Applied Mathematics Letters 23 (2010) 1027–1032

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

A new subclass of the meromorphic harmonic starlike functions

Hakan Bostanci [a,*], Metin Öztürk [b]

[a] Department of Mathematics, Faculty of Science and Art, Karabuk University, Karabük, Turkey

[b] Department of Mathematics, Faculty of Science and Art, Uludag University, Bursa, Turkey

ARTICLE INFO

Article history:
Received 3 June 2009
Accepted 22 April 2010

Keywords:
Harmonic
Meromorphic
Univalent
Starlike

ABSTRACT

In this work, we introduce a new class of meromorphic harmonic starlike functions exterior to the unit disc $\tilde{U} = \{z : |z| > 1\}$. We obtain coefficient inequalities and a distortion theorem. In addition, we investigate some properties of this class.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain D if both u and v are real harmonic in D. For in any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [1]). In [2] Hengartner and Schober investigated functions harmonic in the exterior of the unit disc $\tilde{U} = \{z : |z| > 1\}$. They showed that a complex valued, harmonic, sense preserving, univalent mapping f, defined on $\tilde{U}$ and satisfying $f(\infty) = \infty$, must admit the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z|, \tag{1}$$

where

$$h(z) = \alpha z + \sum_{k=1}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \beta z + \sum_{k=1}^{\infty} b_k z^{-k} \tag{2}$$

$0 \le |\beta| < |\alpha|$, $A \in \mathbb{C}$ and $a = \bar{f}_{\bar{z}}/f_z$ is analytic and satisfies $|a(z)| < 1$ for $z \in \tilde{U}$. After this work, Jahangiri and Silverman [3] gave sufficient coefficient conditions for which functions of the form (1) will be univalent. For under certain restrictions, they also give necessary and sufficient coefficient conditions for functions to be harmonic and starlike. In [3] the following theorem, which we shall use in this work, is also proved.

Theorem 1. *Let $f(z) = h(z) + \overline{g(z)} + A \log |z|$ with $h(z)$ and $g(z)$ of the form of (2). If*

$$\sum_{k=1}^{\infty} k(|a_k| + |b_k|) \le |\alpha| - |\beta| - |A| \tag{3}$$

then $f(z)$ is sense preserving and univalent in $\tilde{U}$.

Also, Jahangiri [4] and Murugusundaramoorthy [5,6] have studied the classes of meromorphic harmonic functions.

* Corresponding author.
E-mail addresses: hakanbostanci16@gmail.com (H. Bostanci), ometin@uludag.edu.tr (M. Öztürk).

0893-9659/$ – see front matter Crown Copyright © 2010 Published by Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2010.04.031

In this work, we define a new operator M for meromorphic harmonic functions in $\tilde{U}$. Also, we define the classes $MH^*(n)$ and $\overline{MH}^*(n)$. Then, we investigate some properties of these classes such as coefficient estimates and a distortion theorem. We define a new operator M for harmonic functions $f(z) = h(z) + \overline{g(z)}$ where $h(z)$ and $g(z)$ are of the form (2), as follows:

$$M^0 f(z) = f(z), \qquad M^1 f(z) = \overline{\frac{(z^2 g(z))'}{z}} - z^3\left(\frac{h(z)}{z^2}\right)'$$

and for $n = 2, \ldots,$

$$M^n f(z) = M(M^{n-1} f(z)).$$

Hence, we obtain for $n = 0,\ 1, \ldots,$

$$M^n f(z) = \alpha z + \sum_{k=1}^{\infty}(k+2)^n a_k z^{-k} + \overline{3^n \beta z + (-1)^n \sum_{k=1}^{\infty}(k-2)^n b_k z^{-k}}.$$

Using the operator M, we now introduce the following classes:
Let $MH^*(n)$ denote the class of harmonic, sense preserving, univalent functions that consist of functions satisfying for $z \in \tilde{U},\ n \in \mathbf{N}_0 = \{0,\ 1,\ 2, \ldots\}$,

$$\operatorname{Re}\left\{2 - \frac{M^{n+1} f(z)}{M^n f(z)}\right\} > 0. \tag{4}$$

Also, let $\overline{MH}^*(n)$ be the subclass of $MH^*(n)$ which consists of meromorphic harmonic functions of the form

$$f_n(z) = h(z) + \overline{g_n(z)} = -\alpha z - \sum_{k=1}^{\infty} a_k z^{-k} + \overline{\beta z - (-1)^n \sum_{k=1}^{\infty} b_k z^{-k}} \tag{5}$$

where $\alpha > \beta \geq 0$, $a_k \geq 0$, $b_k \geq 0$, $b_2 \leq (\alpha - \beta)/2$.
A necessary and sufficient condition for f functions of the form (1) to be starlike in $\tilde{U}$ is that for each z, $|z| = r > 1$, we have

$$\frac{\partial}{\partial\theta}\arg(f(re^{i\theta})) = \operatorname{Im}\frac{\partial}{\partial\theta}(\log f(re^{i\theta})) = \operatorname{Re}\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} > 0, \tag{6}$$

where $z = re^{i\theta}$, $0 \leq \theta < 2\pi$, $r > 1$. This classification (6) for harmonic univalent functions was first used by Jahangiri [7].

Notice that if we take $n = 0$ in the inequalities (4), then we obtain inequalities (6) and $f(z)$ is meromorphic harmonic, sense preserving, univalent starlike in $\tilde{U}$.

2. Coefficient inequalities

In this section we obtain coefficient bounds. Our first theorem gives a sufficient coefficient condition for the class $MH^*(n)$.

Theorem 2. *If $f(z) = h(z) + \overline{g(z)}$ where $|b_2| \leq (|\alpha| - |\beta|)/2$, $h(z)$ and $g(z)$ is of the form (2) and the condition*

$$\sum_{k=1}^{\infty} k(k+2)^n |a_k| + \sum_{k=3}^{\infty} k(k-2)^n |b_k| + |b_1| \leq |\alpha| - 3^n|\beta| \tag{7}$$

is satisfied, then $f(z)$ is univalent, sense preserving in $\tilde{U}$ and $f(z) \in MH^*(n)$.

Proof. In view of Theorem 1, $f(z)$ is sense preserving and univalent in $\tilde{U}$. Now it remains to show that whether the condition (7) is sufficient for $f(z)$ to be in $MH^*(n)$. We use the fact that $\operatorname{Re}\zeta \geq 0$ if and only if $|1+\zeta| \geq |1-\zeta|$ in $\tilde{U}$. Therefore, it is sufficient to show that

$$|p_n(z) + 1| > |p_n(z) - 1|, \quad z \in \tilde{U}, \tag{8}$$

where

$$p_n(z) = \frac{2M^n f(z) - M^{n+1} f(z)}{M^n f(z)}.$$

From (8) we must have

$$\frac{\left|3M^n f(z) - M^{n+1} f(z)\right|}{|M^n f(z)|} - \frac{\left|M^n f(z) - M^{n+1} f(z)\right|}{|M^n f(z)|} > 0. \tag{9}$$

Substituting for $f(z)$ in (9) we obtain

$$\begin{aligned}
&\left|3M^n f(z) - M^{n+1} f(z)\right| - \left|M^n f(z) - M^{n+1} f(z)\right| \\
&= \left|2\alpha z - \sum_{k=1}^{\infty}(k-1)(k+2)^n a_k z^{-k} + (-1)^n \overline{\sum_{k=1}^{\infty}(k+1)(k-2)^n b_k z^{-k}}\right| \\
&\quad - \left|\sum_{k=1}^{\infty}(k+1)(k+2)^n a_k z^{-k} + \overline{23^n \beta z +}(-1)^n \overline{\sum_{k=1}^{\infty}(k-1)(k-2)^n b_k z^{-k}}\right| \\
&= 2|z|\,|\alpha| - \sum_{k=1}^{\infty} 2k(k+2)^n |a_k|\,|z|^{-k} - 23^n|\beta|\,|z| - 2|b_1|\,|z|^{-1} - \sum_{k=3}^{\infty} 2k(k-2)^n |b_k|\,|z|^{-k} \\
&= 2|z|\left[|\alpha| - \sum_{k=1}^{\infty} k(k+2)^n |a_k|\,|z|^{-k-1} - 3^n|\beta| - |b_1|\,|z|^{-2} - \sum_{k=3}^{\infty} k(k-2)^n |b_k|\,|z|^{-k-1}\right] \\
&\geq 2\left\{|\alpha| - \sum_{k=1}^{\infty} k(k+2)^n |a_k| - 3^n|\beta| - |b_1| - \sum_{k=3}^{\infty} k(k-2)^n |b_k|\right\} \geq 0, \quad \text{by (7).}
\end{aligned}$$

We next show that the above sufficient condition for $MH^*(n)$ is also necessary for functions in $\overline{MH}^*(n)$. □

Theorem 3. *Let the function $f_n(z)$ be defined by (3). A necessary and sufficient condition for having $f_n(z) \in \overline{MH}^*(n)$ is that*

$$\sum_{k=1}^{\infty} k(k+2)^n a_k + \sum_{k=3}^{\infty} k(k-2)^n b_k + b_1 \leq \alpha - 3^n \beta. \tag{10}$$

Proof. In view of Theorem 2, it is sufficient to prove the "only if" part, since $\overline{MH}^*(n) \subset MH^*(n)$. Assume that $f_n(z) \in \overline{MH}^*(n)$. Let the z be complex numbers. If $\mathrm{Re}(z) > 0$ then $\mathrm{Re}(1/z) > 0$. Thus from (4) we obtain

$$\begin{aligned}
0 &< \mathrm{Re}\left\{\frac{M^n f(z)}{2M^n f(z) - M^{n+1} f(z)}\right\} \leq \left|\frac{M^n f(z)}{2M^n f(z) - M^{n+1} f(z)}\right| \\
&= \left|\frac{-\alpha z - \sum_{k=1}^{\infty}(k+2)^n a_k z^{-k} + \overline{3^n \beta z - \sum_{k=1}^{\infty}(k-2)^n b_k z^{-k}}}{-\alpha z + \sum_{k=1}^{\infty} k(k+2)^n a_k z^{-k} + \overline{3^n \beta z - \sum_{k=1}^{\infty} k(k-2)^n b_k z^{-k}}}\right| \\
&\leq \frac{\alpha|z| + \sum_{k=1}^{\infty}(k+2)^n a_k |z|^{-k} + 3^n \beta|z| + \sum_{k=3}^{\infty}(k-2)^n b_k |z|^{-k} + b_1|z|^{-1}}{\alpha|z| - \sum_{k=1}^{\infty} k(k+2)^n a_k |z|^{-k} - 3^n\beta|z| - \sum_{k=3}^{\infty} k(k-2)^n b_k |z|^{-k} - b_1|z|^{-1}} \\
&< \frac{\alpha + \sum_{k=1}^{\infty}(k+2)^n a_k + 3^n\beta + \sum_{k=3}^{\infty}(k-2)^n b_k + b_1}{\alpha - \sum_{k=1}^{\infty} k(k+2)^n a_k - 3^n\beta - \sum_{k=3}^{\infty} k(k-2)^n b_k - b_1}.
\end{aligned} \tag{11}$$

From (11), it must be the case that

$$\sum_{k=1}^{\infty} k(k+2)^n a_k + \sum_{k=3}^{\infty} k(k-2)^n b_k + b_1 \leq \alpha - 3^n \beta.$$

Hence, the proof is completed. □

3. Convex combinations

In this section, we show that the class $\overline{MH}^*(n)$ is invariant under convex combinations of its members.

Theorem 4. *The class $\overline{MH}^*(n)$ is a convex set.*

Proof. For $i = 1, 2, \ldots$ suppose that $f_{n,i}(z) \in \overline{MH}^*(n)$ where $f_{n,i}(z)$ is given by

$$f_{n,j}(z) = -\alpha_i z - \sum_{k=1}^{\infty} a_{k,i} z^{-k} + \overline{\beta_i z - (-1)^n \sum_{k=1}^{\infty} b_{k,i} z^{-k}}.$$

Then, by (10),

$$\sum_{k=1}^{\infty} k(k+2)^n a_{k,i} + \sum_{k=3}^{\infty} k(k-2)^n b_{k,i} + b_{1,i} \le \alpha_i - 3^n \beta_i.$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \le t_i \le 1$, the convex combination of $f_{n,i}$ may be written as

$$\sum_{i=1}^{\infty} t_i f_{n,i}(z) = -\left(\sum_{i=1}^{\infty} t_i \alpha_i\right) z - \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{k,i}\right) z^{-k} + \left(\sum_{i=1}^{\infty} t_i \beta_i\right) \bar{z} - (-1)^n \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{k+p-1,i}\right) \bar{z}^{-k}.$$

Hence, $\sum_{i=1}^{\infty} t_i f_{n,i}(z) \in \overline{MH}^*(n)$, since

$$\begin{aligned}
&\sum_{k=1}^{\infty} k(k+2)^n \left(\sum_{i=1}^{\infty} t_i a_{k,i}\right) + \sum_{k=3}^{\infty} k(k-2)^n \left(\sum_{i=1}^{\infty} t_i b_{k,i}\right) + \left(\sum_{i=1}^{\infty} t_i b_{1,i}\right) \\
&\quad = \sum_{i=1}^{\infty} t_i \left[\sum_{k=1}^{\infty} k(k+2)^n a_{k,i} + \sum_{k=3}^{\infty} k(k-2)^n b_{k,i} + b_{1,i}\right] \le \sum_{i=1}^{\infty} t_i \left[\alpha_i - 3^n \beta_i\right] = \left(\sum_{i=1}^{\infty} t_i \alpha_i\right) - 3^n \left(\sum_{i=1}^{\infty} t_i \beta_i\right). \quad \square
\end{aligned}$$

4. A distortion theorem and extreme points

In this section we shall obtain distortion bounds for the functions in $\overline{MH}^*(n)$. We shall also examine the extreme points for functions in $\overline{MH}^*(n)$ when f_n is defined by (5).

Theorem 5. *Let the function $f_n(z)$ be in the class $\overline{MH}^*(n)$. Then, for $|z| = r > 1$, we have*

$$(\alpha - \beta)r - (\alpha - 3^n\beta)r^{-1} \le |f_n(z)| \le (\alpha + \beta)r + (\alpha - 3^n\beta)r^{-1}.$$

Proof. Suppose $f_n(z) \in \overline{MH}^*(n)$. Taking the absolute value of f_n we obtain

$$\begin{aligned}
|f_n(z)| &= \left| -\alpha z - \sum_{k=1}^{\infty} a_k z^{-k} + \overline{\beta z - (-1)^n \sum_{k=1}^{\infty} b_k z^{-k}} \right| \\
&\le \alpha r + \beta r + \sum_{k=1}^{\infty} (a_k + b_k) r^{-k} \le \alpha r + \beta r + \sum_{k=1}^{\infty} (a_k + b_k) r^{-1} \\
&\le \alpha r + \beta r + r^{-1} \left(\sum_{k=1}^{\infty} k(k+2)^n a_k + \sum_{k=3}^{\infty} k(k-2)^n b_k + b_1\right) \le (\alpha + \beta) r + (\alpha - 3^n \beta) r^{-1}, \quad \text{by (10)}.
\end{aligned}$$

The proof for the right hand bound is similar to that given above and we omit it.

$\overline{MH}^*(n)$ is still not compact under the topology of locally uniform convergence. To see this, observe that for each fixed α, for $m = 1, 2, \ldots,$

$$f_m(z) = -\alpha z + \frac{\alpha m}{m+1} \bar{z} \in \overline{MH}^*(n)$$

but

$$\lim_{m \to \infty} f_m(z) = -\alpha z + \alpha \bar{z} \notin \overline{MH}^*(n).$$

Nevertheless, we can still use the coefficient bounds of Theorem 3 to determine the extreme points of the closed convex hull of $\overline{MH}^*(n)$ denoted by $clco\overline{MH}^*(n)$. $\square$

Theorem 6. *Let, for* $z \in \tilde{U}$,

$$h_{n,0}(z) = -z, \qquad g_{n,0}(z) = -z + \frac{\bar{z}}{3^n}, \qquad g_{n,1}(z) = -z - (-1)^n \bar{z}^{-1}, \qquad g_{n,2}(z) = -z - \frac{(-1)^n}{2}\bar{z}^{-2},$$
$$h_{n,k}(z) = -z - \frac{1}{k(k+2)^n} z^{-k}, \quad k \geq 1 \quad \text{and} \quad g_{n,k}(z) = -z - \frac{(-1)^n}{k(k-2)^n}\bar{z}^{-k}, \quad k \geq 3.$$

Then $f_n(z) \in clco\overline{MH}^*(n)$ *if and only if it can be expressed in the form*

$$f_n(z) = \sum_{k=0}^{\infty} \left[x_k h_{n,k}(z) + y_k g_{n,k}(z) \right]$$

where $x_k \geq 0,\ y_k \geq 0$ *and* $\sum_{k=0}^{\infty}(x_k + y_k) = \alpha$.

Proof. Let

$$f_n(z) = \sum_{k=0}^{\infty} \left[x_k h_{n,k}(z) + y_k g_{n,k}(z) \right]$$

with $x_k \geq 0,\ y_k \geq 0$ and $\sum_{k=0}^{\infty}(x_k + y_k) = \alpha$.
Then, we have

$$\begin{aligned} f_n(z) &= \sum_{k=0}^{\infty} \left[x_k h_{n,k}(z) + y_k g_{n,k}(z) \right] \\ &= x_0 h_{n,0}(z) + \sum_{k=1}^{\infty} x_k \left[-z - \frac{1}{k(k+2)^n} z^{-k} \right] + y_0 g_{n,0}(z) + y_1 g_{n,1}(z) \\ &\quad + y_2 g_{n,2}(z) + \sum_{k=3}^{\infty} y_k \left[-z - \frac{(-1)^n}{k(k-2)^n} \bar{z}^{-k} \right] \\ &= -\sum_{k=0}^{\infty}(x_k + y_k) z - \sum_{k=1}^{\infty} \frac{x_k}{k(k+2)^n} z^{-k} + \frac{y_0}{3^n}\bar{z} - (-1)^n y_1 \bar{z}^{-1} - (-1)^n \frac{y_2}{2}\bar{z}^{-2} - (-1)^n \sum_{k=3}^{\infty} \frac{y_k}{k(k-2)^n}\bar{z}^{-k} \\ &= -\alpha z - \sum_{k=1}^{\infty} \frac{x_k}{k(k+2)^n} z^{-k} + \frac{y_0}{3^n}\bar{z} - (-1)^n y_1 \bar{z}^{-1} - (-1)^n \frac{y_2}{2}\bar{z}^{-2} - (-1)^n \sum_{k=3}^{\infty} \frac{y_k}{k(k-2)^n}\bar{z}^{-k}. \end{aligned}$$

Since

$$\begin{aligned} &\sum_{k=1}^{\infty} k(k+2)^n \frac{x_k}{k(k+2)^n} + \sum_{k=3}^{\infty} k(k-2)^n \frac{y_k}{k(k-2)^n} + y_1 = \left(\sum_{k=1}^{\infty} x_k + y_1 + \sum_{k=3}^{\infty} y_k \right) \\ &\quad = \alpha - y_0 - x_0 - y_2 \leq \alpha - 3^n \frac{y_0}{3^n} \end{aligned}$$

by Theorem 3, $f_n(z) \in clco\overline{MH}^*(n)$. Conversely, we suppose that $f_n(z) \in clco\overline{MH}^*(n)$; then we may write

$$f_n(z) = h(z) + \overline{g_n(z)} = -\alpha z - \sum_{k=1}^{\infty} a_k z^{-k} + \overline{\beta z - (-1)^n \sum_{k=1}^{\infty} b_k z^{-k}}$$

where $\alpha > \beta \geq 0$, $a_k \geq 0$, $b_k \geq 0$. We set

$$a_k = \frac{x_k}{k(k+2)^n}, \quad k = 1, 2, \ldots, \qquad \beta = \frac{y_0}{3^n}, \qquad b_1 = y_1, \qquad b_2 = \frac{y_2}{2},$$
$$b_k = \frac{y_k}{k(k-2)^n}, \quad k = 3, 4 \ldots.$$

Hence, we obtain

$$\begin{aligned} f_n(z) &= h(z) + \overline{g_n(z)} = -\alpha z - \sum_{k=1}^{\infty} a_k z^{-k} + \overline{\beta z - (-1)^n \sum_{k=1}^{\infty} b_k z^{-k}} \\ &= -\sum_{k=0}^{\infty}(x_k + y_k) z - \sum_{k=1}^{\infty} \frac{x_k}{k(k+2)^n} z^{-k} + \frac{y_0}{3^n}\bar{z} - (-1)^n y_1 \bar{z}^{-1} - (-1)^n \frac{y_2}{2}\bar{z}^{-2} - \sum_{k=3}^{\infty} \frac{(-1)^n y_k}{k(k-2)^n}\bar{z}^{-k} \end{aligned}$$

$$= x_0\,(-z) + \sum_{k=1}^{\infty} x_k \left[-z - \frac{1}{k(k+2)^n} z^{-k}\right] + y_0 \left(-z + \frac{\bar{z}}{3^n}\right) + y_1(-z - (-1)^n \bar{z}^{-1})$$

$$+ y_2 \left(-z - \frac{(-1)^n}{2} \bar{z}^{-2}\right) + \sum_{k=3}^{\infty} y_k \left[-z - \frac{(-1)^n}{k(k-2)^n} \bar{z}^{-k}\right]$$

$$= \sum_{k=0}^{\infty} \left[x_k h_{n,k}(z) + y_k g_{n,k}(z)\right].$$

This completes the proof of Theorem 6. □

Remark 1. The result of this work, for $n = 0$, coincides with that given by [3].

References

[1] J. Cluine, T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. Al Math. 9 (1984) 3–25.
[2] W. Hengartner, G. Schober, Univalent harmonic functions, Trans. Amer. Math. Soc. 299 (1987) 1–31.
[3] J.M. Jahangiri, H. Silverman, Meromorphic univalent harmonic functions with negative coefficients, Bull. Korean Math. Soc. 36 (1999) 763–770.
[4] J.M. Jahangiri, Harmonic meromorphic starlike functions, Bull. Korean Math. Soc. 37 (2) (2000) 291–301.
[5] G. Murugusundaramoorthy, Starlikeness of multivalent meromorphic harmonic functions, Bull. Korean Math. Soc. 40 (4) (2003) 553–564.
[6] G. Murugusundaramoorthy, Harmonic meromorphic convex functions with missing coefficients, J. Indones. Math. Soc. (MIHMI) 10 (1) (2004) 15–22.
[7] J.M. Jahangiri, Coefficient bounds and univalence criteria for harmonic functions with negative coefficients, Ann. Univ. Mariae Curie-Sklodowska, Sec. A 52 (1998) 57–66.