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A new approach to connect algebra with analysis: relationships and applications between presentations and generating functions

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Abstract

For a minimal group (or monoid) presentation $\mathcal{P}$, let us suppose that $\mathcal{P}$ satisfies the algebraic property of either being efficient or inefficient. Then one can investigate whether some generating functions can be applied to it and study what kind of new properties can be obtained by considering special generating functions. To establish that, we will use the presentations of infinite group and monoid examples, namely the split extensions $\mathbb{Z}_n \rtimes \mathbb{Z}$ and $\mathbb{Z}^2 \rtimes \mathbb{Z}$, respectively. This study will give an opportunity to make a new classification of infinite groups and monoids by using generating functions.

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1 Introduction and preliminaries

In the literature, although there are so many studies about figuring out the relationship between rings (or fields) and special generating functions (*cf.*, for instance, [1–4]), there are no such studies about the relationship between group (or monoid) *presentations* and *generating functions*. In fact, the studies on the efficient and inefficient (but minimal) group and monoid presentations gave very important characterisations for groups and monoids in the branch of combinatorial group theory of mathematics (see, for instance, [5–12]). It is known that generating functions are still interesting for many mathematicians and physicians (see, for instance, [2, 13, 14] in addition to above). Thus, it would be quite interesting for future studies to connect these two important areas and then search for possible properties.

In the light of this thought, in this paper, a connection between special (efficient and inefficient) presentations defined on infinite groups (and monoids) and some generating functions related to the special polynomials and numbers will be investigated. (These special polynomials are chosen by their integer coefficients. Of course, one can choose some other polynomials used in this paper.) Another aim of this paper is to try to make a classification of infinite groups and monoids.

This paper is divided into four sections. Main results are presented specially in Sections 2 and 3. In the remaining parts of this section, we will present some fundamental material related to the group or monoid presentations that will be needed in later sections of this paper.

A *group* (or a *monoid*) *presentation*

$$\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle \tag{1}$$

is a pair where $\mathbf{x}$ is a set (generating symbols) and $\mathbf{r}$ is a set of non-empty, cyclically reduced words (relators) on $\mathbf{x}$. In monoids, each $R \in \mathbf{r}$ is actually an ordered pair (R_+, R_-), where R_+ and R_- are distinct, positive (one of them could be empty) words on $\mathbf{x}$. We say that $\mathcal{P}$ is finite if $\mathbf{x}$ and $\mathbf{r}$ are both finite. Further, all results in this paper are related to split extensions and their presentations. In [15], a split extension is also named a semidirect product and detailed properties of this product can be found in elementary algebra textbooks. Here, we will just remind the *presentation* of a semidirect product of arbitrary groups (or monoids). Therefore, for arbitrary groups (or monoids) A and K with presentations $\mathcal{P}_A = \langle \mathbf{x}; \mathbf{r} \rangle$ and $\mathcal{P}_K = \langle \mathbf{y}; \mathbf{s} \rangle$, the presentation of the group (or monoid) $K \rtimes_\theta A$ is defined by

$$\mathcal{P}_{K \rtimes_\theta A} = \langle \mathbf{x}, \mathbf{y}; \mathbf{r}, \mathbf{s}, \mathbf{t} \rangle, \tag{2}$$

where $\mathbf{t}$ is the set of relators of the form

$$T_{yx} : yx = x(y\theta_x)$$

for all $x \in \mathbf{x}$ and $y \in \mathbf{y}$ (*cf.* [8, 9]). We remind that the homomorphism θ is defined from A to $\mathrm{Aut}(K)$ for the semidirect product of groups, while it is defined from A to $\mathrm{End}(K)$ for the product of monoids. Further, θ_x is an isomorphism of the group K and a homomorphism in a monoid case.

In the next two subsections, we will give some other preliminary material that will be needed for the construction of the results in this paper by considering the presentation $\mathcal{P}$ in (1).

1.1 Efficiency

The subject under this title will be given over a group G with a presentation $\mathcal{P}$ as defined in (1). But we should note that the following material will be completely the same if the group G is replaced by a monoid M.

For the presentation $\mathcal{P}$, the Euler characteristic is defined by $\chi(\mathcal{P}) = 1 - |\mathbf{x}| + |\mathbf{r}|$. By [16–18], there exists a lower bound $\delta(G)$ which is equal to $1 - rk_{\mathbb{Z}}(H_1(G)) + d(H_2(G))$ with the condition $\delta(G) \leq \chi(\mathcal{P})$, where $rk(\cdot)$ denotes the $\mathbb{Z}$-rank of the torsion-free part and $d(\cdot)$ denotes the minimal number of generators. Depending on these numbers, we define

$$\chi(G) = \min\{\chi(\mathcal{P}) : \mathcal{P} \text{ is a finite presentation for } G\}.$$

Therefore a presentation $\mathcal{P}$ is called *minimal* if $\chi(\mathcal{P}) \leq \chi(\mathcal{P}')$ for all presentations $\mathcal{P}'$ of G, or is called *efficient* if $\chi(\mathcal{P}) = \delta(G)$. Moreover, G is called *efficient* if $\chi(G) = \delta(G)$. In [7, 8], Cevik recalled known results for efficiency of groups and monoids. (We should remark that some authors also consider $-|\mathbf{x}| + |\mathbf{r}|$ and call this the *deficiency* of the presentation $\mathcal{P}$.)

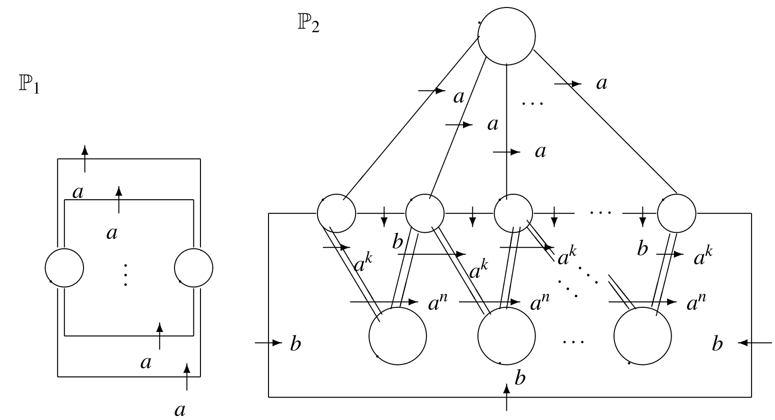


Figure 1 Generating pictures of $\mathcal{P}_G$ as given in (6).

Remark 1 In both group and monoid cases, if the presentation $\mathcal{P}$ in (1) is efficient or inefficient while it is minimal, then it always has a minimal number of generators. So, this fact affects positively the use of generating functions for this type of presentations since we have a great advantage to work with quite a limited number of variables in such a generating function.

1.2 Pictures

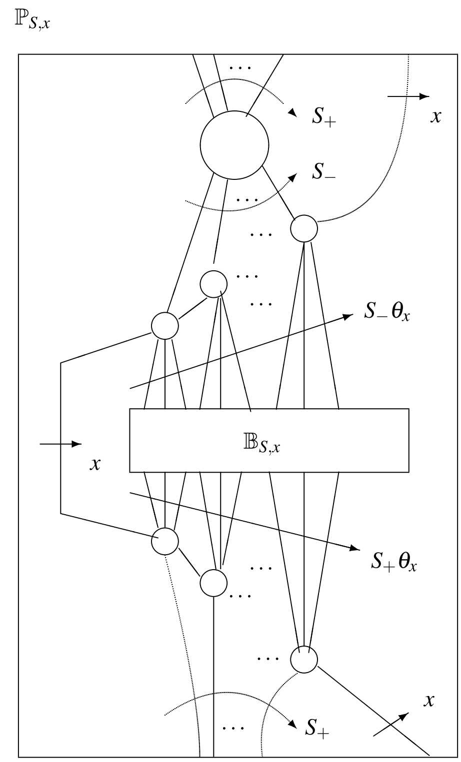
There exists a geometric method called *spherical group* (*or monoid*) *pictures* related to the presentation $\mathcal{P}$ given in (1). This method was constructed and first used by Pride [6, 17–20] for both groups and monoids, and since then it has still been in use for the solution of many important combinatorial problems such as *word problems* (*cf.* [21, 22]). Here, we will recall a brief description of pictures for groups and monoids in separate cases. Before that, we express the following remark.

Remark 2 Similarly to (undirected) graphs, this geometric configuration has a large application area, especially in engineering sciences. For example, the plan of electrical network for a city or the behaviour of DNA molecules in a human body can be figured out with pictures (see Figures 1, 2 and 3).

Pictures for groups: As we depicted in Remark 2, a group picture $\mathbb{P}$ over $\mathcal{P}$ is a geometric configuration consisting of the following:

- A disc D^2 with a basepoint O on the boundary ∂D^2 of D^2.
- Disjoint discs $\Delta_1, \Delta_2, \ldots, \Delta_n$ in the interior of D^2. Each Δ_i has a basepoint O_i on the boundary $\partial\Delta_i$ of Δ_i.
- A finite number of disjoint arcs $\alpha_1, \alpha_2, \ldots, \alpha_m$, where each arc lies in the closure of $D^2 - \bigcup_{i=1}^{n} \Delta_i$ and is either a simple closed curve having trivial intersection with $\partial D^2 \cup \partial\Delta_1 \cup \partial\Delta_2 \cup \cdots \cup \partial\Delta_n$, or is a simple non-closed curve which joins two points of $\partial D^2 \cup \partial\Delta_1 \cup \partial\Delta_2 \cup \cdots \cup \partial\Delta_n$, neither point being a basepoint. Each arc has a normal orientation indicated by a short arrow meeting with the arc transversely and is labelled by an element of $\mathbf{x} \cup \mathbf{x}^{-1}$ which is called the label of the arc.
- If we travel around $\partial\Delta_i$ once in the clockwise direction starting from O_i and read off the labels on arcs encountered (if we cross an arc, labelled x say, in the direction of its

Figure 2 The single generating picture of $\mathcal{P}_M$ given in (9).



normal orientation, then we read x, whereas if we cross the arc in the direction of its opposite orientation, then we read x^{-1}), then we obtain a word which belongs to $\mathbf{r} \cup \mathbf{r}^{-1}$. We call this word the label of Δ_i. If $\mathbf{s}$ is a subset of $\mathbf{r}$, then a disc labelled by an element of $\mathbf{s} \cup \mathbf{s}^{-1}$ is called an $\mathbf{s}$-disc.

When we refer to the discs of $\mathbb{P}$, we in fact mean the discs $\Delta_1, \Delta_2, \dots, \Delta_n$, and not the ambient disc D^2. A closed arc which encircles neither a disc nor an arc of $\mathbb{P}$ is called a floating circle. We define $\partial\mathbb{P}$ to be ∂D^2. The label on $\mathbb{P}$ (denoted by $W(\mathbb{P})$) is the word read off by travelling around $\partial\mathbb{P}$ once in the clockwise direction starting from O. (In fact, this fact on pictures implies the fundamentals of solving the word problem [21, 22].)

Further, $\mathbb{P}$ is called *spherical* if no arcs meet $\partial\mathbb{P}$ (*i.e.* if $\mathbb{P}$ is spherical, then $\partial\mathbb{P}$ is omitted). A *transverse path* γ in a picture $\mathbb{P}$ is a path in the closure of $D^2 - \bigcup_{i=1}^{n} \Delta_i$ which intersects the arcs of $\mathbb{P}$ only finitely many times. Reading off the labels on the arcs encountered while travelling along a transverse path from its initial point to its terminal point gives a word on $\mathbf{x}$ denoted by $W(\gamma)$. Let γ be a simple closed transverse path in $\mathbb{P}$. The part of $\mathbb{P}$ enclosed by γ is called a *subpicture* of $\mathbb{P}$. If γ intersects no arcs, then the part of $\mathbb{P}$ enclosed by γ is called a spherical subpicture of $\mathbb{P}$. A *cancelling pair* in $\mathbb{P}$ is a spherical subpicture with exactly two discs whose basepoints lie in the same region.

A *spray* for $\mathbb{P}$ is a sequence $\underline{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_n)$ of simple transverse paths satisfying the following: for $i = 1, 2, \dots, n$, γ_i starts at O and ends at the basepoint of Δ_i, for $1 \leq i < j \leq n$, γ_i and γ_j intersect only at O; travelling around O clockwise in $\mathbb{P}$, we encounter these transverse paths $\gamma_1, \gamma_2, \dots, \gamma_n$, respectively.

There are some elementary operations (deletion and insertion of a floating circle, deletion and insertion of a cancelling pair, bridge move) on spherical pictures. Then two spherical pictures are called *equivalent* if one can be obtained from the other by a finite number

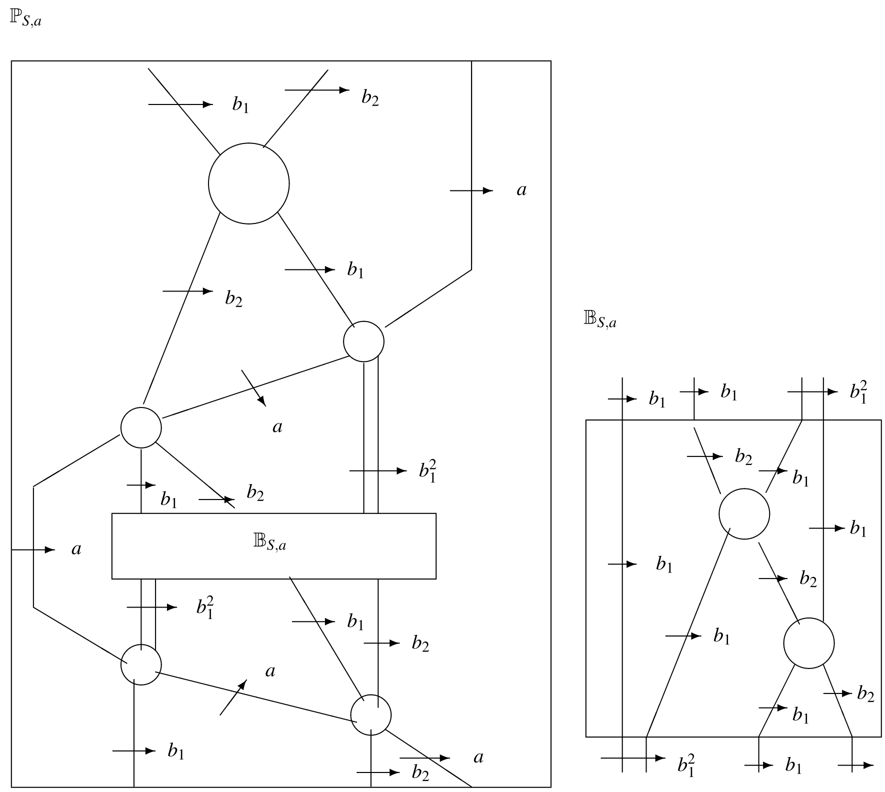


Figure 3 The single generating picture of $\mathcal{P}_M$ in (12).

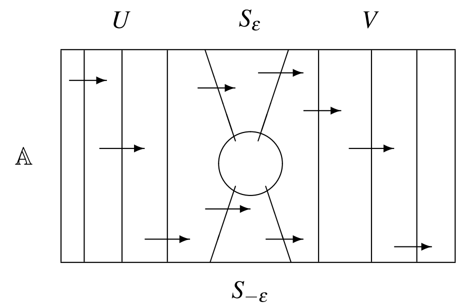
of above operations. These operations imply an equivalence relation and the equivalence class containing $\mathbb{P}$ which is denoted by $\langle\mathbb{P}\rangle$. The set of all equivalence classes of spherical pictures over $\mathcal{P}$ forms an abelian group. In addition, for a word W on $\mathbf{x}$, a new spherical picture over $\mathcal{P}$ denoted by $\mathbb{P}^W$ can also be obtained from W by surrounding $\mathbb{P}$ with a collection of concentric arcs with total label W. Hence, there is a well-defined $G(\mathcal{P})$-action on equivalence classes of spherical pictures given by $\overline{W} \cdot \langle\mathbb{P}\rangle = \langle\mathbb{P}^W\rangle$ (where $\overline{W} \in G(\mathcal{P})$). We then obtain a $\mathbb{Z}G(\mathcal{P})$-module $\pi_2(\mathcal{P})$ called the *second homotopy module* of $\mathcal{P}$. Let $\mathbf{X}$ be a set of spherical pictures. Then we say that $\mathbf{X}$ generates $\pi_2(\mathcal{P})$ (or $\mathbf{X}$ is a set of generating pictures) if the elements $\langle\mathbb{P}\rangle$ (where $\mathbb{P} \in \mathbf{X}$) generate $\pi_2(\mathcal{P})$.

For any picture $\mathbb{P}$ over $\mathcal{P}$ and for any $R \in \mathbf{r}$, the *exponent sum* of R in $\mathbb{P}$, denoted by $\exp_R(\mathbb{P})$, is the number of discs of $\mathbb{P}$ labelled by R minus the number of discs labelled by R^{-1}. We remind that if pictures $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent, then $\exp_R(\mathbb{P}_1) = \exp_R(\mathbb{P}_2)$ for all $R \in \mathbf{r}$. Depending on the exponent sum, we have the following definition.

Definition 1 For a non-negative integer n, $\mathcal{P}$ is said to be n-Cockcroft if $\exp_R(\mathbb{P}) \equiv 0 \pmod{n}$ (where congruence $(\bmod\, 0)$ is taken to be equality) for all $R \in \mathbf{r}$ and for all spherical pictures $\mathbb{P}$ over $\mathcal{P}$. Moreover, a group G is said to be n-Cockcroft if it admits an n-Cockcroft presentation.

Actually, to verify that the n-Cockcroft property holds, it is enough to check it only for pictures $\mathbb{P} \in \mathbf{X}$, where $\mathbf{X}$ is a set of generating pictures. Also, the 0-Cockcroft property is usually just called Cockcroft, and in practice, n is taken as a prime p or 0. By [16, 19],

Figure 4 An atomic monoid picture.



the presentation $\mathcal{P}$ is efficient if and only if it is p-Cockcroft for some prime p. So, this connection between efficiency and p-Cockcroft property will be one of the main ideas during the construction of this paper.

There is an embedding μ of $\pi_2(\mathcal{P})$ into the free module $\bigoplus_{R\in\mathbf{r}} \mathbb{Z}G(\mathcal{P})e_R$ defined as follows (see [6, 19]): Let $\langle\mathbb{P}\rangle \in \pi_2(\mathcal{P})$ and suppose that $\mathbb{P}$ has discs $\Delta_1, \Delta_2, \ldots, \Delta_n$ with the labels $R_1^{\varepsilon_1}, R_2^{\varepsilon_2}, \ldots, R_n^{\varepsilon_n}$, respectively ($R_i \in \mathbf{r}$, $\varepsilon_i = \pm 1$, $i = 1, 2, \ldots, n$). Let $\underline{\gamma} = (\gamma_1, \ldots, \gamma_n)$ be a spray defined previously. Then

$$\mu\big(\langle\mathbb{P}\rangle\big) = \sum_{i=1}^{n} \varepsilon_i \overline{W(\gamma_i)} e_{R_i}. \tag{3}$$

For simplicity, the notation $\mu(\mathbb{P})$ will be preferred instead of $\mu(\langle\mathbb{P}\rangle)$. For each spherical picture $\mathbb{P}$ over $\mathcal{P}$ and for each $R \in \mathbf{r}$, let $\lambda_{\mathbb{P},R}$ be the coefficients of e_R in $\mu(\mathbb{P})$. Let $I_2(\mathcal{P})$ be the two-sided ideal in $\mathbb{Z}G$ generated by the set $\{\lambda_{\mathbb{P},R} : \mathbb{P} \text{ is a spherical picture}, R \in \mathbf{r}\}$. This ideal is called the *second Fox ideal* of $\mathcal{P}$. The concept of Fox ideals has been discussed in [23]. In fact, we need this concept for our studies in this paper as our main goal in this paper is to establish a relationship between generating functions and presentations. For the group case in Section 2, the generating functions will be labelled by $\varepsilon_i \overline{W(\gamma_i)}$ defined in (3).

Pictures for monoids. As we pointed out in the beginning of this section, some of the following material may also be found in [11, 17, 18, 20]. For a monoid M, let $\mathcal{P}$ be a monoid presentation as in (1); and let $F(\mathbf{x})$ be a free monoid on $\mathbf{x}$. If we have an element $W = US_\varepsilon V$ (where $U, V \in F(\mathbf{x})$, $S \in \mathbf{r}$, $\varepsilon = \pm 1$) of $F(\mathbf{x})$, then we can replace S_ε by $S_{-\varepsilon}$ to get a word $W' = US_{-\varepsilon}V$. This can be represented by a geometric object called an *atomic* (*monoid*) *picture* $\mathbb{A} = (U, S, \varepsilon, V)$ as depicted in Figure 4.

We remark that the disc labelled by S in an atomic picture $\mathbb{A}$ is said to be *positive* if $\varepsilon = 1$ and is said to be *negative* if $\varepsilon = -1$.

We have a graph Γ ($= \Gamma(\mathcal{P})$) associated with $\mathcal{P}$, called the *Squier graph*, which is defined as follows: The *vertex set* is $F(\mathbf{x})$, and the *edge set* is the collection of all atomic monoid pictures. For an orientation of Γ, we will take all edges $(U, S, +1, V)$. For an atomic picture $\mathbb{A}$, as in Figure 4, the word we read off by travelling along the top of the atomic picture from left to right gives the initial function, denoted by $\iota(\mathbb{A}) = US_\varepsilon V$, and the word we read off by travelling along the bottom gives the terminal function, denoted by $\tau(\mathbb{A}) = US_{-\varepsilon}V$. Also, the mirror image of $\mathbb{A}$ is denoted by $\mathbb{A}^{-1} = (U, S, -\varepsilon, V)$. A *path* $\mathbb{P} = \mathbb{A}_1\mathbb{A}_2\cdots\mathbb{A}_n$ (where each $\mathbb{A}_i$ is an atomic picture for $i = 1, 2, \ldots, n$) in Γ will also be called a monoid picture over $\mathcal{P}$. If $\iota(\mathbb{A}_1) = \tau(\mathbb{A}_n)$, then $\mathbb{P}$ is called a *spherical monoid picture* over $\mathcal{P}$. Note that we also have the term *subpicture* of monoid pictures.

There is a left action of $F(\mathbf{x})$ on Γ defined as follows. Let $C \in F(\mathbf{x})$.

(i) Let W be a vertex of Γ. Then we define CW to be $C \cdot W$ (product in $F(\mathbf{x})$).

(ii) Let $\mathbb{A}$, as in Figure 4, be an edge of Γ. Then $C \cdot \mathbb{A} = (CU, S, \varepsilon, V)$.

We can define a similar right action of $F(\mathbf{x})$ on Γ. The left and right actions of $F(\mathbf{x})$ on Γ extend to actions on pictures. That is, if $\mathbb{P}$ is a picture and $W, V \in F(\mathbf{x})$, then $W \cdot \mathbb{P} \cdot V = (W \cdot \mathbb{A}_1 \cdot V)(W \cdot \mathbb{A}_2 \cdot V) \cdots (W \cdot \mathbb{A}_n \cdot V)$.

For atomic monoid pictures $\mathbb{A}$ and $\mathbb{B}$, one can introduce some operations (deletion and insertion of inverse pairs of atomic pictures and a replacement operation (*cf.* [17, 18])) on spherical monoid pictures. These operations imply an equivalence relation on paths. Therefore the graph Γ with this equivalence relation on paths is called the *Squier complex* of $\mathcal{P}$ denoted by $\mathcal{D}(\mathcal{P})$. Let $\mathbf{Y}$ be a set of spherical monoid pictures. Two spherical monoid pictures will be said to be *equivalent* (*relative to* $\mathbf{Y}$) if one can be transformed into the other by a finite number of above operations. By [20], the set $\mathbf{Y}$ is called a *trivializer* of $\mathcal{D}(\mathcal{P})$ if every spherical picture is equivalent to an empty picture (relative to $\mathbf{Y}$). Some examples and the details of the trivializer can be found, for instance, in [11, 12]. Similarly as in the group case, for any monoid picture $\mathbb{P}$ over $\mathcal{P}$ and for any $S \in \mathbf{r}$, the *exponent sum* of S in $\mathbb{P}$ is the number of positive discs labelled by S minus the number of negative discs labelled by S. Then the monoid version of Definition 1 can be obtained in completely the same way by replacing the term group with monoid. To verify that the n-Cockcroft (in fact n is taken as a prime p or 0) property holds, it is enough to check it for pictures $\mathbb{P} \in \mathbf{Y}$, where $\mathbf{Y}$ is a trivializer of $\mathcal{D}(\mathcal{P})$.

Let M be a monoid with the presentation $\mathcal{P}$ as in (1). Let

$$P^{(l)} = \bigoplus_{S \in \mathbf{r}} \mathbb{Z}Me_S$$

be a free left $\mathbb{Z}M$-module with basis $\{e_S : S \in \mathbf{r}\}$. For an atomic picture $\mathbb{A} = (U, S, \varepsilon, V)$ with $U, V \in F(\mathbf{x})$, $S \in \mathbf{r}$, $\varepsilon = \pm 1$, we define $\mathrm{eval}^{(l)}(\mathbb{A}) = \varepsilon \overline{U} e_S \in P^{(l)}$, where $\overline{U} \in M(\mathcal{P})$ as in Figure 4. For any spherical monoid picture $\mathbb{P}$, we define

$$\mathrm{eval}^{(l)}(\mathbb{P}) = \sum_{i=1}^{n} \mathrm{eval}^{(l)}(\mathbb{A}_i) \in P^{(l)}. \tag{4}$$

Let $\lambda_{\mathbb{P},S}$ be the coefficient of e_S in $\mathrm{eval}^{(l)}(\mathbb{P})$. So, we can write

$$\mathrm{eval}^{(l)}(\mathbb{P}) = \sum_{S \in \mathbf{r}} \lambda_{\mathbb{P},S} e_S \in P^{(l)}. \tag{5}$$

Let $I_2^{(l)}(\mathcal{P})$ be the two-sided ideal of $\mathbb{Z}M$ generated by the elements $\lambda_{\mathbb{P},S}$, where $\mathbb{P}$ is a spherical monoid picture and $S \in \mathbf{r}$. Then this ideal is called the *second Fox ideal* of $\mathcal{P}$. More specifically, for a trivializer $\mathbf{Y}$ of $\mathcal{D}(\mathcal{P})$, the set $I_2^{(l)}(\mathcal{P})$ is generated (as two-sided ideal) by the elements $\lambda_{\mathbb{P},S}$, where $\mathbb{P} \in \mathbf{Y}$ and $S \in \mathbf{r}$. We note that all this above material given with the consideration 'left' can also be applied to 'right' for a monoid M.

In Section 3, the generating functions will be connected to the $\sum_{i=1}^{n} \varepsilon \overline{U_i}$ part in (4) or, equivalently, to the $\lambda_{\mathbb{P},S}$ in (5).

2 The group case $\mathbb{Z}_n \rtimes_{\theta_1} \mathbb{Z}$

Let us consider the split extension $G = \mathbb{Z}_n \rtimes_{\theta_1} \mathbb{Z}$, where $\mathbb{Z} = \langle b \rangle$ is the a group with rank one, $\mathbb{Z}_n = \langle a \rangle$ is a cyclic group of order n and $\theta_1 : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}_n)$ is a homomorphism. Then, by (2), G has the presentation

$$\mathcal{P}_G = \langle a, b; a^n, aba^{-k}b^{-1} \rangle, \tag{6}$$

where $k \in \mathbb{Z}^+$, $\gcd(k,n) = 1$ and $k < n$. In [24, Theorem 3.2.1], the generating set of the second homotopy module $\pi_2(\mathcal{P}_G)$ has been constructed as drawn in Figure 1. In this generating set, there are two spherical pictures $\mathbb{P}_1$ and $\mathbb{P}_2$. In $\mathbb{P}_1$, we have two a^n-discs (one of them is positive and the other is negative), and in $\mathbb{P}_2$ we have a negative a^n-disc and k-times positive a^n-discs. Furthermore, again in $\mathbb{P}_2$, there is a total of n-times $aba^{-k}b^{-1}$-discs. Then, by considering the number of discs in these pictures, Baik [24, Theorem 3.3.3] proved the following result.

Proposition 1 *The presentation $\mathcal{P}_G$ in* (6) *is efficient* (*equivalently, p-Cockcroft for any prime p*) *if and only if* $\gcd(k-1,n) \neq 1$.

Therefore, if we suppose $\gcd(k-1,n) = 1$, then we obtain an inefficient presentation. Clearly, n must be an odd prime and the $\mathcal{P}_G$ given in (6) be an inefficient presentation. Otherwise, by setting $n = 2$ in this inefficient case, we obtain the direct product $\mathbb{Z}_n \times \mathbb{Z}$ which is a special case of the semidirect products and will not be considered in this paper. By Remark 1, it is always true that efficient presentations (even for groups or monoids) are minimal. But to check the minimality of a presentation while it is inefficient is important, because in this case we obtain the inefficiency of the related group that has this presentation (see [8–10]). For the group case, this important subject is investigated by the following 'minimality test' due to Lustig [23].

Lemma 1 ([23]) *For any group G with a presentation $\mathcal{P}$ as in* (1), *suppose there is a ring homomorphism ψ from $\mathbb{Z}G$ into the matrix ring of all $m \times m$-matrices* ($m \geq 1$) *over some commutative ring $\mathcal{R}$ with* 1. *Suppose also that $\psi(1) = I_{m\times m}$. If ψ maps the second Fox ideal $I_2(\mathcal{P})$ to* 0 (*in other words, if $I_2(\mathcal{P})$ is contained in the kernel of ψ*), *then $\mathcal{P}$ is minimal.*

By considering Proposition 1, the first main result of this paper is presented as follows.

Theorem 1 *Let us consider the presentation $\mathcal{P}_G$ as in* (6) *for the group $G = \mathbb{Z}_n \rtimes_{\theta_1} \mathbb{Z}$, where $k < n$ and* $\gcd(k,n) = 1$ *but* $\gcd(k-1,n) \neq 1$. *Then $\mathcal{P}_G$ has a set of generating functions*

$$p_1(a) = a - 1, \qquad p_2(b) = kb - 1, \qquad p_3(a) = \phi_n(a),$$

where ϕ_n denotes the nth cyclotomic polynomial over $\mathbb{Q}$ defined by

$$\phi_n(x) = \frac{x^n - 1}{x - 1} \tag{7}$$

having a degree $n - 1$.

Proof We first note that since $\mathcal{P}_G$ is presented as in (6), the action in $\mathrm{Aut}(\mathbb{Z}_n)$ is defined by $a \xrightarrow{\theta_1(b)} \theta_1(b) = a^k$, where $hcf(k,n) = 1$ and $k < n$.

Assume that $hcf(k-1,n) \neq 1$. Then, by Proposition 1, $\mathcal{P}_G$ is an efficient presentation and so, by Remark 1, is minimal (*i.e.* has a minimal number of generators). Let us consider the pictures $\mathbb{P}_1$ and $\mathbb{P}_2$ in Figure 1. Now, by (3), we have

$$\begin{aligned}
\mu(\mathbb{P}_1) &= (\overline{a} - \overline{1})e_{a^n} \quad \text{and} \\
\mu(\mathbb{P}_2) &= (-\overline{1} + \overline{b} + \overline{b} + \cdots + \overline{b})e_{a^n} + \big(\overline{1} + \overline{a} + \overline{a^2} + \cdots + \overline{a^{n-1}}\big)e_{aba^{-k}b^{-1}} \\
&= (-\overline{1} + k\overline{b})e_{a^n} + \big(\overline{1} + \overline{a} + \overline{a^2} + \cdots + \overline{a^{n-1}}\big)e_{aba^{-k}b^{-1}},
\end{aligned}$$

where $hcf(k,n) = 1$, but $hcf(k-1,n) \neq 1$ and $k < n$. For simplicity, by omitting the overlines on the elements in the above equalities, we obtain that the second Fox ideal is generated by the polynomial elements $a-1$, $kb-1$ and $1+a+a^2+\cdots+a^{n-1}$. Now, we can reformulate these polynomial elements as generating functions. It is clear that $p_1(a)$ has the root $a = 1$. On the other hand, since we have

$$\begin{aligned}
p_2(b) = kb - 1 \equiv 0 (\operatorname{mod} n) \quad &\Rightarrow \quad t(kb-1) \equiv 0 (\operatorname{mod} n) \\
&\Rightarrow \quad b - t \equiv 0 (\operatorname{mod} n) \\
&\Rightarrow \quad b \equiv t (\operatorname{mod} n),
\end{aligned}$$

$p_2(b)$ has a root t, where t is the multiplicative inverse of k.

Finally, $p_3(a) = 1 + a + a^2 + \cdots + a^{n-1} \equiv 0 (\operatorname{mod} n)$ has a root $a = 1$ modulo n which gives (7) directly. □

Let us take n as an odd prime p. Then, by Proposition 1 and Lemma 1, we get an inefficient but minimal presentation. Thus we have the following corollary.

Corollary 1 *For an odd prime p and a positive integer $k < p$, the presentation $\mathcal{P}_G$ in* (6) *has a set of generating functions*

$$p_1(a) = a - 1, \qquad p_2(b) = kb - 1, \qquad p_3(a) = \phi_p(a) = \frac{x^p - 1}{x - 1},$$

where ϕ_p has a degree of an even number $p - 1$.

Remark 3 Theorem 1 and Corollary 1 imply that by choosing the efficient or inefficient minimal presentations, we can get different constants (*i.e.* the cases of k in both results) and different powers (*i.e.* n to be a positive integer or an odd prime) in the set of generating functions. Therefore, the structure of the presentation (*i.e.* efficient or inefficient) affects getting different types of generating functions.

The following consequence of Theorem 1 points out another connection between the presentation in (6) as defined in either Theorem 1 or Corollary 1 and generating functions.

Corollary 2 *The polynomial $p_3(a)$ in Theorem* 1 (*or Corollary* 1) *is actually a 'locally constant function'.*

Proof We recall that the family of locally constant functions [14] is defined as

$$f(x) = \zeta^x \quad \text{and} \quad \zeta^n = 1 \quad (n \in \mathbb{Z})$$

for which $f'(x) = 0$ holds. Moreover, in the meaning of group homomorphisms, each function in this family satisfies

$$f : (\mathbb{R}, +) \longrightarrow (\mathbb{C}, \cdot), \quad f(x+y) = f(x) \cdot f(y).$$

Now, by replacing ζ^x with

$$p_3(\zeta) = \frac{\zeta^n - 1}{\zeta - 1}, \tag{8}$$

it is clear that we get a locally constant function, as required. □

After Theorem 1, Corollary 1 and Corollary 2, we can express the following connection between the generating functions and (twisted) Bernoulli numbers.

Remark 4 The locally constant function corresponding to the generating function $p_3(a)$ of the presentation $\mathcal{P}_G$ given in (6) is related to the twisted Bernoulli numbers and polynomials. (We may refer the reader, for example, to [3, 14, 25, 26] for the twisted Bernoulli numbers and polynomials.) In the next paragraph, we give a brief description.

According to [4, 13, 14], for each integer $N \geq 0$, C_{p^N} denotes the multiplicative group of the primitive p^Nth roots of unity in $\mathbb{C}_p^* = \mathbb{C}_p - \{0\}$. Let

$$\mathbb{T}_p = \{\xi \in \mathbb{C}_p : \xi^{p^N} = 1, \text{ for } N \geq 0\} = \bigcup_{N \geq 0} C_{p^N} = \lim_{N \to \infty} C_{p^N}.$$

The dual of $\mathbb{Z}_p$ in the sense of p-adic Pontryagin duality is $\mathbb{T}_p = C_{p^\infty}$, the direct limit (under inclusion) of cyclic groups C_{p^N} of order p^N with $N \geq 0$, with discrete topology. The $\mathbb{T}_p$ admits a natural $\mathbb{Z}_p$-module structure which is written as ξ^x for $\xi \in \mathbb{T}_p$ and $x \in \mathbb{Z}_p$. Moreover, $\mathbb{T}_p$ can be embedded discretely in $\mathbb{C}_p$ as the multiplicative p-torsion subgroup. If $\xi \in \mathbb{T}_p$, then $\omega : (\mathbb{Z}_p, +) \longrightarrow (\mathbb{C}_p, \cdot)$, $x \longmapsto \xi^x$, is a locally constant character which is actually a locally analytic character if $\xi \in \{\xi \in \mathbb{C}_p : v_p(\xi - 1) > 0\}$. Then, by [4, 13, 14, 27, 28], ω_ξ has a continuation to a continuous group homomorphism from $(\mathbb{Z}_p, +)$ to $(\mathbb{C}_p, \cdot)$. We further remind that if $\xi \in \mathbb{C}$, then ξ will be assumed to have an rth root of unity with $r \in \mathbb{Z}^+$.

3 The monoid case $\mathbb{Z}^2 \rtimes_{\theta_2} \mathbb{Z}$

Before presenting this special case, let us first discuss a more general situation for the p-Cockcroft property of semidirect products of monoids. In [8, 9], by considering a similar version of the picture $\mathbb{P}_{S,x}$ in Figure 2, the second author investigated the p-Cockcroft property by using the trivializer for the semidirect product $M = K \rtimes_{\theta_2} A$, where K and A are arbitrary monoids. (It is seen that there is a single non-spherical subpicture $\mathbb{B}_{S,x}$ in $\mathbb{P}_{S,x}$. In fact, $\mathbb{B}_{S,x}$ contains only S-discs. For an illustration, see Figure 3.) As a special case of it, let us assume that K is a one-relator monoid and A is an infinite cyclic monoid $\mathbb{Z}$ with presentations

$$\mathcal{P}_K = \langle \mathbf{y}; S_+ = S_- \rangle \quad \text{and} \quad \mathcal{P}_A = \langle x; : \rangle,$$

respectively. Suppose ψ is an endomorphism of K. Then the mapping $x \mapsto \psi$ induces a homomorphism $\theta_2 : A \to \mathrm{End}(K)$, and we can form the semidirect product $M = K \rtimes_\theta A$. By (2), this product has a presentation

$$\mathcal{P}_M = \langle \mathbf{y}, x; S_+ = S_-, \mathbf{t} \rangle, \tag{9}$$

where, for all $y \in \mathbf{y}$, the set $\mathbf{t}$ is the set of relators

$$T_{yx} : yx = x(y\theta_x)$$

such that the relator S satisfies the condition $\iota(S_+) \neq \iota(S_-)$ (or $\tau(S_+) \neq \tau(S_-)$). In [8], the necessary and sufficient conditions for $\mathcal{P}_M$ to be efficient are determined.

In the special case above, let us take K as a free abelian monoid of rank two (*i.e.* $K = \mathbb{Z}^2$) presented by $\mathcal{P}_K = \langle y_1, y_2; y_1y_2 = y_2y_1 \rangle$, and let ψ be the endomorphism $\psi_{\mathbf{M}}$, where $\mathbf{M}$ is the matrix $\left[\begin{smallmatrix} \alpha & \alpha' \\ \beta & \beta' \end{smallmatrix}\right]$ ($\alpha, \alpha', \beta, \beta' \in \mathbb{Z}^+$) given by $[y_1] \longmapsto [y_1^{\alpha} y_2^{\alpha'}]$ and $[y_2] \longmapsto [y_1^{\beta} y_2^{\beta'}]$. As a special case of the presentation in (9), we obtain

$$\mathcal{P}_M = \left\langle y_1, y_2, x; y_1y_2 = y_2y_1, y_1x = xy_1^{\alpha} y_2^{\alpha'}, y_2x = xy_1^{\beta} y_2^{\beta'} \right\rangle \tag{10}$$

for the monoid $M = \mathbb{Z}^2 \rtimes_{\theta_2} \mathbb{Z}$ (see [9]). Again, in the same reference, the second author figured out the efficiency of the above presentation as in the following proposition.

Proposition 2 ([9]) *For any prime p, the presentation $\mathcal{P}_M$ in* (10) *is p-Cockcroft if and only if* $\det \mathbf{M} \equiv 1 (\mathrm{mod}\, p)$.

According to Proposition 2, in particular, $\mathcal{P}_M$ is not efficient if $\det \mathbf{M} = 0$ or 2. Therefore the following proposition is proved in the same manner.

Proposition 3 ([9]) *The presentation $\mathcal{P}_M$ in* (10) *is minimal but inefficient if* $\det \mathbf{M} = 2$.

The proof of Proposition 3 is based on the following Pride result, which is a monoid version of Lemma 1. Although this result has not been published yet, it has been used in many papers (see, for instance, [8–10]).

Lemma 2 (Pride) *For any monoid M with a presentation $\mathcal{P}$ as in* (1), *let ψ be a ring homomorphism from $\mathbb{Z}M$ into the ring of all $m \times m$-matrices* ($m \geq 1$) *over some commutative ring $\mathcal{R}$ with* 1, *and suppose $\psi(1) = I_{m\times m}$. If the second Fox ideal $I_2^{(l)}(\mathcal{P})$ is contained in the kernel of ψ, then $\mathcal{P}$ is minimal.*

From now on, by considering Propositions 2 and 3, we will reach our main aim of this paper for monoids.

Our first result in this section gives the connection between a monoid presentation and array polynomials. In fact the *array polynomials* $S_k^n(x)$ are defined by means of the following generating function:

$$\frac{(e^t - 1)^k e^{tx}}{x!} = \sum_{n=0}^{\infty} S_k^n(x) \frac{t^n}{n!},$$

(*cf.* [29–31]). According to the same references, array polynomials can also be defined in the form

$$S_k^n(x) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(x+j)^n. \tag{11}$$

Since the coefficients of array polynomials are integers, these polynomials find a very large application area, especially in engineering. Array polynomials are used, for instance, in system control (*cf.* [32]).

In fact these integer coefficients give us an opportunity to use these polynomials in our case. We should note that there also exist some other polynomials, namely Dickson, Bell, Abel, Mittag-Leffler *etc.*, which have integer coefficients. But, since array polynomials have a larger application area in science, we have preferred them. Hence, by considering Proposition 3, we obtain the following theorem as another main result.

Theorem 2 *Let us consider the monoid* $M = \mathbb{Z}^2 \rtimes_{\theta_2} \mathbb{Z}$ *with a presentation*

$$\mathcal{P}_M = \langle b_1, b_2, a; b_1b_2 = b_2b_1, b_1a = ab_1^2, b_2a = ab_1b_2\rangle. \tag{12}$$

Then $\mathcal{P}_M$ *has a set of generating functions*

$$\left.\begin{aligned} p_1(a) &= S_n^n(a) - 2S_0^1(a) \\ p_2(b_1) &= S_n^n(b_1) - S_0^1(b_1) \\ p_3(b_2) &= S_0^1(b_2) - S_n^n(b_2) \end{aligned}\right\}, \tag{13}$$

where $S_k^n(x)$ *is defined as in* (11).

Proof Let us consider the spherical picture $\mathbb{P}_{S,a}$ with its non-spherical subpicture $\mathbb{B}_{S,a}$ as drawn in Figure 3. In fact, by [9], this is the only picture in the trivializer of $\mathcal{D}(\mathcal{P}_M)$.

In presentation (12), let us label the relators $b_1b_2 = b_2b_1$, $b_1a = ab_1^2$ and $b_2a = ab_1b_2$ by S, $T_{b_1,a}$ and $T_{b_2,a}$, respectively. It is clear that $\exp_S(\mathbb{P}_{S,a}) = 1 - 2 = -1$, $\exp_{T_{b_1,a}}(\mathbb{P}_{S,a}) = 1 - 1 = 0$ and $\exp_{T_{b_2,a}}(\mathbb{P}_{S,a}) = 1 - 1 = 0$. In the calculation of these exponent sums, we included the exponent sums of S-discs in the non-spherical picture $\mathbb{B}_{S,a}$. Actually, a simple calculation shows that $\det \mathbf{M} = \exp_S(\mathbb{B}_{S,a})$ and so, by our assumption about $\mathcal{P}_M$ that is not efficient, we expect $\exp_S(\mathbb{B}_{S,a})$ to be 2.

Now, by (4) and (5), the evaluation of $\mathbb{P}_{S,a}$ is determined as follows:

$$\operatorname{eval}^{(l)}(\mathbb{P}_{S,a}) = (1 - \overline{2a})e_S + (1 - \overline{b_1})e_{T_{b_1,a}} + (\overline{b_2} - 1)e_{T_{b_2,a}}.$$

Therefore, by the definition, the second Fox ideal $I_2^{(l)}(\mathcal{P}_M)$ of the presentation $\mathcal{P}_M$ in (12) is generated by the polynomial elements

$$1 - \overline{2a}, \qquad 1 - \overline{b_1}, \qquad \overline{b_2} - 1.$$

For simplicity, let us replace each of $\overline{2a}$, $\overline{b_1}$ and $\overline{b_2}$ by $2a$, b_1 and b_2, respectively. In [9], by considering Lemma 2, it has been showed that this presentation in (12) is minimal.

Now, by using (11) and keeping in our mind that the coefficients of array polynomials are integer, we clearly have

$$S_k^n(x) = \begin{cases} x^n; & k = 0, \\ x; & k = 0 \text{ and } n = 1, \\ 1; & k = n \text{ or } n = k = 0. \end{cases}$$

Then, by reformulating the elements of the second Fox ideal $I_2^{(l)}(\mathcal{P}_M)$, we arrive at the functions in (13) as desired. □

By considering Proposition 2, if we take $\det \mathbf{M} \neq 2$, then we get an efficient presentation. So, for an even prime p, let $\det \mathbf{M} = 3$. Then one of the presentations of the similar form $\mathcal{P}_M$ as in (12) can be taken as

$$\mathcal{P}_M = \langle b_1, b_2, a; b_1b_2 = b_2b_1, b_1a = ab_1^3, b_2a = ab_1b_2 \rangle, \tag{14}$$

which will be efficient. The same procedure in the proof of Theorem 2 gives us the set of generating functions of $\mathcal{P}_M$ in (14) in the form $p_1(a)$, $p_2(b_1)$ and $p_3(b_2)$, where $p_1(a) = S_n^n(a) - 3S_0^1(a)$ and the others are defined in (13) such that $S_k^n(x)$ is given in (11). Nevertheless, by induction steps, we can generalise this last presentation as follows:

$$\mathcal{P}_M = \langle b_1, b_2, a; b_1b_2 = b_2b_1, b_1a = ab_1^{\det \mathbf{M}}, b_2a = ab_1b_2 \rangle. \tag{15}$$

Hence we get the following version of Theorem 2 which deals with efficient presentations.

Theorem 3 *Let us consider the presentation $\mathcal{P}_M$ in* (15) *for the monoid $M = \mathbb{Z}^2 \rtimes_{\theta_2} \mathbb{Z}$. Then $\mathcal{P}_M$ has a set of generating functions*

$$\left.\begin{aligned} p_1(a) &= S_n^n(a) - \det \mathbf{M} S_0^1(a) \\ p_2(b_1) &= S_n^n(b_1) - S_0^1(b_1) \\ p_3(b_2) &= S_0^1(b_2) - S_n^n(b_2) \end{aligned}\right\}, \tag{16}$$

where $\det \mathbf{M} \neq 2$ *and $S_k^n(x)$ is defined as in* (11).

Remark 5 According to the expression in Remark 1, presentations given in (12), (14) or (15) have a minimal number of generators. But we classified these presentations according to their efficiency status separately in Theorem 2 and Theorem 3. The aim of this separation is to find a solution for a general remark depicted in the final section about obtaining a method for a minimality test by using generating functions (see Section 4 below).

At this point, we should note that for $t_1 \neq t_2 \in \mathbb{R}^+$, $\lambda \in \mathbb{C}$, $k \in \mathbb{N}_0$, generalised array type polynomials $\mathcal{S}_k^n(x; t_1, t_2; \lambda)$ which are related to the non-negative real parameters have been recently developed and some elementary properties including recurrence relations of these polynomials have been obtained [30]. In fact, by setting $t_1 = \lambda = 1$ and $t_2 = e$, the equation (11) is obtained.

Remark 6 One can try to study the generalisation of Theorem 2 by using $\mathcal{S}_k^n(x;t_1,t_2;\lambda)$.

The remaining goal of this section is to establish a connection between the presentation $\mathcal{P}_M$ in (12) or (15) and Stirling numbers of the second kind (*cf.* [3, 30, 33–36]). In fact, *Stirling numbers* of the second kind $S(n,k)$ are defined by means of the following generating function:

$$\frac{(e^t-1)^k}{k!}=\sum_{n=0}^{\infty}S(n,k)\frac{t^n}{n!}$$

(see [3, 36]). According to [30, Theorem 1, Remark 2], Stirling numbers can also be defined by

$$S(n,k)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n.$$

We remind that these numbers satisfy the well-known properties

$$S(n,k)=\begin{cases}1; & k=1 \text{ or } k=n,\\ \binom{n}{2}; & k=n-1,\\ \delta_{n,0}; & k=0,\end{cases}$$

where $\delta_{n,0}$ denotes the Kronecker symbol (see [3, 36]). It is known that Stirling numbers are used in combinatorics, in number theory, in discrete probability distributions for finding higher-order moments, *etc.* We finally note that since $S(n,k)$ is the number of ways to partition a set of n objects into k groups, these numbers find an application area in combinatorics and in the theory of partitions.

In addition to the above formulas for $S(n,k)$, by [30, 35, 36], we have

$$x^n=\sum_{k=0}^{n}\binom{x}{k}k!S(n,k) \tag{17}$$

as a formula for Stirling numbers. Therefore, in equation (17) by replacing x with a, b_1 and b_2, respectively, and taking $n=1$, $n=0$, the polynomial elements of the second Fox ideal $I_2^{(l)}(\mathcal{P}_M)$ of the presentation $\mathcal{P}_M$ in (12) can be restated as follows:

$$\left.\begin{aligned} a^0-2a^1&=\sum_{k=0}^{0}\binom{a}{k}k!S(0,k)-2\sum_{k=0}^{1}\binom{a}{k}k!S(1,k)\\ b_1^0-b_1^1&=\sum_{k=0}^{0}\binom{b_1}{k}k!S(0,k)-\sum_{k=0}^{1}\binom{b_1}{k}k!S(1,k)\\ b_2^1-b_2^0&=\sum_{k=0}^{1}\binom{b_2}{k}k!S(1,k)-\sum_{k=0}^{0}\binom{b_2}{k}k!S(0,k)\end{aligned}\right\}. \tag{18}$$

As a different version of Theorem 2, we express the following corollary.

Corollary 3 *The presentation* $\mathcal{P}_M$ *in* (12) *has a set of generating functions in terms of Stirling numbers as given in* (18).

We note that the above corollary can also be stated for the presentation $\mathcal{P}_M$ in (15).

Furthermore, in a recent work, Simsek [30] has constructed the *generalised λ-Stirling numbers of the second kind* $\mathcal{S}(n,v;a,b;\lambda)$ related to non-negative real parameters ($a,b \in \mathbb{R}^+$, $a \neq b$, λ is a complex number and $v \in \mathbb{N}_0$). In fact, this new generalisation is defined by the generating function

$$f_{S,v}(t;a,b;\lambda) = \frac{(\lambda b^t - a^t)^v}{v!} = \sum_{n=0}^{\infty} \mathcal{S}(n,v;a,b;\lambda)\frac{t^n}{n!}. \tag{19}$$

By setting $a = 1$ and $b = e$ in (19), one can obtain the λ-Stirling numbers of the second kind $S(n,v;\lambda)$ which are defined by the generating function

$$\frac{(\lambda e^t - 1)^v}{v!} = \sum_{n=0}^{\infty} S(n,v;\lambda)\frac{t^n}{n!}$$

(see [3, 36]). By substituting $\lambda = 1$ into the above equation, the Stirling numbers of the second kind $S(n,v)$are obtained.

By considering this new generalisation $\mathcal{S}(n,v;a,b;\lambda)$, in [30, Theorem 1], it has been obtained that

$$\mathcal{S}(n,v;a,b;\lambda) = \frac{1}{v!}\sum_{j=0}^{n}(-1)^j\binom{v}{j}\lambda^{v-j}\bigl(j\ln a + (v-j)\ln b\bigr)^n, \tag{20}$$

for λ-Stirling numbers of the second kind. In fact, by setting $a = 1$ and $b = e$ in (20), one can get the following equality on λ-Stirling numbers:

$$S(n,v;\lambda) = \frac{1}{v!}\sum_{j=0}^{v}\binom{v}{j}\lambda^{(v-j)}(-1)^j(v-j)^n \tag{21}$$

(see [3, 36]).

Hence we can present the following notes about this section:

Remark 7 It is clearly seen that in Theorems 2, 3 and Corollary 3, only Stirling numbers are considered. However, one can also study the λ-Stirling numbers $S(n,v;\lambda)$ defined in (21) and generalised λ-Stirling numbers $\mathcal{S}(n,v;a,b;\lambda)$ defined in (20) as stated in these theorems and corollaries.

Remark 8 For a suitable $\mathbf{M}_{n\times n}$ matrix, it is possible to define the presentation $\mathcal{P}_M$ in (9) (or in (10)) for the monoid $\mathbb{Z}^n \rtimes_{\theta_2} \mathbb{Z}$. Thus one can try to transform all studies in Section 3 to this general case.

4 Final remarks

In this section we will express some other remarks depicted in the previous sections. We hope that the following material will be used as new study areas:

- The first general note would be as follows. The studies here can be thought of as the initial step of a general idea, namely constructing a new method (or a test) for the minimality (while the inefficiency holds) of group (in Section 2) and monoid (in

Section 3) presentations other than the methods presented in Lemma 1 and Lemma 2, respectively. Especially for the monoid case, although Lemma 2 has not been published, the theory of it has been used widely in last ten years. Therefore, by using generating functions, to obtain a new test on the minimality of monoids would be an interesting and important result.

- As we noted in Remark 1, to study with the minimal presentations has an advantage for our aim in this paper. Conversely, the use of generating functions to obtain a presentation with a minimal number of generators is still an open question.
- Until now, any result to check whether a *semigroup* presentation is minimal while it is inefficient has not been published. Therefore the whole idea of this paper can also be used for this case.
- It is known that the *chemical energy* is one of most important application areas of graph theory (*cf.* [37]). So, as a next step of the expressions in Remark 2, it is worth studying whether this chemical energy can also be obtained from pictures.
- We believe that the same approximation between presentations and generating functions as done in this paper can also be applied to some other special cases of groups and monoids other than $\mathbb{Z}_n \rtimes_{\theta_1} \mathbb{Z}$ and $\mathbb{Z}^2 \rtimes_{\theta_2} \mathbb{Z}$. Moreover, one can investigate which type of polynomials (other than depicted in here) can be used for the general case.
- Here we used exponent sums of pictures as a method to obtain constants of functions. What other methods other than this geometric way can be used could be studied.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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