CHEN INEQUALITIES FOR SUBMANIFOLDS OF A LOCALLY CONFORMAL ALMOST COSYMPLECTIC MANIFOLD WITH A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract

In this paper we prove Chen inequalities for submanifolds of a locally conformal almost cosymplectic manifold $N^{2m+1}(c)$ of constant φ-sectional curvature c endowed with a semi-symmetric metric connection, i.e., relations between the mean curvature associated with the semi-symmetric metric connection, scalar and sectional curvatures, Ricci curvatures and the sectional curvature of the ambient space.

1 Introduction

In [10], Friedmann and Schoutenn introduced the notion of a semi-symmetric linear connection on a differentiable manifold. Later in [11], H. A. Hayden defined a semi-symmetric metric connection on a Riemannian manifold. In [23], K. Yano studied some properties of a Riemannian manifold endowed with a semi-symmetric metric connection. In the case of hypersurfaces, in [12] and [13], T. Imai found some properties of a Riemannian manifold and a hypersurface of a Riemannian manifold with a semi-symmetric metric connection. In [19], Z. Nakao studied submanifolds of a Riemannian manifold with a semi-symmetric metric connection.

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To establish simple relationships between the main intrinsic invariants and the main extrinsic invariants of a submanifold is one of the most fundamental problems in submanifold theory as recalled by B.-Y. Chen [6]. The main intrinsic invariants include Chen's δ-invariant, scalar curvature, Ricci curvature and k-Ricci curvature. The main extrinsic invariants are squared mean curvature and shape operator. There are also other important modern intrinsic invariants of submanifolds introduced by B.-Y. Chen [9]. Many famous results in differential geometry can be regarded as results in this respect.

Following B.-Y. Chen, many geometers have studied similar problems for different submanifolds in various ambient spaces, for example see [2], [3], [15], [16] and [20].

In [4], [14], [22] and [24], submanifolds of locally conformal almost cosymplectic manifolds of pointwise constant φ-sectional curvature c satisfying Chen's inequalities were studied.

Recently, in [17] and [18], the first author and A. Mihai proved Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection and Chen inequalities for submanifolds of complex space forms and Sasakian space forms endowed with semi-symmetric metric connections, respectively.

Motivated by the studies of the above authors, in this study, we consider Chen inequalities for submanifolds in a locally conformal almost cosymplectic manifold $N^{2m+1}(c)$ of pointwise constant φ-sectional curvature c endowed with a semi-symmetric metric connection.

2 Semi-symmetric metric connection

Let N^{n+p} be an $(n+p)$-dimensional Riemannian manifold and $\widetilde{\nabla}$ a linear connection on N^{n+p}. If the torsion tensor $\widetilde{T}$ of $\widetilde{\nabla}$, defined by

$$\widetilde{T}\left(\widetilde{X},\widetilde{Y}\right) = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} \; - \; \widetilde{\nabla}_{\widetilde{Y}}\widetilde{X} \; - \; [\widetilde{X},\widetilde{Y}],$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on N^{n+p}, satisfies

$$\widetilde{T}\left(\widetilde{X},\widetilde{Y}\right) = \omega(\widetilde{Y})\widetilde{X} - \omega(\widetilde{X})\widetilde{Y}$$

for a 1-form ω, then the connection $\widetilde{\nabla}$ is called a *semi-symmetric connection.*

Let g be a Riemannian metric on N^{n+p}. If $\widetilde{\nabla}g = 0$, then $\widetilde{\nabla}$ is called a *semi-symmetric metric connection* on N^{n+p}.

A semi-symmetric metric connection $\widetilde{\nabla}$ on N^{n+p} is given by

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} + \omega(\widetilde{Y})\widetilde{X} - g(\widetilde{X},\widetilde{Y})U,$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on N^{n+p}, where $\overset{\circ}{\widetilde{\nabla}}$ denotes the Levi-Civita connection with respect to the Riemannian metric g and U is a vector field defined by $g(U,\widetilde{X}) = \omega(\widetilde{X})$, for any vector field $\widetilde{X}$ [23].

We will consider a Riemannian manifold N^{n+p} endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection denoted by $\overset{\circ}{\widetilde{\nabla}}$.

Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold N^{n+p}. On the submanifold M^n we consider the induced semi-symmetric metric connection denoted by ∇ and the induced Levi-Civita connection denoted by $\overset{\circ}{\nabla}$.

Let $\widetilde{R}$ be the curvature tensor of N^{n+p} with respect to $\widetilde{\nabla}$ and $\overset{\circ}{\widetilde{R}}$ the curvature tensor of N^{n+p} with respect to $\overset{\circ}{\widetilde{\nabla}}$. We also denote by R and $\overset{\circ}{R}$ the curvature tensors of ∇ and $\overset{\circ}{\nabla}$, respectively, on M^n.

The Gauss formulas with respect to ∇, respectively $\overset{\circ}{\nabla}$ can be written as:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad X, Y \in \chi(M),$$

$$\overset{\circ}{\widetilde{\nabla}}_X Y = \overset{\circ}{\nabla}_X Y + \overset{\circ}{h}(X,Y), \quad X, Y \in \chi(M),$$

where $\overset{\circ}{h}$ is the second fundamental form of M^n in N^{n+p} and h is a $(0,2)$-tensor on M^n. According to the formula (7) from [19] h is also symmetric. The Gauss equation for the submanifold M^n into an $(n+p)$-dimensional Riemannian manifold N^{n+p} is

$$\overset{\circ}{\widetilde{R}}(X,Y,Z,W) = \overset{\circ}{R}(X,Y,Z,W) + g(\overset{\circ}{h}(X,Z),\overset{\circ}{h}(Y,W)) - g(\overset{\circ}{h}(X,W),\overset{\circ}{h}(Y,Z)). \tag{1}$$

One denotes by $\overset{\circ}{H}$ the mean curvature vector of M^n in N^{n+p}.

Then the curvature tensor $\widetilde{R}$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ on N^{n+p} can be written as (see [13])

$$\widetilde{R}(X,Y,Z,W) = \overset{\circ}{\widetilde{R}}(X,Y,Z,W) - \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) - \tag{2}$$
$$-\alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z),$$

for any vector fields $X, Y, Z, W \in \chi(M^n)$, where α is a $(0,2)$-tensor field defined by

$$\alpha(X,Y) = \left(\overset{\circ}{\widetilde{\nabla}}_X \omega\right) Y - \omega(X)\omega(Y) + \frac{1}{2}\omega(P)g(X,Y), \quad \forall X, Y \in \chi(M).$$

Denote by λ the trace of α.

Let $\pi \subset T_xM^n$, $x \in M^n$, be a 2-plane section. Denote by $K(\pi)$ the sectional curvature of M^n with respect to the induced semi-symmetric metric connection ∇. For any orthonormal basis $\{e_1, ..., e_m\}$ of the tangent space T_xM^n, the scalar curvature τ at x is defined by

$$\tau(x) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j).$$

Recall that the *Chen first invariant* is given by

$$\delta_M(x) = \tau(x) - \inf\left\{K(\pi) \mid \pi \subset T_xM^n, x \in M^n, \dim \pi = 2\right\},$$

(see for example [9]), where M^n is a Riemannian manifold, $K(\pi)$ is the sectional curvature of M^n associated with a 2-plane section, $\pi \subset T_xM^n, x \in M^n$ and τ is the scalar curvature at x.

The following algebraic Lemma is well-known.

Lemma 2.1. [6] *Let $a_1, a_2, ..., a_n, b$ be $(n+1)$ $(n \geq 2)$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right).$$

Then $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3 = ... = a_n$.

Let M^n be an n-dimensional Riemannian manifold, L a k-plane section of T_xM^n, $x \in M^n$, and X a unit vector in L.

We choose an orthonormal basis $\{e_1, ..., e_k\}$ of L such that $e_1 = X$.

One defines [8] the *Ricci curvature* (or *k-Ricci curvature*) of L at X by

$$Ric_L(X) = K_{12} + K_{13} + ... + K_{1k},$$

where K_{ij} denotes, as usual, the sectional curvature of the 2-plane section spanned by e_i, e_j. For each integer k, $2 \leq k \leq n$, the Riemannian invariant Θ_k on M^n is defined by:

$$\Theta_k(x) = \frac{1}{k-1} \inf_{L,X} Ric_L(X), \quad x \in M^n,$$

where L runs over all k-plane sections in T_xM^n and X runs over all unit vectors in L.

3 Chen first inequality for submanifolds of locally conformal almost cosymplectic manifolds

Let N^{2m+1} be a $(2m+1)$-dimensional almost contact manifold endowed with an almost contact structure (φ, ξ, η), that is, φ is a $(1,1)$-tensor field, ξ is a vector field and η is 1-form such that $\varphi^2 X = -X + \eta(X)\xi$, $\eta(\xi) = 1$. Then, $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. The almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $N \times \mathbb{R}$ defined by $J(X, a\frac{d}{dt}) = (\varphi X - a\xi, \eta(X)\frac{d}{dt})$ is integrable, where X is tangent to N, t the coordinate of $\mathbb{R}$ and a a smooth function on $N \times \mathbb{R}$. The condition for being normal is equivalent to vanishing of the torsion tensor $[\varphi, \varphi] + 2d\eta \otimes \xi$, where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ.

Let g be a compatible Riemannian metric with (φ, ξ, η), that is, $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ or equivalently, $\Phi(X, Y) = g(X, \varphi Y) = -g(\varphi X, Y)$ and $g(X, \xi) = \eta(X)$ for all $X, Y \in TN$. Then N becomes an almost contact metric manifold equipped with an almost contact metric structure (φ, ξ, η, g) [5].

If the fundamental 2-form Φ and 1-form η are closed then N is said to be an *almost cosymplectic manifold*. A normal almost cosymplectic manifold is cosymplectic. N is called a *locally conformal almost cosymplectic manifold* if there exist a 1-form ω such that $d\Phi = 2w \wedge \Phi$, $d\eta = w \wedge \eta$ and $dw = 0$ [21].

A necessary and sufficient condition for a structure to be normal locally conformal almost cosymplectic is

$$\left(\overset{\circ}{\widetilde{\nabla}}_X \varphi\right) Y = f\left(g(X, \varphi Y)\xi - \eta(Y)\varphi X\right), \tag{3}$$

where $\overset{\circ}{\widetilde{\nabla}}$ is the Levi-Civita connection of the Riemannian metric g and $\omega = f\eta$. From formula (3) it follows that

$$\overset{\circ}{\widetilde{\nabla}}_X \xi = f\left(X - \eta(X)\xi\right),$$

(see [21]).

A locally conformal almost cosymplectic manifold N^{2m+1} of dimension ≥ 5 is of pointwise constant φ-sectional curvature c if and only if its Riemannian curvature tensor $\overset{\circ}{\widetilde{R}}$ is of the form

$$\overset{\circ}{\widetilde{R}}(X, Y, Z, W) = \frac{c - 3f^2}{4}[g(X, W)g(Y, Z) - g(X, Z)g(Y, W)] +$$

$$+\frac{c + f^2}{4}\left[g(X, \varphi W)g(Y, \varphi Z) - g(X, \varphi Z)g(Y, \varphi W) - 2g(X, \varphi Y)g(Z, \varphi W)\right]$$

$$-\left(\frac{c+f^2}{4}+f'\right)[\eta(Y)\eta(Z)g(X,W)-\eta(Y)\eta(W)g(X,Z)+ \qquad (4)$$

$$+\eta(X)\eta(W)g(Y,Z)-\eta(X)\eta(Z)g(Y,W)],$$

where f is the function such that $\omega = f\eta$, $f' = \xi f$ [21].

If $N^{2m+1}(c)$ is a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$, from (2) and (4) it follows that the curvature tensor $\widetilde{R}$ of $N^{2m+1}(c)$ can be expressed as

$$\widetilde{R}(X,Y,Z,W)=\frac{c-3f^2}{4}[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)]+$$

$$+\frac{c+f^2}{4}[g(X,\varphi W)g(Y,\varphi Z)-g(X,\varphi Z)g(Y,\varphi W)-2g(X,\varphi Y)g(Z,\varphi W)]$$

$$-\left(\frac{c+f^2}{4}+f'\right)[\eta(Y)\eta(Z)g(X,W)-\eta(Y)\eta(W)g(X,Z)+ \qquad (5)$$

$$+\eta(X)\eta(W)g(Y,Z)-\eta(X)\eta(Z)g(Y,W)]$$

$$-\alpha(Y,Z)g(X,W)+\alpha(X,Z)g(Y,W)-\alpha(X,W)g(Y,Z)+\alpha(Y,W)g(X,Z).$$

Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $N^{n+p}(c)$ of constant φ-sectional curvature c. For any tangent vector field X to M^n, we put

$$\varphi X = PX + FX,$$

where PX and FX are tangential and normal components of φX, respectively and we decompose

$$\xi = \xi^{\top} + \xi^{\perp},$$

where $\xi^{\top}$ and $\xi^{\perp}$denotes the tangential and normal parts of ξ.

Denote by $\Theta^2(\pi) = g^2(Pe_1, e_2)$, where $\{e_1, e_2\}$ is an orthonormal basis of a 2-plane section π, is a real number in $[0,1]$, independent of the choice of e_1, e_2 (see [1]).

For submanifolds of locally conformal almost cosymplectic manifold $N^{2m+1}(c)$ of constant φ-sectional curvature c endowed with a semi-symmetric metric connection we establish the following optimal inequality.

Theorem 3.1. *Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature $N^{2m+1}(c)$ endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. We have:*

$$\tau(x) - K(\pi) \leq (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{c-3f^2}{8} - \lambda\right] + \tag{6}$$

$$+\frac{3(c+f^2)}{4}\left(\frac{1}{2}\|P\|^2 - \Theta^2(\pi)\right) + \left(\frac{c+f^2}{4} + f'\right)\left[-(n-1)\|\xi^\top\|^2 + \|\xi_\pi\|^2\right] -$$

$$-trace\left(\alpha_{|_{\pi^\perp}}\right),$$

where π is a 2-plane section of $T_xM^n, x \in M^n$.

Proof. From [19], the Gauss equation with respect to the semi-symmetric metric connection is

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(Y,Z),h(X,W)). \tag{7}$$

Let $x \in M^n$ and $\{e_1, e_2, ..., e_n\}$ and $\{e_{n+1}, ..., e_{2m+1}\}$ be orthonormal basis of T_xM^n and $T_x^\perp M^n$, respectively. For $X = W = e_i, Y = Z = e_j$, $i \neq j$, from the equation (5) it follows that:

$$\tilde{R}(e_i,e_j,e_j,e_i) = \frac{c-3f^2}{4} + \frac{3(c+f^2)}{4}g^2(Pe_j,e_i) - \tag{8}$$

$$-\left(\frac{c+f^2}{4} + f'\right)\left\{\eta(e_i)^2 + \eta(e_j)^2\right\} - \alpha(e_i,e_i) - \alpha(e_j,e_j).$$

From (7) and (8) we get

$$\frac{c-3f^2}{4} + \frac{3(c+f^2)}{4}g^2(Pe_j,e_i) - \left(\frac{c+f^2}{4} + f'\right)\left\{\eta(e_i)^2 + \eta(e_j)^2\right\} - \alpha(e_i,e_i) -$$

$$-\alpha(e_j,e_j) = R(e_i,e_j,e_j,e_i) + g(h(e_i,e_j),h(e_i,e_j)) - g(h(e_i,e_i),h(e_j,e_j)).$$

By summation after $1 \leq i,j \leq n$, it follows from the previous relation that

$$2\tau + \|h\|^2 - n^2\|H\|^2 = -2(n-1)\lambda + (n^2-n)\left(\frac{c-3f^2}{4}\right) + \frac{3(c+f^2)}{4}\|P\|^2 - \tag{9}$$

$$-2\left(\frac{c+f^2}{4} + f'\right)(n-1)\|\xi^\top\|^2.$$

We take

$$\varepsilon = 2\tau - \frac{n^2(n-2)}{n-1}\left\|H\right\|^2 + 2(n-1)\lambda - (n^2-n)\left(\frac{c-3f^2}{4}\right) - \tag{10}$$

$$-\frac{3(c+f^2)}{4}\left\|P\right\|^2 + 2\left(\frac{c+f^2}{4}+f'\right)(n-1)\|\xi^\top\|^2.$$

Then, from (9) and (10) we get

$$n^2\left\|H\right\|^2 = (n-1)\left(\left\|h\right\|^2 + \varepsilon\right). \tag{11}$$

Let $x \in M^n$, $\pi \subset T_xM^n$, $\dim\pi = 2$, $\pi = sp\{e_1, e_2\}$. We define $e_{n+1} = \frac{H}{\|H\|}$ and from the relation (11) we obtain:

$$(\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n-1)(\sum_{i,j=1}^{n}\sum_{r=n+1}^{2m+1}(h_{ij}^r)^2 + \varepsilon),$$

or equivalently,

$$(\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n-1)\left[\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^r)^2 + \varepsilon\right].$$

By using the algebraic Lemma we have from the previous relation

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^r)^2 + \varepsilon.$$

If we denote by $\xi_\pi = pr_\pi\xi$ we can write (see [18])

$$\eta(e_1)^2 + \eta(e_2)^2 = \left\|\xi_\pi\right\|^2.$$

The Gauss equation for $X = W = e_1, Y = Z = e_2$ gives

$$K(\pi) = R(e_1, e_2, e_2, e_1) = \frac{c-3f^2}{4} + \frac{3(c+f^2)}{4}g^2(Pe_1, e_2) - \left(\frac{c+f^2}{4} + f'\right)\left\|\xi_\pi\right\|^2 -$$

$$-\alpha(e_1, e_1) - \alpha(e_2, e_2) + \sum_{r=n+1}^{2m+1}[h_{11}^r h_{22}^r - (h_{12}^r)^2] \geq$$

$$\geq \frac{c-3f^2}{4} + \frac{3(c+f^2)}{4}g^2(Pe_1, e_2) - \left(\frac{c+f^2}{4} + f'\right)\left\|\xi_\pi\right\|^2 - \alpha(e_1, e_1) - \alpha(e_2, e_2) +$$

$$+\frac{1}{2}[\sum_{i\neq j}(h_{ij}^{n+1})^2+\sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^r)^2+\varepsilon]+\sum_{r=n+2}^{2m+1}h_{11}^r h_{22}^r-\sum_{r=n+1}^{2m+1}(h_{12}^r)^2=$$

$$=\frac{c-3f^2}{4}+\frac{3(c+f^2)}{4}g^2(Pe_1,e_2)-\left(\frac{c+f^2}{4}+f'\right)\|\xi_\pi\|^2-\alpha(e_1,e_1)-\alpha(e_2,e_2)+$$

$$+\frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2+\frac{1}{2}\sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^r)^2+\frac{1}{2}\varepsilon+\sum_{r=n+2}^{2m+1}h_{11}^r h_{22}^r-\sum_{r=n+1}^{2m+1}(h_{12}^r)^2=$$

$$=\frac{c-3f^2}{4}+\frac{3(c+f^2)}{4}g^2(Pe_1,e_2)-\left(\frac{c+f^2}{4}+f'\right)\|\xi_\pi\|^2-\alpha(e_1,e_1)-\alpha(e_2,e_2)+$$

$$+\frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2+\frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i,j>2}(h_{ij}^r)^2+\frac{1}{2}\sum_{r=n+2}^{2m+1}(h_{11}^r+h_{22}^r)^2+\sum_{j>2}[(h_{1j}^{n+1})^2+(h_{2j}^{n+1})^2]+\frac{1}{2}\varepsilon\geq$$

$$\geq\frac{c-3f^2}{4}+\frac{3(c+f^2)}{4}g^2(Pe_1,e_2)-\left(\frac{c+f^2}{4}+f'\right)\|\xi_\pi\|^2-\alpha(e_1,e_1)-\alpha(e_2,e_2)+\frac{\varepsilon}{2},$$

which implies

$$K(\pi)\geq\frac{c-3f^2}{4}+\frac{3(c+f^2)}{4}g^2(Pe_1,e_2)-\left(\frac{c+f^2}{4}+f'\right)\|\xi_\pi\|^2-\alpha(e_1,e_1)-\alpha(e_2,e_2)+\frac{\varepsilon}{2}.$$

Denote by

$$\alpha(e_1,e_1)+\alpha(e_2,e_2)=\lambda-trace\left(\alpha_{|_{\pi^\perp}}\right),$$

(see [18]). From (10) it follows

$$K(\pi)\geq\tau-(n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2+(n+1)\frac{c-3f^2}{8}-\lambda\right]+$$

$$+\frac{3(c+f^2)}{4}\left(\Theta^2(\pi)-\frac{1}{2}\|P\|^2\right)+\left(\frac{c+f^2}{4}+f'\right)\left[(n-1)\|\xi^\top\|^2-\|\xi_\pi\|^2\right]+trace\left(\alpha_{|_{\pi^\perp}}\right),$$

which represents the inequality to prove.

Corollary 3.2. *Under the same assumptions as in Theorem* 3.1 *if* ξ *is tangent to* M^n*, we have*

$$\tau(x)-K(\pi)\leq(n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2+(n+1)\frac{c-3f^2}{8}-\lambda\right]+$$

$$+\frac{3(c+f^2)}{4}\left(\frac{1}{2}\|P\|^2-\Theta^2(\pi)\right)+\left(\frac{c+f^2}{4}+f'\right)\left[-(n-1)+\|\xi_\pi\|^2\right]-trace\left(\alpha_{|_{\pi^\perp}}\right).$$

If ξ is normal to M^n, we have

$$\tau(x) - K(\pi) \leq (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{c-3f^2}{8} - \lambda\right] + \\ +\frac{3(c+f^2)}{4}\left(\frac{1}{2}\|P\|^2 - \Theta^2(\pi)\right) - trace\left(\alpha_{|_{\pi^\perp}}\right).$$

Recall the following important result (Proposition 1.2) from [12].

Proposition 3.3. *The mean curvature H of M^n with respect to the semi-symmetric metric connection coincides with the mean curvature $\overset{\circ}{H}$ of M^n with respect to the Levi-Civita connection if and only if the vector field U is tangent to M^n.*

Remark 3.4. *According to the formula* (7) *from [19] (see also Proposition* 3.3*), it follows that $h = \overset{\circ}{h}$ if U is tangent to M^n. In this case inequality* (6) *becomes*

$$\tau(x) - K(\pi) \leq (n-2)\left[\frac{n^2}{2(n-1)}\left\|\overset{\circ}{H}\right\|^2 + (n+1)\frac{c-3f^2}{8} - \lambda\right] + \\ +\frac{3(c+f^2)}{4}\left(\frac{1}{2}\|P\|^2 - \Theta^2(\pi)\right) + \left(\frac{c+f^2}{4} + f'\right)\left[\|\xi_\pi\|^2 - (n-1)\right] - \\ -trace\left(\alpha_{|_{\pi^\perp}}\right).$$

Theorem 3.5. *If the vector field U is tangent to M^n, then the equality case of inequality* (6) *holds at a point $x \in M^n$ if and only if there exists an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of T_xM^n and an orthonormal basis $\{e_{n+1}, ..., e_{n+p}\}$ of $T_x^\perp M^n$ such that the shape operators of M^n in $N^{2m+1}(c)$ at x have the following forms:*

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a + b = \mu,$$

$$A_{e_r} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad n+2 \leq i \leq 2m+1,$$

where we denote by $h_{ij}^r = g(h(e_i, e_j), e_r)$, $1 \leq i, j \leq n$ *and* $n+2 \leq r \leq 2m+1$.

Proof. The equality case holds at a point $x \in M^n$ if and only if it achieves the equality in all the previous inequalities and we have the equality in the Lemma.

$$h_{ij}^{n+1} = 0, \forall i \neq j, i, j > 2,$$

$$h_{ij}^r = 0, \forall i \neq j, i, j > 2, r = n+1, ..., 2m+1,$$

$$h_{11}^r + h_{22}^r = 0, \forall r = n+2, ..., 2m+1,$$

$$h_{1j}^{n+1} = h_{2j}^{n+1} = 0, \forall j > 2,$$

$$h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = ... = h_{nn}^{n+1}.$$

We may chose $\{e_1, e_2\}$ such that $h_{12}^{n+1} = 0$ and we denote by $a = h_{11}^r, b = h_{22}^r, \mu = h_{33}^{n+1} = ... = h_{nn}^{n+1}$.

It follows that the shape operators take the desired forms.

4 Ricci curvature for submanifolds of locally conformal almost cosymplectic manifolds

We first state a relationship between the sectional curvature of a submanifold M^n of a locally conformal almost cosymplectic manifold $N^{2m+1}(c)$ of constant φ-sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\widetilde{\nabla}}$ and the squared mean curvature $\|H\|^2$. Using this inequality, we prove a relationship between the k-Ricci curvature of M^n (intrinsic invariant) and the squared mean curvature $\|H\|^2$ (extrinsic invariant), as another answer of the basic problem in submanifold theory which we have mentioned in the introduction.

In this section we suppose that the vector field U is tangent to M^n.

Theorem 4.1. *Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $N^{2m+1}(c)$ of pointwise constant φ-sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ such that the vector field U is tangent to M^n. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} + \frac{2}{n}\lambda - \frac{c-3f^2}{4} - \frac{3}{4n(n-1)}(c+f^2)\|P\|^2 + $$
$$+\frac{2}{n}\left(\frac{c+f^2}{4} + f'\right)\|\xi^\top\|^2. \qquad (12)$$

Proof. Let $x \in M^n$ and $\{e_1, e_2, ..., e_n\}$ and orthonormal basis of T_xM^n. The relation (9) is equivalent with

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 + 2(n-1)\lambda - (n^2-n)\left(\frac{c-3f^2}{4}\right) - \frac{3(c+f^2)}{4}\|P\|^2 + \quad (13)$$

$$+2\left(\frac{c+f^2}{4} + f'\right)(n-1)\|\xi^\top\|^2.$$

We choose an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{n+p}\}$ at x such that e_{n+1} is parallel to the mean curvature vector $H(x)$ and $e_1, ..., e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$A_{e_{n+1}} \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix},$$

$$A_{e_r} = (h_{ij}^r),\ i, j = 1, ..., n; r = n+2, ..., 2m+1, \operatorname{trace} A_{e_r} = 0.$$

From (13), we get

$$n^2 \|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + 2(n-1)\lambda - \quad (14)$$

$$-(n^2-n)\left(\frac{c-3f^2}{4}\right) - \frac{3(c+f^2)}{4}\|P\|^2 + 2\left(\frac{c+f^2}{4} + f'\right)(n-1)\|\xi^\top\|^2.$$

Since

$$\sum_{i=1}^{n} a_i^2 \geq n\|H\|^2,$$

hence we obtain

$$\begin{aligned} n^2\|H\|^2 \quad &\geq \quad 2\tau + n\|H\|^2 + 2(n-1)\lambda - (n^2-n)\left(\frac{c-3f^2}{4}\right) \\ &-\frac{3(c+f^2)}{4}\|P\|^2 + 2\left(\frac{c+f^2}{4} + f'\right)(n-1)\|\xi^\top\|^2. \end{aligned}$$

Last inequality represents (12).

Using Theorem 4.1, we obtain the following

Theorem 4.2. *Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $N^{2m+1}(c)$ of pointwise constant φ-sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$, such that the vector field U is tangent to M^n. Then, for any integer k, $2 \leq k \leq n$, and any point $x \in M^n$, we have*

$$\|H\|^2(x) \geq \Theta_k(x) + \frac{2}{n}\lambda - \frac{c-3f^2}{4} - \frac{3}{4n(n-1)}(c+f^2)\|P\|^2 + \\ + \frac{2}{n}\left(\frac{c+f^2}{4} + f'\right)\|\xi^\top\|^2. \tag{15}$$

Proof. Let $\{e_1, ...e_n\}$ be an orthonormal basis of T_xM. Denote by $L_{i_1...i_k}$ the k-plane section spanned by $e_{i_1}, ..., e_{i_k}$. By the definitions, one has

$$\tau(L_{i_1...i_k}) = \frac{1}{2}\sum_{i\in\{i_1,...,i_k\}} Ric_{L_{i_1...i_k}}(e_i), \tag{16}$$

$$\tau(x) = \frac{1}{C_{n-2}^{k-2}}\sum_{1\leq i_1<...<i_k\leq n} \tau(L_{i_1...i_k}). \tag{17}$$

From (12), (16) and (17), one derives

$$\tau(x) \geq \frac{n(n-1)}{2}\Theta_k(x),$$

which implies (15).

References

[1] P. Alegre, A. Carriazo, Y. H. Kim, D. W. Yoon, *B.Y. Chen's inequality for submanifolds of generalized space forms*, Indian J. Pure Appl. Math., **38** (2007), no. 3, 185-201.

[2] K. Arslan, R. Ezentaş, I. Mihai, C. Murathan and C. Özgür, *B. Y. Chen inequalities for submanifolds in locally conformal almost cosymplectic manifolds*, Bull. Inst. Math., Acad. Sin., **29** (2001), 231-242.

[3] K. Arslan, R. Ezentaş, I. Mihai, C. Murathan and C. Özgür, *Certain inequalities for submanifolds in (k,μ)-contact space forms*, Bull. Aust. Math. Soc., **64** (2001), 201-212.

[4] K. Arslan, R. Ezentaş, I. Mihai, C. Murathan and C. Özgür, *Ricci curvature of submanifolds in locally conformal almost cosymplectic manifolds*, Math. J. Toyama Univ., **26** (2003), 13-24.

[5] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics, Vol. 509. Springer-Verlag, Berlin-New York, 1976.

[6] B-Y. Chen, *Some pinching and classification theorems for minimal submanifolds,* Arch. Math. (Basel), **60** (1993), no. 6, 568-578.

[7] B-Y. Chen, *Strings of Riemannian invariants, inequalities, ideal immersions and their applications*, The Third Pacific Rim Geometry Conference (Seoul, 1996), 7–60, Monogr. Geom. Topology, 25, Int. Press, Cambridge, MA, 1998.

[8] B-Y. Chen, *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions*, Glasg. Math. J., **41** (1999), no. 1, 33-41.

[9] B-Y. Chen, *δ -invariants, Inequalities of Submanifolds and Their Applications, in Topics in Differential Geometry*, Eds. A. Mihai, I. Mihai, R. Miron, Editura Academiei Romane, Bucuresti, 2008, 29-156.

[10] A. Friedmann and J. A. Schouten, *Über die Geometrie der halbsymmetrischen Übertragungen*, Math. Z., **21** (1924), no. 1, 211-223.

[11] H. A. Hayden, *Subspaces of a space with torsion*, Proc. London Math. Soc., **34** (1932), 27-50.

[12] T. Imai, *Hypersurfaces of a Riemannian manifold with semi-symmetric metric connection*, Tensor (N.S.), **23** (1972), 300-306.

[13] T. Imai, *Notes on semi-symmetric metric connections*, Vol. I. Tensor (N.S.), **24** (1972), 293-296.

[14] J-S. Kim, M. M. Tripathi and J. Choi, *Ricci curvature of submanifolds in locally conformal almost cosymplectic manifolds*, Indian J. Pure Appl. Math., 35 (2004), no. 3, 259-271

[15] K. Matsumoto, I. Mihai and A. Oiaga, *Ricci curvature of submanifolds in complex space forms,* Rev. Roumaine Math. Pures Appl., **46** (2001), no. 6, 775-782.

[16] A. Mihai, *Modern Topics in Submanifold Theory*, Editura Universitatii Bucuresti, Bucharest, 2006.

[17] A. Mihai and C. Özgür, *Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection*, Taiwanese J. Math., to appear.

[18] A. Mihai and C. Özgür, Chen inequalities for submanifolds of complex space forms and Sasakian space forms endowed with semi-symmetric metric connections, to appear in Rocky Mountain J. Math.

[19] Z. Nakao, *Submanifolds of a Riemannian manifold with semisymmetric metric connections*, Proc. Amer. Math. Soc., **54** (1976), 261-266.

[20] A. Oiaga and I. Mihai, *B. Y. Chen inequalities for slant submanifolds in complex space forms*, Demonstratio Math., **32** (1999), no. 4, 835-846.

[21] Z. Olszak, *Locally conformal almost cosymplectic manifolds*, Colloq. Math., **57** (1989), no. 1, 73-87.

[22] M. M. Tripathi, J-S. Kim, S-B. Kim, *A basic inequality for submanifolds in locally conformal almost cosymplectic manifolds*, Proc. Indian Acad. Sci. Math. Sci., **112** (2002), no. 3, 415-423.

[23] K. Yano, *On semi-symmetric metric connection*, Rev. Roumaine Math. Pures Appl., **15** (1970), 1579-1586.

[24] D. W. Yoon, *Certain inequalities for submanifolds in locally conformal almost cosymplectic manifolds*, Bull. Inst. Math. Acad. Sinica, **32** (2004), no. 4, 263-283.

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