are not necessarily constant and ∇ is the Levi-Civita connection [12]. *For more details see also* [1].

Definition 3. *The unit speed Frenet curve of rank r is called general helix of order $(r-2)$ if*

$$\sum_{i=1}^{r-2} H_i^2 = c, \tag{4}$$

where c is constant [12].

By the use of (3) and (4) we get the following result.

Proposition 5. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^{2n+1}$ be a regular curve of rank r with unit speed. If γ has constant harmonic curvature, then*

$$\begin{aligned} H_{2r} &= 0, \quad 1 \leq r \leq n, \\ H_{2r-1} &= \frac{\kappa_1}{\kappa_2} \cdot \frac{\kappa_3}{\kappa_4} \cdot ... \cdot \frac{\kappa_{2r-1}}{\kappa_{2r}}, \quad 1 \leq r \leq n. \end{aligned}$$

Definition 4. *Let γ be a smoothly immersed curve in $\mathbb{R}^{2n+1}$, $n \geqslant 1$, with curvatures $\kappa_1, \kappa_2, ..., \kappa_{2n-1}, \kappa_{2n}$, where $\kappa_{2n} \neq 0$. Let us denote*

$$\begin{aligned} a_0 &= \kappa_2\kappa_4...\kappa_{2n}, \\ a_1 &= \frac{\kappa_1}{\kappa_2} a_0, \\ &... \quad ... \\ a_j &= \frac{\kappa_{2j-1}}{\kappa_{2j}} a_{j-1}, \\ a_n &= \frac{\kappa_{2n-1}}{\kappa_{2n}} a_{n-1} = \kappa_1\kappa_3...\kappa_{2n-1}. \end{aligned}$$

The Darboux vector in $\mathbb{R}^{2n+1}$ is defined by

$$\tilde{d}(s) = a_0 t + a_1 n_2 + ... + a_n n_{2n},$$

where $\left\{t = \gamma'(s), n_1, n_2, ..., n_{2n}\right\}$ is the Frenet frame of γ [15].

Lemma 6. [15] *The derivative of $\tilde{d}(s)$ is*

$$\tilde{d}'(s) = a_0' t + a_1' n_2 + ... + a_n' n_{2n}.$$

Definition 5. (*Darboux vertex*): *The point $\gamma(s_0)$ is called Darboux vertex of γ if the first derivative of the Darboux vector $\widetilde{d}(s)$ is vanishing at that point.*

Theorem 7. [15] *Let γ be a smoothly immersed curve in $\mathbb{R}^{2n+1}$ ($n \geqslant 1$), with $\kappa_1, \kappa_2, ..., \kappa_{2n}$ for its curvatures. The curve has a Darboux vertex at point $\gamma(s_0)$ if and only if*

$$\left(\frac{\kappa_1}{\kappa_2}\right)' = 0, \ \left(\frac{\kappa_3}{\kappa_4}\right)' = 0, \ ..., \ \left(\frac{\kappa_{2n-1}}{\kappa_{2n}}\right)' = 0. \tag{5}$$

Proof. Let γ be a smoothly immersed curve in $\mathbb{R}^{2n+1}$. If γ has a Darboux vertex at $\gamma(s_0)$, then by Lemma 6 we get $a_0' = 0$, $a_1' = 0, ..., a_n' = 0$. By Definition 4 we get the result. □

By Proposition 5 and Theorem 7 we get the following results.

Corollary 8. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^{2n+1}$ be a regular curve of rank $2n$ with unit speed. If the harmonic curvatures H_i are constant at the point $\gamma(s_0)$, then γ has a Darboux vertex at that point.*

Proof. If the harmonic curvatures H_i are constant at the point $\gamma(s_0)$, then by Proposition 5 all the ratios $\frac{\kappa_1}{\kappa_2}, \frac{\kappa_3}{\kappa_4}, ..., \frac{\kappa_{2n-1}}{\kappa_{2n}}$ are constant. So, taking the derivatives of the ratios $\frac{\kappa_1}{\kappa_2}, \frac{\kappa_3}{\kappa_4}, ..., \frac{\kappa_{2n-1}}{\kappa_{2n}}$ with respect to s, we obtain (5). Using Theorem 7, we complete the proof. □

Corollary 9. *If $\gamma : I \subset \mathbb{R} \to \mathbb{R}^{2n+1}$ has a Darboux vertex at the point $\gamma(s_0)$, then γ is a general helix of order $(2n-1)$.*

5. CURVES WITH CONSTANT CURVATURE RATIOS

A curve $\gamma = \gamma(s) : I \to \mathbb{R}^m$ is said to have constant curvature ratios (ccr-curve) if all the quotients $\frac{\kappa_{i+1}}{\kappa_i}$ are constant [11].

As is well known, generalized helices in $\mathbb{R}^3$ are characterized by the fact that the quotient $\frac{\tau}{\kappa}$ is constant (Lancret's theorem). It is in this sense that ccr-curves are generalization to $\mathbb{R}^m$ of generalized helices in $\mathbb{R}^3$.

In [9] a generalized helix in the m-dimensional space (m odd) is defined as a curve satisfying that the ratios $\frac{\kappa_2}{\kappa_1}, \frac{\kappa_4}{\kappa_3}, ...$ are constant. It is also proven that the curve is a generalized helix if and only if there exists a fixed direction which makes constant angles with all the vectors of the Frenet frame.

Obviously, ccr-curves are a subset of generalized helices in the sense of [9].

Corollary 10. *Every W-curve is a ccr-curve.*

Lemma 11. [11] *Let β be a ccr-curve with non-constant curvature. Then Frenet's formulae of β are reduced to a linear system of first order differential equations with constant coefficients*

$$\begin{pmatrix} \overrightarrow{e}'_1(t) \\ \overrightarrow{e}'_2(t) \\ \overrightarrow{e}'_3(t) \\ \cdots \\ \overrightarrow{e}'_{n-1}(t) \\ \overrightarrow{e}'_n(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & c_2 & 0 & \cdots & 0 & 0 \\ 0 & -c_2 & 0 & c_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & & & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & c_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & -c_{n-1} & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{e}_1(t) \\ \overrightarrow{e}_2(t) \\ \overrightarrow{e}_3(t) \\ \cdots \\ \overrightarrow{e}_{n-1}(t) \\ \overrightarrow{e}_n(t) \end{pmatrix} \tag{6}$$

for some constants $c_2, ..., c_{n-1}$.

Lemma 12. [16] *Let $\frac{\mathrm{dx}}{\mathrm{dt}} = Ax(t)$ be the linear system of first order differential equations with constant coefficients. Then the homogeneous solutions of the system are given by*

$$x = \sum_{i=1}^{n} d_i u_i e^{\lambda_i t},$$

where u_i are the eigenvectors, λ_i are the eigenvalues of the constant coefficient matrix of the system, and d_i are arbitrary constants.

We prove the following main result.

Theorem 13. (Main Result) *Every closed ccr-curve is of finite type.*

Proof. Let A be the matrix of constant coefficient of system (6). Due to the skewsymmetry of matrix A, it can have no real eigenvalues other than zero. Due to the fact that the determinant of A vanishes only for odd n, we can say that for odd dimensions, 0 is an eigenvalue, whereas for even dimensions, 0 is an eigenvalue only if $k_{n-1} = 0$.

From now on, we shall consider that all the curvatures, and all the constants c_i are not zero. Therefore, the eigenvalues are all of multiplicity 1.

Let $\lambda_l = \alpha_l \pm i\mu_l$, $l = 1, ..., \left[\frac{n}{2}\right]$, with $\alpha_l, \mu_l \in \mathbb{R}$ be the nonzero eigenvalues of the coefficient matrix A. For $n = 2k$, from Lemma 12 the general solution of the system for the first vector becomes

$$\overrightarrow{e}_1(u) = \sum_{l=1}^{k} d_l u_l e^{\alpha_l u} \cos(\mu_l u) + f_l u_l e^{\alpha_l u} \sin(\mu_l u)$$

and similarly for $n = 2k+1$, the general solution of the system for the first vector is

$$\overrightarrow{e}_1(u) = a_0 + \sum_{l=1}^{k} d_l u_l e^{\alpha_l u} \cos(\mu_l u) + f_l u_l e^{\alpha_l u} \sin(\mu_l u),$$

where $a_0, .u_1, ..., u_k$ are vectors in $\mathbb{R}^m$ and d_l, f_l are arbitrary constants.

Condition $\|\overrightarrow{e}_1(u)\| = 1$ for all u implies that all the real parts of the eigenvalues are zero. Therefore, for $n = 2k$, the general solution of the system for the first vector is

$$\overrightarrow{e}_1(u) = \sum_{l=1}^{k} d_l u_l \cos(\mu_l u) + f_l u_l \sin(\mu_l u).$$

Similarly for $n = 2k+1$, the general solution of the system for the first vector is

$$\overrightarrow{e}_1(u) = a_0 + \sum_{l=1}^{k} d_l u_l \cos(\mu_l u) + f_l u_l \sin(\mu_l u),$$

where $a_0, u_1, ..., u_k$ are vectors in $\mathbb{R}^m$ and d_l, f_l are arbitrary constants.

Since $\dot{\beta}(u) = \overrightarrow{e}_1(u)$, for $n = 2k$,

$$\beta(s) = a_0 + \sum_{l=1}^{k} \overrightarrow{D}_l \cos(\mu_l s) + \overrightarrow{E}_l \sin(\mu_l s).$$

Similarly for $n = 2k+1$,

$$\beta(s) = b_0 + a_0 s + \sum_{l=1}^{k} \overrightarrow{D}_l \cos(\mu_l s) + \overrightarrow{E}_l \sin(\mu_l s),$$

where $\overrightarrow{E}_l = \frac{d_l}{\mu_l} u_l$ and $\overrightarrow{D}_l = -\frac{f_l}{\mu_l} u_l$ are vectors which are not necessarily constant. So, using Corollary 2 and Corollary 3, we complete the proof of the theorem. □

Example 4. The Frenet curve $\alpha : I \to \mathbb{R}^4$ given by the parameterization

$$\alpha(s) = \left(0, -\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right) + \int_0^s \overrightarrow{e}_1(\arcsin(2u))\mathrm{d}u, \qquad s \in \left]-\frac{1}{2}, \frac{1}{2}\right[$$

is a spherical ccr-curve with the centre at the origin of coordinates, with radius 1 and non-constant curvatures (see [11]), where

$$\overrightarrow{e}_1(t) = \frac{1}{\sqrt{2}}\left(\cos\left(\sqrt{\frac{3}{2}}t\right), \sin\left(\sqrt{\frac{3}{2}}t\right), \cos\left(\frac{1}{\sqrt{2}}t\right), \sin\left(\frac{1}{\sqrt{2}}t\right)\right).$$

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Ruumi $\mathbb{R}^m$ ccr-kõverate iseloomustamine

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On uuritud ruumi $\mathbb{R}^m$ kõveraid, mille järjestikuste naaberkõveruste suhted on konstantsed (ccr-kõverad). On näidatud, et kinnine ccr-kõver eukleidilises ruumis on lõplikku tüüpi. On käsitletud ka Frenet' kõveraid konstantsete harmooniliste kõverustega ja näidatud, et sellisel kõveral on ruumis $\mathbb{R}^{2n+1}$ Darboux' tipp uuritavas punktis.