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SOME SPECIAL TYPES OF DEVELOPABLE RULED SURFACE

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Abstract

In this study we consider the focal curve C_γ of a space curve γ and its focal curvatures. We characterize some special types of ruled surface, choosing one of the base curves or director curves as the focal curve of the space curve γ. Finally we construct new types of ruled surface and calculate their distinguished parameters. We give necessary and sufficient conditions for these types of ruled surface to become developable.

Keywords: Frenet formulas, Helix, Focal curvature, Focal curve, Ruled surface.

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1. Introduction

The differential geometry of space curves is a classical subject which usually relates geometrical intuition with analysis and topology. For any unit speed curve $\gamma = \gamma(s) : I \to \mathbb{E}^3$, the focal curve C_γ is defined as the centers of the osculating spheres of γ. Since the center of any sphere tangent to γ at a point lies on the normal plane to γ at that point, the focal curve of γ may be parameterized using the Frenet frame $(t(s),\, n_1(s),\, n_2(s))$ of γ as follows:

$$C_\gamma(s) = (\gamma + c_1 n_1 + c_2 n_2)(s),$$

where the coefficients c_1, c_2 are smooth functions that are called *focal curvatures* of γ [15].

Recently, ruled surfaces have been studied by many authors (see,[7, 8, 9, 10]). A ruled surface in $\mathbb{E}^3$ is (locally) the map $F_{(\gamma,\delta)} : I \times \mathbb{R} \to \mathbb{E}^3$ defined by

$$F_{(\gamma,\delta)}(s,u) = \gamma(s) + u\delta(s)$$

where $\gamma : I \to \mathbb{E}^3$, $\delta : I \to \mathbb{E}^3 \setminus \{0\}$ are smooth mappings and I is an open interval or the unit circle S^1, We call γ the *base curve* and δ the *director curve*. The straight lines

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$u \to \gamma(s) + u\delta(s)$ are called *rulings*. The ruled surface $F_{(\gamma,\delta)}$ is called *developable* if the Gaussian curvature of the regular part of $F_{(\gamma,\delta)}$ vanishes. This is equivalent to the fact that $F_{(\gamma,\delta)}$ is developable if and only if the distinguished parameter

$$P_{(\gamma,\delta)} = \frac{\langle \gamma', \delta \wedge \delta' \rangle}{\langle \delta', \delta' \rangle}$$

of $F_{(\gamma,\delta)}$ vanishes identically. In [7], S. Izumiya and N. Takeuchi studied a special type of ruled surface with Darboux vector $\widetilde{D}(s) = \delta(s)$. They called the ruled surface $F_{(\gamma,\widetilde{D})}$ the *rectifying developable surface* of the space curve γ.

In this study we use the properties of the focal curvatures to obtain some results for the curve γ and its focal curve C_γ. Further, we characterize some special types of ruled surface obtained by choosing either the base curve or director curve as the focal curve of the space curve γ. Finally, we characterize the ruled surfaces related with the distinguished parameter of the focal surface.

In § 2 we describe basic notions and properties of space curves. In § 3 we review the basic notions and properties of the focal curve $C_\gamma(s)$ of a space curve γ. We prove that γ is a cylindrical helix if and only if the focal curve $C_\gamma(s)$ has constant length. Further, we also prove that γ is a conical geodesic curve if and only if the ratio of its torsion and curvature is a nonzero linear function in the arclength function s. In the final section we define new types of ruled surface, and calculate their distinguished parameters. We give necessary and sufficient conditions for these types of ruled surface to become developable.

All manifolds and maps considered here are of class C^∞ unless otherwise stated.

2. Basic notation and properties

We now review some basic concepts on the classical differential geometry of space curves in Euclidean space. Let $\gamma = \gamma(s) : I \to \mathbb{E}^3$ be a curve parametrized by the arc-length parameter s with $\gamma'(s) \neq 0$, where $\gamma'(s) = \frac{d\gamma(s)}{ds}$. The tangent vector $t(s) = \gamma'(s)$ is unitary and it is orthogonal to $t'(s) = \gamma''(s)$. If $\gamma''(s) \neq 0$, these vectors span the osculating plane of γ at s.

Define the first curvature of γ by $\kappa_1(s) = \|\gamma''(s)\|$. If $\kappa_1(s) \neq 0$, the unit principal normal vector $n_1(s)$ of the curve γ at s is given by $t'(s) = \kappa_1(s)n_1(s)$. The unit vector $n_2(s) = t(s) \times n_1(s)$ is called the unit binormal vector of γ at s. Then the Serret-Frenet formulae of γ are

$$\begin{aligned} t'(s) &= \kappa_1(s)n_1(s), \\ n_1'(s) &= -\kappa_1(s)t(s) + \kappa_2(s)n_2(s), \\ n_2'(s) &= -\kappa_2(s)n_1(s), \end{aligned} \tag{2.1}$$

where $\kappa_2(s)$ is the second curvature of the curve γ at s. The radius of the osculating circle of γ at s is given by $R(s) = \frac{1}{\kappa_1(s)}$, and is called the *radius of curvature* of γ at s [3].

The Serret-Frenet formulae can be interpreted kinematically as follows: If a moving point traverses the curve in such a way that s is the time parameter, then the moving frame $\{t(s), n_1(s), n_2(s)\}$ moves in accordance with (2.1). This motion contains, apart from an instantaneous translation, an instantaneous rotation with angular velocity vector given by the Darboux vector $D(s) = \kappa_2(s)t(s) + \kappa_1(s)n_2(s)$. The direction of the Darboux vector is that of instantaneous axis of rotation and its length $\sqrt{\kappa_1^2(s) + \kappa_2^2(s)}$ is the scalar angular velocity (cf. [12, p. 12]).

For any unit speed curve $\gamma : I \to \mathbb{E}^3$ we define a vector field

$$\widetilde{D}(s) = \left(\frac{\kappa_2}{\kappa_1}\right)(s)t(s) + n_2(s), \tag{2.2}$$

along γ under the condition that $\kappa_1(s) \neq 0$, and we call it the *modified Darboux vector field along* γ. We also denote the unit Darboux vector field by

$$\overline{D}(s) = \left(\frac{1}{\sqrt{\kappa_1^2 + \kappa_2^2}}\right)(s)(\kappa_2(s)t(s) + \kappa_1(s)n_2(s)),$$

(cf. [11, Section 5.2]).

A curve $\gamma : I \to \mathbb{E}^3$ with $\kappa_1(s) \neq 0$ is called a *generalized helix* if the tangent lines of γ make a constant angle with a fixed direction. It is known that the curve γ is a generalized helix if and only if $\left(\frac{\kappa_2}{\kappa_1}\right)(s)$ is constant. If both of $\kappa_1(s) \neq 0$ and $\kappa_2(s)$ are constant, the curve γ is called *circular helix*.

3. Focal curve of a space curve

For a unit speed curve $\gamma = \gamma(s) : I \to \mathbb{E}^3$, the curve consisting of the centers of the osculating spheres of γ is called the *parametrized focal curve of* γ. The hyperplanes normal to γ at a point consist of the set of centers of all spheres tangent to γ at that point. Hence the center of the osculating spheres at that point lies in such a normal plane. Therefore, denoting the focal curve by C_γ we can write

$$C_\gamma(s) = (\gamma + c_1 n_1 + c_2 n_2)(s), \tag{3.1}$$

where the coefficients c_1, c_2 are smooth functions of the parameter of the curve γ, called the first and second focal curvatures of γ, respectively. Further, the focal curvatures c_1, c_2 are defined by

$$c_1 = \frac{1}{\kappa_1},\ c_2 = \frac{c_1'}{\kappa_2};\ \kappa_1 \neq 0,\ \kappa_2 \neq 0. \tag{3.2}$$

The focal curvatures c_1, c_2 of γ satisfy the following Frenet equations:

$$\begin{bmatrix} 1 \\ c_1' \\ c_2' - \frac{(R^2)'}{2c_2} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ -\kappa_1 & 0 & \kappa_2 \\ 0 & -\kappa_2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ c_1 \\ c_2 \end{bmatrix}, \tag{3.3}$$

where R is the radius of the osculating sphere of γ. If the curve γ is spherical, i.e., lies on a sphere, then the last component of the left hand side vector of equation only consists of c_2' [15]. We give some classical results for the spherical curves:

3.1. Proposition. [13] *A curve $\gamma : I \to \mathbb{E}^3$ is spherical, i.e., it is contained in a sphere of radius R, if and only if*

$$\frac{1}{\kappa_1^2} + \left(\frac{\kappa_1'}{\kappa_1^2 \kappa_2}\right) = R^2. \tag{3.4}$$

This means that the curve γ is spherical if and only if the equality $c_2' + c_1\kappa_2 = 0$ holds. □

Further, the derivative of the focal curve with respect to the arclength parameter is $C_\gamma' = (c_2' + c_1\kappa_2)n_2$, where $(R^2)' = 2c_2(c_2' + c_1\kappa_2)$ and $R^2 = c_1^2 + c_2^2$.

3.2. Lemma. [2, 15] *Let K_1 and K_2 (resp. κ_1 and κ_2) be the curvatures of C_γ (resp. of γ). Then*

$$\frac{K_1}{|\kappa_2|} = \frac{|K_2|}{\kappa_1} = \frac{1}{|c_2' + c_1\kappa_2|} = \frac{2c_2}{|(R^2)'|}. \tag{3.5}$$ □

By the use of Lemma 3.2 we obtain the following results.

3.3. Proposition. *Let $\gamma \subset \mathbb{E}^3$ be a unit speed curve and C_γ its focal curve. Then γ is a generalized helix if and only if*

$$|C_\gamma'| = |c_2' + c_1\kappa_2|$$

is a nonzero constant. □

3.4. Definition. [7] A curve $\gamma : I \to \mathbb{E}^3$ with $\kappa_1(s) \neq 0$ is called a *conical geodesic* if $\left(\frac{\kappa_2}{\kappa_1}\right)(s)$ is a constant function.

3.5. Theorem. *Let $\gamma \subset \mathbb{E}^3$ be a unit speed curve and C_γ its focal curve. Then γ is a conical geodesic if and only if*

$$\left|C'_\gamma\right| = \frac{1}{as+b}$$

for some real constants a, b with $a \neq 0$.

Proof. $\Longrightarrow$ Suppose γ is a conical geodesic curve and C_γ its focal curve. Then from Lemma 3.2 and Definition 3.4, we get $(\frac{|\kappa_2|}{\kappa_1})''(s) = 0$. Further, by the use of

$$\frac{|\kappa_2|}{\kappa_1} = \frac{1}{|c'_2 + c_1\kappa_2|},$$

and after some computations we obtain

$$\left|C'_\gamma\right| = \left|c'_2 + c_1\kappa_2\right| = \frac{1}{as+b}. \tag{3.6}$$

$\Longleftarrow$ Conversely, if (3.6) holds then by the use of (3.5) we get $(\frac{|\kappa_2|}{\kappa_1})'' = 0$, which means that γ is a conical geodesic. □

3.6. Definition. A curve $\gamma : I \to \mathbb{E}^3$ is called *rectifying* if the position vector of γ lies in its rectifying plane, i.e. the position vector satisfies

$$\gamma(s) = \lambda(s)t(s) + \mu(s)n_2(s) \tag{3.7}$$

for some functions λ and μ [1].

By taking the derivative of (3.7) with respect to the parameter s and applying the Serret-Frenet equations (2.1), we get

$$\lambda(s) = 1,\ \mu(s) = 0,\ \lambda(s)\kappa_1 = \mu(s)\kappa_2(s). \tag{3.8}$$

3.7. Theorem. *Let $\gamma \subset \mathbb{E}^3$ be a non-spherical unit speed curve and C_γ its focal curve. Then γ is a conical geodesic if and only if γ is congruent to a rectifying curve.*

Proof. $\Longrightarrow$ Suppose γ is a conical geodesic. Then by Theorem 3.5, the ratio of the torsion and curvature of γ is a nonzero linear function of s, i.e. $\frac{|\kappa_2|}{\kappa_1} = as + b$ for some real constants a, b with $a \neq 0$. So, by B. Y. Chen [1, Theorem 2], γ is congruent to a rectifying curve.

$\Longleftarrow$ Conversely, if γ is congruent to a rectifying curve then by (3.7), the ratio of the torsion and curvature of γ is a nonzero linear function of s. □

4. Developable surfaces associated with a space curve

In this section we consider developable surfaces associated with a space curve. A ruled surface in $\mathbb{E}^3$ is (locally) the map $F_{(\gamma,\delta)}(s,u) : I \times \mathrm{R} \to \mathbb{E}^3$ defined by $F_{(\gamma,\delta)}(s,u) = \gamma(s) + u\delta(s)$, where $\gamma : I \to \mathbb{E}^3$, $\delta : I \to \mathbb{E}^3 \setminus \{0\}$ are smooth mappings and I is an open interval or the unit circle S^1. We call γ the base curve and δ the director curve. The straight lines $u \to \gamma(s) + u\delta(s)$ are called rulings of $F_{(\gamma,\delta)}$ (see, [5]).

Let $F_{(\gamma,\delta)}$ be a ruled surface. We say that $F_{(\gamma,\delta)}$ is *developable* if the Gaussian curvature of the regular part of $F_{(\gamma,\delta)}$ vanishes. From now on, we may assume that $\|\delta(s)\| = 1$. It is easy to show that Gaussian curvature of $F_{(\gamma,\delta)}$ is

$$K(s,u) = \frac{-\left(\det\left(\gamma'(s), \delta'(s), \delta(s)\right)\right)}{\left(EG - F^2\right)^2}, \tag{4.1}$$

where $E = E(s,u) = \|\gamma'(s) + u\delta'(s)\|^2$, $F = F(s,u) = \langle \gamma'(s), \delta(s)^2 \rangle$, and $G = G(s,u) = 1$ (see [4]).

For a given ruled surface $F_{(\gamma,\delta)}$, the distinguished parameter P_δ of $F_{(\gamma,\delta)}$ is defined by

$$P_{(\gamma,\delta)} = \frac{\langle \gamma', \delta \wedge \delta' \rangle}{\langle \delta', \delta' \rangle}. \tag{4.2}$$

Comparing (4.1) with (4.2), it is easy to see that the ruled surface $F_{(\gamma,\delta)}$ is developable if and only if P_δ vanishes identically [6]. See also [14] for the Lorentzian case.

For a ruled surface $F_{(\gamma,\delta)}$, we can find the following equality;

$$\frac{\partial F_{(\gamma,\delta)}}{\partial t}(s,u) \times \frac{\partial F_{(\gamma,\delta)}}{\partial u}(s,u) = \gamma'(s) \times \delta(s) + u\delta'(s) \times \delta(s). \tag{4.3}$$

Therefore, (s_0, u_0) is a singular point of $F_{(\gamma,\delta)}$ if and only if $\gamma(s_0) \times \delta(s_0) + u_0\delta'(s_0) \times \delta(s_0) = 0$. We say that a ruled surface is *cylindrical* if the equality $\delta'(s) \times \delta(s) \equiv 0$ holds. Thus, we can say that the ruled surface $F_{(\gamma,\delta)}$ is non-cylindrical if $\delta'(s) \times \delta(s) \neq 0$.

In [8], S. Izumiya and N. Tekauchi studied the rectifying developable surfaces of a unit speed space curve γ with $\kappa_1(s) \neq 0$ using

$$F_{(\gamma,\widetilde{D})}(s,u) = \gamma(s) + u\widetilde{D}(s). \tag{4.4}$$

From Equation (2.2), an easy calculation gives

$$\widetilde{D}'(s) = (\frac{\kappa_2}{\kappa_1})'(s)t(s), \tag{4.5}$$

so that (s_0, u_0) is singular point of $F_{(\gamma,\widetilde{D})}$ if and only if $(\frac{\kappa_2}{\kappa_1})'(s_0) \neq 0$ and $u_0 = \frac{-1}{(\frac{\kappa_2}{\kappa_1})'(s_0)}$.

Moreover, they proved the following results.

4.1. Proposition. [8] *For a unit speed curve $\gamma : I \to \mathbb{E}^3$ with $\kappa_1(s) \neq 0$, the following are equivalent:*

(1) *The rectifying developable surface $F_{(\gamma,\widetilde{D})} : I \times \mathbb{R} \to \mathbb{E}^3$ of γ is a non-singular surface.*
(2) *γ is a cylindrical helix.*
(3) *The rectifying developable surface $F_{(\gamma,\widetilde{D})}$ of γ is a cylindrical surface.* □

4.2. Proposition. [7] *For a unit speed curve $\gamma : I \to \mathbb{E}^3$ with $\kappa_1(s) \neq 0$, the following are equivalent:*

(1) *The rectifying developable surface $F_{(\gamma,\widetilde{D})} : I \times \mathbb{R} \to \mathbb{E}^3$ of γ is a conical surface.*
(2) *γ is a conical geodesic curve.* □

We now consider a curve $\tau(s)$ on the ruled surface $F_{(\gamma,\delta)}$ with the property that

$$\langle \tau'(s), \delta'(u) \rangle \neq 0.$$

We call such a curve a line of striction [7]. Let γ be a geodesic of a rectifying developable surface $F_{(\gamma,\widetilde{D})}$. The locus of the singular points of the rectifying developable surface of γ is given by

$$\tau(s) = \gamma(s) - \frac{1}{\left(\frac{\kappa_2}{\kappa_1}\right)'(s)}\widetilde{D}(s), \tag{4.6}$$

where $\widetilde{D}$ is the modified Darboux vector defined by the equation (2.2). An easy calculation gives

$$\tau'(s) = \frac{\left(\frac{\kappa_2}{\kappa_1}\right)''(s)}{\left(\frac{\kappa_2}{\kappa_1}\right)'(s)}\widetilde{D}(s), \tag{4.7}$$

so $\tau(s)$ is a regular space curve which is a generalized helix [8].

4.3. Definition. Let $\gamma \subset \mathbb{E}^3$ be a unit speed curve. We define the following ruled surfaces;

$$F_{(\gamma,C_\gamma)}(s,u) = \gamma(s) + uC_\gamma(s), \tag{4.8}$$

$$F_{(C_\gamma,\gamma)}(s,u) = C_\gamma(s) + u\gamma'(s), \tag{4.9}$$

$$F_{(\widetilde{D},C_\gamma)}(s,u) = \widetilde{D}(s) + uC_\gamma(s), \tag{4.10}$$

$$F_{(C_\gamma,\widetilde{D})}(s,u) = C_\gamma(s) + u\widetilde{D}(s), \tag{4.11}$$

where C_γ and $\widetilde{D}$ are the focal curve and modified Darboux vector field of γ, respectively.

Now, we have the following results:

4.4. Lemma. *Let $\gamma \subset \mathbb{E}^3$ be a unit speed curve and C_γ its focal curve. Then the equalities*

$$C_\gamma(s) \wedge C_\gamma'(s) = \left\|C_\gamma'(s)\right\| (\gamma \wedge n_2(s) + c_1 t(s)), \tag{4.12}$$

$$\widetilde{D}(s) \wedge \widetilde{D}(s) = (\frac{\kappa_2}{\kappa_1})(s)(\frac{\kappa_2}{\kappa_1})'(s)n_1(s), \tag{4.13}$$

$$\gamma(s) \wedge \gamma''(s) = -\kappa_2(s)n_1(s), \tag{4.14}$$

$$\langle \widetilde{D}(s), \widetilde{D}(s) \rangle = \left[(\frac{\kappa_2}{\kappa_1})'(s) \right]^2 \tag{4.15}$$

hold. □

4.5. Proposition. *For a unit speed curve $\gamma \subset \mathbb{E}^3$ the distinguished parameter of the ruled surfaces given by the equations (4.8)–(4.11) are, respectively,*

$$P_{(\gamma,C_\gamma)} = \frac{\langle \gamma \wedge n_2, t\rangle + c_1}{\left\|C_\gamma'(s)\right\|},$$
$$P_{(C_\gamma,\gamma)} = 0,$$
$$P_{(\widetilde{D},C_\gamma)} = \frac{(\frac{\kappa_2}{\kappa_1})'(s)(\langle \gamma \wedge n_2, t\rangle + c_1)}{\left\|C_\gamma'(s)\right\|},$$
$$P_{(C_\gamma,\widetilde{D})} = 0.$$

Proof. By using (4.2) we obtain the results. □

Finally, we obtain following results.

4.6. Corollary.

i) *The ruled surfaces $F_{(C_\gamma,\gamma)}$ and $F_{(C_\gamma,\widetilde{D})}$ are developable.*

ii) *If γ is a generalized helix then the ruled surface $F_{(C_\gamma,\widetilde{D})}$ is cylindrical and also developable.*

Proof. i) Since $P_{(C_\gamma,\gamma)} = 0 = P_{(C_\gamma,\widetilde{D})}$, then the ruled surfaces $F_{(C_\gamma,\gamma)}$ and $F_{(C_\gamma,\widetilde{D})}$ are developable.

ii) Suppose γ is a generalized helix. Then $\left(\frac{\kappa_2}{\kappa_1}\right)' = 0$. So, the distinguished parameter $P_{(\widetilde{D},C_\gamma)}$ of the ruled surface vanishes identically. Hence, $F_{(\widetilde{D},C_\gamma)}$ becomes a developable surface. □

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References

[1] Chen, B. Y. *When does the position vector of a space curve always lie in its rectifying plane?*, Am. Math Monthly **110**, 147–152, 2003.

[2] Eisenhart, L. P. *A Treatise on the Differential Geometry of Curves and Surfaces* (Dover Pub. Inc., New York, 1909).

[3] Gluck, H. *Higher curvatures of curves in Euclidean space*, Am. Math. Monthly **73**, 699–704, 1966.

[4] Gray, A. *Modern Differential Geometry of Curves and Surfaces* (CRC Press, Boca Raton, FL, 1993).

[5] Gursoy, O. *Some results on closed ruled surfaces and closed space curves*, Mech. Mach. Theory **27**, 323–330, 1990.

[6] Hacısalihoğlu, H. H. *Diferensiyel Geometri* (Gazi Üniversitesi Basın-Yayın Yüksekokulu Basımevi, Ankara, 1983) (In Turkish).

[7] Izumiya, S. and Takeuchi, N. *New special curves and developable surfaces*, Turkish J. Math. **28**, 153–163, 2004.

[8] Izumiya, S. and Takeuchi, N. *Special curves and ruled surfaces*, Cont. to Alg. and Geo. **44**, 200–212, 2003.

[9] Izumiya, S. and Takeuchi, N. *Geometry of Ruled Surfaces* (Applicable mathematics in the Goldon Age (ed. J. C. Misra), 305–308, Narosa Pulishing House, New Delhi, 2003).

[10] Izumiya, S., Katsumi, H. and Yamashi, T. *The rectifying developable and spherical Darboux image of a space curve*, Geom. and Top. of caustics - Caustics'98 Banach Center Publications **50**, 137–149, 1999.

[11] Koenderink, J. *Solid Shape* (MIT Press, Cambridge MA, 1990).

[12] Laugwitz, D. *Differential and Riemannian Geometry* (Academic Press, New York, 1965).

[13] Monterde, J. *Curves with constant curvatures ratios*, `arXiv:math/0412323v1`.

[14] Ozyilmaz, E. and Yayli, Y. *On the closed space-like developable ruled surface*, Hadronic J. **23** (4), 439–456, 2000.

[15] Uribe-Vargas, R. *On vertices, focal curvatures and differential geometry of space curves*, Bull. Brazilian Math. Soc. **36** (3), 285–307, 2005.