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Nonlinear self adjointness, conservation laws and exact solutions of ill-posed Boussinesq equation

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Abstract: In this work, we consider the ill-posed Boussinesq equation which arises in shallow water waves and non-linear lattices. We prove that the ill-posed Boussinesq equation is nonlinearly self-adjoint. Using this property and Lie point symmetries, we construct conservation laws for the underlying equation. In addition, the generalized solitonary, periodic and compact-like solutions are constructed by the exp-function method.

Keywords: ill posed Boussinesq equation; conservation laws; nonlinear self-adjointness; exp-function method; exact solutions

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1 Introduction

The nonlinear evolution equations (NLEEs) are extensively used as models to describe physical phenomena in various disciplines of the sciences, especially in fluid mechanics, solid state physics, plasma physics, plasma waves and chemical physics. When a NLEE is analysed, one of the most important question is the construction of the exact solutions for equation [1]. In the open literature, quite a few methods for obtaining explicit travelling and solitary wave solutions to NLEEs have been suggested such as the inverse scattering method [2], the bilinear transformation method [3], the tanh–sech method [4, 5], the extended tanh method [6, 7], the sine–cosine method [8–10], the homogeneous balance method [11, 12], the pseudo spectral method [13], the (G'/G)-expansion method [14–16], exp-function method [17], variational iteration method [18], homotopy perturbation method [19], the Jacobi elliptic function method [20], Lie group analysis method [21] and so on.

It is well known that, conservation laws are very important tools in the study of differential equations from a mathematical as well as a physical point of view. A variety of powerful methods, such as Noether's method [22], the multiplier approach [21], [23–25], symmetry conditions method on the conserved quantities [26], partial Lagrangian method [27, 28], nonlocal conservation method [29–31] have been used to investigate conservation laws of PDEs.

The well known and celebrated Korteweg-de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

was derived by Korteweg and de Vries in 1895, and which described weakly nonlinear shallow water waves.

The ill posed Boussinesq (sometimes also called as bad Boussinesq) equation

$$u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxxx} \tag{2}$$

was described in 1870 by the French scientist J. Boussinesq, for the propagation of long waves on the surface of water with a small amplitude in one-dimensional nonlinear lattices and in non-linear strings [33–35]. The well posed Boussinesq equation was also described in this context. It differs only in the sign of the last dispersive term of the Equation (2). The Equation (2) is used to describe two-dimensional flow of shallow-water waves having small amplitudes [36]. In the weakly nonlinear limit, the shallow water wave equation for long waves reduces to the KdV equation. The main difference between the KdV equation and Boussinesq equations are the shape of the waves. The Boussinesq equations allows bidirectional waves while KdV only unidirectional waves.

Very recently, the analytical and numerical solutions of the ill posed Boussinesq equation were examined intensively in the literature. In [36], the authors studied the explicit homoclinic orbits solutions for Equation (2) with periodic boundary condition and even constraint. In [37], Jafari *et al.* obtained the solitary wave solutions of Equation (2) by sine-cosine and extended tanh func-

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tion method. In [35] and [38], the authors used the theory of Lie groups and obtained the symmetry reductions and group invariant traveling wave solutions. In [39], filtering and regularization techniques were applied for obtaining the approximate solutions and to control growth of the errors. Gomes and Valls in [40] shows that the dynamics in the centre manifold of the ill-posed equation tracks the dynamics of the well-posed equation. Their results give partial justification to the long-wave perturbation theory. There exist also some literatures around the numerical and analytical studies for the singulaly perturbed Boussinesq equation [41, 42]. Meanwhile, we observe some important studies on the local fractional Boussineq equations (see, [43] and also [44]).

In the present study, we first intended to study the exact traveling wave solutions including periodic, solitonary and compact-like solutions of Equation (2). For this aim, we implemented the exp-function method which was developed by He and Wu [17]. Then we investigated nonlinear self-adjointness and local conservation laws of Equation (2) by Ibragimov's nonlocal conservation method.

The plan of the paper is organized as follows : In Section 2, we give briefly the description of the exp-function method.Then, we apply this method to Equation (2). Section 3 is devoted to the nonlinear self adjointness, multiplier functions and conservation laws of Equation (2). In Section **??**, some concluding remarks are given.

2 Nonlinear self-adjointness and conservation laws for Equation (2)

We briefly present notation to be used and recall basic definitons and theorems that appear in [29, 30] (see also [25]).

Consider the k^{th} order system of PDEs of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$

$$E^{\alpha}\left(x, u, u_{(1)}, \ldots, u_{(k)}\right) = 0, \quad \alpha = 1, \ldots, m, \tag{3}$$

where $u_{(i)}$ is the collection of ith-order partial derivatives and the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^{\alpha}\frac{\partial}{\partial u^{\alpha}} + u_{ij}^{\alpha}\frac{\partial}{\partial u_j^{\alpha}} + \ldots, \quad i = 1, \ldots, n \tag{4}$$

in which the summation convention is used. The adjoint equations to Equation (3) are given by

$$E^{\alpha^{\star}}(x, u, w, u_{(1)}, w_{(1)}, \ldots, u_{(k)}, w_{(k)}) = 0, \qquad \alpha = 1, \ldots, m \tag{5}$$

with

$$E^{\alpha^{\star}}(x, u, w, u_{(1)}, w_{(1)}, \ldots, u_{(k)}, w_{(k)}) = \frac{\delta L}{\delta u^{\alpha}}, \tag{6}$$

where L is the formal Lagrangian for Equation (3) defined by

$$L = w^{\alpha}E^{\alpha} \equiv \sum_{\alpha=1}^{m} w^{\alpha}E^{\alpha} \tag{7}$$

Here $w = (w^1, \ldots, w^m)$ are adjoint variables and $w_{(1)}, \ldots, w_{(k)}$ are their derivatives. Here $\frac{\delta}{\delta u}$ is the Euler-Lagrange operator and defined by

$$\frac{\delta}{\delta u^{\alpha}} = \frac{\partial}{\partial u^{\alpha}} + \sum_{k\geq 1}^{\infty}(-1)^k D_{i_1}\ldots D_{i_k}\frac{\partial}{\partial u^{\alpha}_{i_1\ldots i_k}}, \quad \alpha = 1, \ldots, m. \tag{8}$$

so that

$$\frac{\delta L}{\delta u^{\alpha}} = \frac{\delta(w^{\alpha}E^{\alpha})}{\delta u^{\alpha}} = \frac{\partial(w^{\alpha}E^{\alpha})}{\partial u^{\alpha}} - D_i\left(\frac{\partial(w^{\alpha}E^{\alpha})}{\partial u_i^{\alpha}}\right) + D_iD_k\left(\frac{\partial(w^{\alpha}E^{\alpha})}{\partial u_{ik}^{\alpha}}\right) - \ldots$$

Definition 1. *[30] Equation (3) is said to be strictly self-adjoint if the adjoint Equation (5) becomes equivalent to the original Equation (3) by the substitution $w = u$.*

Definition 2. *[30] Equation (3) is said to be nonlinearly self-adjoint if its adjoint equation (5) becomes equivalent to the original equation after the substitution*

$$w = \phi \tag{9}$$

where ϕ is a nonzero function depending on the independent variables, the dependent variable as well as the partial derivatives of the dependent variable. In other words the following identities holding for undetermined coefficients λ_{α}^{β},

$$E^{\alpha^{\star}}(x, u, w, u_{(1)}, w_{(1)}, \ldots, u_{(k)}, w_{(k)}) = \lambda_{\alpha}^{\beta}E_{\beta}(x, u, u_{(1)}, \ldots, u_{(s)}), \alpha, \beta = 1, \ldots, m \tag{10}$$

which will be applicable in the computations.

Theorem 3. *[29] Every Lie point, Lie-Bäcklund and non-local symmetry*

$$X = \xi^i(x, u, u_{(1)}, \ldots)\partial_{x^i} + \eta^{\alpha}(x, u, u_{(1)}, \ldots)\partial_{u^{\alpha}} \tag{11}$$

of Equation (3) leads to a conservation law $D_i(T^i) = 0$ constructed by the formula

$$\begin{aligned} T^i &= \xi^i L \\ &+ W^\alpha \left[\frac{\partial L}{\partial u_i} - D_j \left(\frac{\partial L}{\partial u_{ij}} \right) + D_j D_k \left(\frac{\partial L}{\partial u_{ijk}} \right) \right. \\ &- \left. D_j D_k D_m \left(\frac{\partial L}{\partial u_{ijkm}} \right) \right] \\ &+ D_j \left(W^\alpha \right) \left[\frac{\partial L}{\partial u_{ij}} - D_k (\frac{\partial L}{\partial u_{ijk}}) + D_k D_m \left(\frac{\partial L}{\partial u_{ijkm}} \right) \right] \\ &+ D_j D_k \left(W^\alpha \right) \left[\frac{\partial L}{\partial u_{ijk}} - D_m \left(\frac{\partial L}{\partial u_{ijkm}} \right) \right] \\ &+ D_j D_k D_m \left(W^\alpha \right) \left[\left(\frac{\partial L}{\partial u_{ijkm}} \right) \right], \end{aligned} \quad (12)$$

where $W^\alpha = \eta^\alpha - \xi^i u_i^\alpha$ and ξ^i, η^α are the coefficient functions of the associated generator (11).

Theorem 4. *The ill posed Boussinesq Equation (2) becomes nonlinearly self-adjoint if and only if there exists a differentiable function*

$$w = c_1 xt + c_2 x + c_3 t + c_4,$$

where c_i are arbitrary constants.

Proof. The formal Lagrangian for the ill posed Boussinesq Equation (2)

$$L = w(u_{tt} - u_{xx} - (u^2)_{xx} - u_{xxxx}), \quad (13)$$

where w is a new dependent variable. The adjoint Equation to (2) has the form

$$F^\star \equiv \frac{\delta L}{\delta u} = 0,$$

where the variational derivative of the Lagrangian in our case is

$$\begin{aligned} \frac{\delta L}{\delta u} &= \frac{\partial (wF)}{\partial u} - D_x \left(\frac{\partial (wF)}{\partial u_x} \right) + D_x^2 \left(\frac{\partial (wF)}{\partial u_{xx}} \right) \\ &+ D_t^2 \left(\frac{\partial (wF)}{\partial u_{tt}} \right) + D_x^4 \left(\frac{\partial (wF)}{\partial u_{xxxx}} \right) \end{aligned} \quad (14)$$

and the operators D_t and D_x denote the total derivatives in t, x. From the (14) equation we find the adjoint equation.

$$F^\star \equiv w_{tt} - (1 + 2u) w_{xx} - w_{xxxx} = 0. \quad (15)$$

If one substitutes u instead of w in Equation (15), Equation (2) is not recoverd. Consequently, Equation (2) is not strictly self-adjoint. According to Definition 2, Equation (2) is nonlinearly self-adjoint if the identity

$$F^\star \mid_{w=\phi(x,t,u)} = \lambda \left[u_{tt} - u_{xx} - (u^2)_{xx} - u_{xxxx} \right],$$

holds the following conditions and λ is a regular undetermined coefficient. The required derivatives of adjoint Equation (15),

$$w_x = \phi_x + \phi_u u_x, \quad w_t = \phi_t + \phi_u u_t,$$

$$w_{xx} = \phi_{xx} + 2\phi_{xu} u_x + \phi_{uu} u_x^2 + \phi_u u_{xx},$$

$$w_{tt} = \phi_{tt} + 2\phi_{tu} u_t + \phi_{uu} u_t^2 + \phi_u u_{tt},$$

$$\begin{aligned} w_{xxxx} &= 12 u_x u_{xx} \phi_{xuu} + 6 u_x^2 u_{xx} \phi_{uuu} + 4 u_x u_{xxx} \phi_{uu} \\ &+ \phi_{xxxx} + 4 u_x \phi_{xxxu} + 6 u_{xx} \phi_{xxu} + 4 u_{xxx} \phi_{xu} + 4 u_x^3 \phi_{xuuu} \\ &+ u_x^4 \phi_{uuuu} + u_{xxxx} \phi_u + 6 u_x^2 \phi_{xxuu} + 3 u_{xx}^2 \phi_{uu} \end{aligned}$$

then the condition (10) is written as follows:

$$\begin{aligned} &\phi_{tt} + 2\phi_{tu} u_t + \phi_{uu} u_t^2 + \phi_u u_{tt} \\ &-(1 + 2u)(\phi_{xx} + 2\phi_{xu} u_x + \phi_{uu} u_x^2 + \phi_u u_{xx}) - 12 u_x u_{xx} \phi_{xuu} \\ &-6 u_x^2 u_{xx} \phi_{uuu} - 4 u_x u_{xxx} \phi_{uu} - \phi_{xxxx} - 4 u_x \phi_{xxxu} \\ &-6 u_{xx} \phi_{xxu} - 4 u_{xxx} \phi_{xu} - u_{xxxx} \phi_u - 6 u_x^2 \phi_{xxuu} \\ &-4 u_x^3 \phi_{xuuu} - u_x^4 \phi_{uuuu} - 3 u_{xx}^2 \phi_{uu} \\ &= \lambda \left[u_{tt} - u_{xx} - (u^2)_{xx} - u_{xxxx} \right] \end{aligned}$$

Comparing the coefficients of derivatives u, we construct determining equations system and solving this system we obtain the adjoint variable as

$$w = c_1 xt + c_2 x + c_3 t + c_4, \quad (16)$$

with c_1, c_2, c_3, c_4 arbitrary constants. □

Taking into account the form of the substitution (16), we have a four parameter family of the substitution

$$\phi_1 = xt, \qquad \phi_2 = x, \qquad \phi_3 = t, \qquad \phi_4 = 1$$

which allows us to get local conservation laws.

We note that the Lie point symmetry generators of Equation (2)

$$X_1 = \frac{\partial}{\partial x}, \; X_2 = \frac{\partial}{\partial t}, \; X_3 = \frac{x}{2} \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} + \left(-\frac{1}{2} - u \right) \frac{\partial}{\partial u}$$

obtained in [35] and [38]. We now construct the corresponding local conserved vectors:

Case 5. *For symmetry operator $X = \frac{\partial}{\partial x}$, the components of the conserved vector $T = (T^t, T^x)$ are given by*

– *Substitution $\phi_1 = xt$:*

$$T^t = x u_x - xt u_{tx}, \quad T^x = xt u_{tt} - t u_x - 2 t u u_x - t u_{xxx}.$$

– *Substitution $\phi_2 = x$:*

$$T^t = -x u_{tx}, \quad T^x = x u_{tt} - u_x - 2 u_x u - u_{xxx}.$$

– *Substitution* $\phi_3 = t$:

$$T^t = u_x - tu_{tx}, \quad T^x = tu_{tt}.$$

– *Substitution* $\phi_4 = 1$:

$$T^t = -u_{tx}, \quad T^x = u_{tt}.$$

It is readily seen that using the divergence condition we obtain the null conserved vectors corresponding to the substitutions $\phi_3 = t$ *and* $\phi_4 = 1$.

Case 6. *For symmetry operator* $X = \frac{\partial}{\partial t}$, *the components of the conserved vector* $T = (T^t, T^x)$ *are given by*

– *Substitution* $\phi_1 = xt$:

$$\begin{aligned} T^t &= -xtu_{xx} - 2xtu_x^2 - 2txuu_{xx} - xtu_{xxxx}, \\ T^x &= 2xtu_xu_t - tu_t - 2tuu_t + xtu_{tx} + 2txuu_{tx} \\ &- tu_{xxt} + xtu_{xxxt}. \end{aligned}$$

– *Substitution* $\phi_2 = x$:

$$T^t = -xu_{xx} - 2xu_x^2 - 2xuu_{xx} - xu_{xxxx},$$

$$\begin{aligned} T^x &= 2xu_tu_x - u_t - 2uu_t + xu_{tx} + 2xuu_{tx} \\ &- u_{txx} + xu_{txxx}. \end{aligned}$$

– *Substitution* $\phi_3 = t$

$$\begin{aligned} T^t &= -tu_{xx} - 2tu_x^2 - 2tuu_{xx} - tu_{xxxx} + u_t, \\ T^x &= 2tu_xu_t + tu_{tx} + 2tuu_{tx} + tu_{txxx}. \end{aligned}$$

– *Substitution* $\phi_4 = 1$:

$$\begin{aligned} T^t &= -u_{xx} - 2u_x^2 - 2uu_{xx} - u_{xxxx}, \\ T^x &= 2u_tu_x + u_{tx} + 2u_{tx}u + u_{txxx}. \end{aligned}$$

It is readily seen that using the divergence condition we obtain the null conserved vectors corresponding to the substitutions $\phi_1 = xt$, $\phi_2 = x$ *and* $\phi_4 = 1$.

Case 7. *For symmetry operator* $X = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \left(-\frac{1}{2} - u\right)\frac{\partial}{\partial u}$, *the components of the conserved vector* $T = (T^t, T^x)$ *are given by*

– *Substitution* $\phi_1 = xt$:

$$\begin{aligned} T^t &= -\frac{1}{2}x\Big(2t^2u_{xx} + 4t^2u_x^2 + 4t^2uu_{xx} + 2t^2u_{xxx} - 1 \\ &- 2u - xu_x + 2tu_t + xtu_{tx}\Big), \end{aligned}$$

$$T^x = \frac{1}{2}t \begin{pmatrix} -1 + 4xu_x + x^2u_{tt} - 4tu_tu + 2txu_{tx} + 4xu_{xxx} + 2txu_{xxxt} \\ +8uxu_x + 4txu_tu_x + 4txu_{tx}u - 4u - 4u^2 - 2tu_t - 4u_{xx} - 2t_{txx} \end{pmatrix}.$$

– *Substitution* $\phi_2 = x$:

$$T^t = -\tfrac{1}{2}x\left(2tu_{xx} + 4tu_x^2 + 4tuu_{xx} + 2u_{xxx} + 4u_t + xu_{tx}\right),$$

$$\begin{aligned} T^x = {} & 2xu_x - tu_t + 2xu_{xxx} - tu_{txx} + \tfrac{1}{2}x^2u_{tt} - 2u^2 \\ & -2tu_tu + xtu_{tx} - \tfrac{1}{2} - 2u - 2u_{xx} + 2txu_xu_t \\ & +2xtu_{tx}u + 4uxu_x. \end{aligned}$$

– *Substitution* $\phi_3 = t$:

$$\begin{aligned} T^t &= -t^2u_{xx} - 2t^2u_x^2 - 2t^2uu_{xx} - 2tu_{xxx} + \frac{1}{2} + u \\ &+ \frac{1}{2}xu_x - tu_t - \frac{1}{2}txu_{tx}, \end{aligned}$$

$$\begin{aligned} T^x &= \frac{1}{2}t\big(xu_{tt} + 5u_x + 10u_xu + 4tu_xu_t + 2tu_{tx} \\ &+ 4tu_{tx}u + 5u_{xxx}\big). \end{aligned}$$

– *Substitution* $\phi_4 = 1$:

$$T^t = -tu_{xx} - 2tu_x^2 - 2tuu_{xx} - u_{xxx} - 2u_t - \tfrac{1}{2}xu_{tx},$$

$$\begin{aligned} T^x = {} & \tfrac{1}{2}xu_{tt} + \tfrac{5}{2}u_x + 5u_xu + 2tu_tu_x + tu_{tx} + 2tu_{tx}u \\ & +\tfrac{5}{2}u_{xxx}. \end{aligned}$$

It is readily seen that using the divergence condition we obtain the null conserved vector corresponding to the substitution $\phi_1 = xt$.

Remark 8. *With the aid of package program Maple14, we have checked that the above vectors* (T^t, T^x) *are the conservation vector of Equation (2).*

3 Exact solutions of Equation (2) with exp function method

Let us consider the Equation (2). Introducing a wave variable ξ defined as

$$\xi = kx + wt, \tag{17}$$

where k and w are nonzero constants. Replacing (17) into (2), we have the following ordinary differential equation (ODE):

$$\left(w^2 - k^2\right)u'' - 2k^2\left(u'\right)^2 - 2k^2u''u - k^4u^{(4)} = 0, \tag{18}$$

where prime denotes the differential with respect to ξ.

The Exp-function method which was developed by He and Wu [17] is very simple and straightforward. The method systematically studied for a plenty of NLEEs. It is based on the assumption that traveling wave solutions can be expressed in the following form

$$u(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{n=-p}^{q} b_m \exp(m\xi)}, \tag{19}$$

where c, d, p and q are positive integers which are unknown to be further determined, a_n and b_m are unknown constants.

We suppose that the solution of Equation (18) can be expressed as

$$u(\xi) = \frac{a_c \exp(c\xi) + \ldots + a_{-d} \exp(-d\xi)}{b_p \exp(p\xi) + \ldots + b_{-q} \exp(-q\xi)}. \tag{20}$$

This well-matched formulation plays a important and basic part for obtaining the exact solution of mathematical problems. To determine values of c and p, we balance term of highest order in Equation (18) with the highest order nonlinear term. By simple calculation, we have

$$u^{(4)} = \frac{c_1 \exp[(c + 15p)\xi] + \ldots}{c_2 \exp[16p\xi] + \ldots} \tag{21}$$

and

$$\begin{aligned} u'' u &= \frac{c_3 \exp[(2c + 3p)\xi] + \ldots}{c_4 \exp[5p\xi] + \ldots} \\ &= \frac{c_3 \exp[(2c + 14p)\xi] + \ldots}{c_4 \exp[16p\xi] + \ldots}, \end{aligned} \tag{22}$$

where c_i are determined coefficients only for simplicity.

Balancing highest order of Exp-function in Equations (21) and (22), we have

$$2c + 14p = c + 15p, \tag{23}$$

which leads to the result

$$p = c. \tag{24}$$

Similarly to determine values of d and q, we balance the linear term of lowest order in Equation (18)

$$u^{(4)} = \frac{\ldots + d_1 \exp[-(d + 15q)\xi]}{\ldots + d_2 \exp[-16q\xi]} \tag{25}$$

and

$$\begin{aligned} u'' u &= \frac{\ldots + d_3 \exp[-(2d + 3q)\xi]}{\ldots + d4 \exp[-5q\xi]} \\ &= \frac{\ldots + d_3 \exp[-(2d + 14q)\xi]}{\ldots + d_4 \exp[-16q\xi]} \end{aligned} \tag{26}$$

where d_i are determined coefficients only for simplicity.

Balancing highest order of Exp-function in Equations (25) and (26), we have

$$-(d + 15q) = -(2d + 14q), \tag{27}$$

which leads to the result

$$q = d. \tag{28}$$

For simplicity, we set $p = c = 1$ and $q = d = 1$, so Equation (20) reduces to

$$u(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \tag{29}$$

Substituting Equation (29) into Equation (18), and by the help of Maple, we have

$$\begin{aligned} 0 = \frac{1}{A} \Big[&(R_4 \exp(4\xi) + R_3 \exp(3\xi) + R_2 \exp(2\xi) \\ &+ R_1 \exp(\xi) + R_0 + R_{-1} \exp(-\xi) + R_{-2} \exp(-2\xi) \\ &+ R_{-3} \exp(-3\xi) + R_{-4} \exp(-4\xi) \Big], \end{aligned} \tag{30}$$

where

$$\begin{aligned}
R_4 &= -k^2 a_0 b_1^4 - k^4 a_0 b_1^4 - w^2 a_1 b_1^3 b_0 + k^2 a_1 b_1^3 b_0 \\
&+ 2k^2 a_1^2 b_1^2 b_0 - 2k^2 a_1 a_0 b_1^3 + k^4 a_1 b_1^3 b_0 + w^2 a_0 b_1^4 \\
R_3 &= 6k^2 a_1 b_0 a_0 b_1^2 - w^2 a_1 b_0^2 b_1^2 - k^2 a_0 b_1^3 b_0 \\
&- 4w^2 a_1 b_1^3 b_{-1} + 4k^2 a_1 b_1^3 b_{-1} + 8k^2 a_1^2 b_1^2 b_{-1} \\
&- 8k^2 a_1 a_{-1} b_1^3 + 16k^4 a_1 b_1^3 b_{-1} + w^2 a_0 b_1^3 b_0 \\
&+ k^2 a_1 b_0^2 b_1^2 - 2k^2 a_1^2 b_0^2 b_1 - 11k^4 a_1 b_1^2 b_0^2 \\
&+ 11k^4 a_0 b_1^3 b_0 - 16k^4 a_{-1} b_1^4 - 4k^2 a_{-1} b_1^4 \\
&- 4k^2 a_0^2 b_1^3 + 4w^2 a_{-1} b_1^4 \\
R_2 &= 7k^2 a_1 b_1^2 b_{-1} b_0 + 26k^2 a_1 b_{-1} a_0 b_1^2 \\
&- 4k^2 a_1 b_0 a_{-1} b_1^2 - 4k^2 a_1^2 b_0 b_{-1} b_1 \\
&- 77k^4 a_1 b_1^2 b_0 b_{-1} - 7w^2 a_1 b_1^2 b_{-1} b_0 \\
&+ 6k^2 a_1 b_0^2 a_0 b_1 - w^2 a_0 b_1^2 b_0^2 - k^2 a_1 b_0^3 b_1 \\
&+ 11w^2 a_{-1} b_1^3 b_0 - 4w^2 a_0 b_1^3 b_{-1} - 11k^2 a_{-1} b_1^3 b_0 \\
&+ 4k^2 a_0 b_1^3 b_{-1} - 18k^2 a_{-1} b_1^3 a_0 + 76k^4 a_0 b_1^3 b_{-1} \\
&+ w^2 a_1 b_0^3 b_1 + k^2 a_0 b_1^2 b_0^2 - 2k^2 a_0^2 b_1^2 b_0 + 11k^4 a_1 b_1 b_0^3 \\
&- 11k^4 a_0 b_1^2 b_0^2 + k^4 a_{-1} b_1^3 b_0 - 4k^2 a_1^2 b_0^3 \\
R_1 &= -w^2 a_0 b_1 b_0^3 + k^2 a_0 b_1 b_0^3 - 8k^2 a_1^2 b_{-1}^2 b_1 \\
&- 2k^2 a_0 a_1 b_0^3 + 2k^2 a_0^2 b_1 b_0^2 + k^4 a_0 b_1 b_0^3 - 18k^2 a_1^2 b_0^2 b_{-1} \\
&+ w^2 a_1 b_0^4 - k^2 a_1 b_0^4 - 16k^2 a_{-1}^2 b_1^3 - k^4 a_1 b_0^4 \\
&+ 24k^2 a_1 b_{-1} a_{-1} b_1^2 - 26k^2 a_{-1} b_1^2 a_0 b_0 \\
&+ 4k^2 a_1 b_0^2 a_{-1} b_1 + 28k^2 a_1 b_{-1} a_0 b_1 b_0 \\
&- 13w^2 a_0 b_1^2 b_{-1} b_0 - 2k^2 a_1 b_1 b_{-1} b_0^2 + 13k^2 a_0 b_1^2 b_{-1} b_0 \\
&+ 58k^4 a_1 b_1 b_0^2 b_{-1} - 47k^4 a_0 b_1^2 b_{-1} b_0 \\
&+ 2w^2 a_1 b_1 b_{-1} b_0^2 - 4w^2 a_1 b_1^2 b_{-1}^2 + 11w^2 a_{-1} b_1^2 b_0^2 \\
&+ 4w^2 a_{-1} b_1^3 b_{-1} + 4k^2 a_1 b_1^2 b_{-1}^2 - 11k^2 a_{-1} b_1^2 b_0^2 \\
&- 4k^2 a_{-1} b_1^3 b_{-1} + 12k^2 a_0^2 b_1^2 b_{-1} - 176k^4 a_1 b_1^2 b_{-1}^2 \\
&- 11k^4 a_{-1} b_1^2 b_0^2 + 176k^4 a_{-1} b_1^3 b_{-1} \\
R_0 &= 40k^2 a_1 b_{-1} a_{-1} b_1 b_0 - 30k^2 a_{-1}^2 b_1^2 b_0 \\
&- 30k^2 a_1^2 b_{-1}^2 b_0 + 5w^2 a_1 b_0^3 b_{-1} + 5w^2 a_{-1} b_1 b_0^3 \\
&- 10w^2 a_0 b_1^2 b_{-1}^2 - 5k^2 a_1 b_0^3 b_{-1} - 5k^2 a_{-1} b_1 b_0^3 \\
&+ 10k^2 a_0 b_1^2 b_{-1}^2 - 230k^4 a_0 b_1^2 b_{-1}^2 \\
&- 5k^4 a_1 b_0^3 b_{-1} - 5k^4 a_{-1} b_1 b_0^3 - 10w^2 a_0 b_1 b_{-1} b_0^2 \\
&- 5k^2 a_1 b_1 b_{-1}^2 b_0 - 5k^2 a_{-1} b_1^2 b_0 b_{-1}
\end{aligned}$$

$$
\begin{aligned}
&+ && 10k^2 a_0 b_1 b_{-1} b_0^2 - 10k^2 a_1 b_0^2 a_0 b_{-1} \\
&- && 10k^2 a_{-1} b_0^2 a_0 b_1 + 20k^2 a_0^2 b_1 b_{-1} b_0 \\
&+ && 10k^2 a_1 a_0 b_1 b_{-1}^2 + 10k^2 a_0 a_{-1} b_1^2 b_{-1} \\
&+ && 115k^4 a_1 b_1 b_0 b_{-1}^2 + 115k^4 a_{-1} b_1^2 b_0 b_{-1} \\
&+ && 10k^4 a_0 b_1 b_{-1} b_0^2 + 5w^2 a_{-1} b_1^2 b_0 b_{-1} \\
&+ && 5w^2 a_1 b_1 b_{-1}^2 b_0
\end{aligned}
$$

$$
\begin{aligned}
R_{-1} &= && 28k^2 a_{-1} b_1 a_0 b_{-1} b_0 - k^2 a_{-1} b_0^4 + w^2 a_{-1} b_0^4 \\
&- && 16k^2 a_1^2 b_{-1}^3 - k^4 a_{-1} b_0^4 + 24k^2 a_1 b_{-1}^2 a_{-1} b_1 \\
&- && 26k^2 a_1 b_{-1}^2 a_0 b_0 + 4k^2 a_1 b_{-1} a_{-1} b_0^2 \\
&+ && 2w^2 a_{-1} b_1 b_0^2 b_{-1} - 13w^2 a_0 b_1 b_{-1}^2 b_0 \\
&- && 2k^2 a_{-1} b_1 b_0^2 b_{-1} + 13k^2 a_0 b_1 b_{-1}^2 b_0 \\
&+ && 58k^4 a_{-1} b_1 b_0^2 b_{-1} - 47k^4 a_0 b_1 b_{-1}^2 b_0 \\
&- && w^2 a_0 b_{-1} b_0^3 + k^2 a_0 b_{-1} b_0^3 \\
&- && 18k^2 a_{-1}^2 b_1 b_0^2 - 8k^2 a_{-1}^2 b_1^2 b_{-1} \\
&- && 2k^2 a_0 a_{-1} b_0^3 + 2k^2 a_0^2 b_{-1} b_0^2 + k^4 a_0 b_{-1} b_0^3 \\
&+ && 4w^2 a_1 b_1 b_{-1}^3 + 11w^2 a_1 b_0^2 b_{-1}^2 - 4w^2 a_{-1} b_1^2 b_{-1}^2 \\
&- && 4k^2 a_1 b_1 b_{-1}^3 - 11k^2 a_1 b_0^2 b_{-1}^2 + 4k^2 a_{-1} b_1^2 b_{-1}^2 \\
&+ && 12k^2 a_0^2 b_{-1}^2 b_1 - 11k^4 a_1 b_0^2 b_{-1}^2 + 176k^4 a_1 b_1 b_{-1}^3 \\
&- && 176k^4 a_{-1} b_1^2 b_{-1}^2
\end{aligned}
$$

$$
\begin{aligned}
R_{-2} &= && -7w^2 a_{-1} b_1 b_0 b_{-1}^2 + 7k^2 a_{-1} b_1 b_0 b_{-1}^2 \\
&- && 4k^2 a_1 a_{-1} b_0 b_{-1}^2 + 26k^2 a_0 a_{-1} b_1 b_{-1}^2 \\
&- && 4k^2 a_{-1}^2 b_0 b_{-1} b_1 - 77k^4 a_{-1} b_1 b_0 b_{-1}^2 \\
&+ && 6k^2 a_{-1} b_0^2 a_0 b_{-1} - w^2 a_0 b_{-1}^2 b_0^2 - k^2 a_{-1} b_0^3 b_{-1} \\
&+ && 11w^2 a_1 b_0 b_{-1}^3 - 4w^2 a_0 b_1 b_{-1}^3 - 11k^2 a_1 b_0 b_{-1}^3 \\
&+ && 4k^2 a_0 b_1 b_{-1}^3 - 18k^2 a_1 a_0 b_{-1}^3 + 76k^4 a_0 b_1 b_{-1}^3 \\
&+ && w^2 a_{-1} b_0^3 b_{-1} + k^2 a_0 b_{-1}^2 b_0^2 - 2k^2 a_0^2 b_{-1}^2 b_0 \\
&+ && 11k^4 a_{-1} b_0^3 b_{-1} - 11k^4 a_0 b_{-1}^2 b_0^2 + k^4 a_1 b_0 b_{-1}^3 \\
&- && 4k^2 a_{-1}^2 b_0^3
\end{aligned}
$$

$$
\begin{aligned}
R_{-3} &= && 6k^2 a_{-1} b_0 a_0 b_{-1}^2 - w^2 a_{-1} b_0^2 b_{-1}^2 - k^2 a_0 b_{-1}^3 b_0 \\
&- && 4w^2 a_{-1} b_1 b_{-1}^3 + 4k^2 a_{-1} b_1 b_{-1}^3 + 8k^2 a_{-1}^2 b_1 b_{-1}^2 \\
&- && 8k^2 a_{-1} a_1 b_{-1}^3 + 16k^4 a_{-1} b_1 b_{-1}^3 + w^2 a_0 b_{-1}^3 b_0 \\
&+ && k^2 a_{-1} b_0^2 b_{-1}^2 - 2k^2 a_{-1}^2 b_0^2 b_{-1} - 11k^4 a_{-1} b_0^2 b_{-1}^2 \\
&+ && 11k^4 a_0 b_{-1}^3 b_0 - 16k^4 a_1 b_{-1}^4 - 4k^2 a_0^2 b_{-1}^3 \\
&+ && 4w^2 a_1 b_{-1}^4 - 4k^2 a_1 b_{-1}^4
\end{aligned}
$$

$$
\begin{aligned}
R_{-4} &= && -k^4 a_0 b_{-1}^4 - k^2 a_0 b_{-1}^4 - w^2 a_{-1} b_0 b_{-1}^3 \\
&+ && k^2 a_{-1} b_0 b_{-1}^3 - 2k^2 a_{-1} a_0 b_{-1}^3 + 2k^2 a_{-1}^2 b_0 b_{-1}^2 \\
&+ && k^4 a_{-1} b_0 b_{-1}^3 + w^2 a_0 b_{-1}^4.
\end{aligned}
$$

Equating the coefficients of $\exp(n\xi)$ to be zero, we have

$$
\left\{
\begin{array}{c}
R_4 = 0,\ R_3 = 0,\ R_2 = 0,\ R_1 = 0 \\
R_0 = 0, \\
R_{-1} = 0,\ R_{-2} = 0,\ R_{-3} = 0,\ R_{-4} = 0.
\end{array}
\right. \tag{31}
$$

Solving the system, Equation (31), simultaneously, we obtain

$$
\begin{aligned}
a_0 &= \frac{1}{2}\frac{b_0\left(-k^2 + 5k^4 + w^2\right)}{k^2} \\
a_1 &= -\frac{1}{2}\frac{\left(k^2 + k^4 - w^2\right) b_1}{k^2}, \\
a_{-1} &= -\frac{1}{8}\frac{b_0^2\left(k^2 + k^4 - w^2\right)}{k^2 b_1}, \\
& b_0 = b_0\,,\ b_1 = b_1,\ k = k, w = w.
\end{aligned} \tag{32}
$$

Therefore, we obtain the following solution :

$$
\begin{aligned}
u(x,t) &= -\frac{1}{2k^2}\left(k^4 + k^2 - w^2\right) \\
&+ \frac{12 b_0 b_1 k^2}{4b_1^2 \exp(kx + wt) + 4b_0 b_1 + b_0^2 \exp(-kx - wt)}.
\end{aligned} \tag{33}
$$

Generally b_0, b_1, k and w are real numbers, and the obtained solution is a generalized solitonary solution.

In case k and w are imaginary number, the obtained solitonary solution can be transformed into periodic solution or compact-like solution. If we write $k = iK$ and $w = iW$ and use the following equality

$$
\begin{aligned}
\exp(kx + wt) &= \exp(i(Kx + Wt)) = \cos(Kx + Wt) \\
&+ i\sin((Kx + Wt))
\end{aligned}
$$

and

$$
\begin{aligned}
\exp(-kx - wt) &= \exp(-i(Kx + Wt)) = \cos(Kx + Wt) \\
&- i\sin((Kx + Wt))\,.
\end{aligned}
$$

Equation (33) becomes

$$
\begin{aligned}
u(x,t) &= \frac{1}{2K^2}\left(K^4 - K^2 + W^2\right) \\
&+ \frac{(-12)\, b_0 b_1 K^2}{\left(4b_1^2 + b_0^2\right)\cos(Kx + Wt) + 4b_0 b_1}. \\
& \overline{+i\left(4b_1^2 - b_0^2\right)\sin(Kx + Wt)}
\end{aligned} \tag{34, 35}
$$

If we search for a periodic solution or compact-like solution, the imaginary part in the denomitor of Equation (34) must be zero, that requires that

$$4b_1^2 - b_0^2 = 0. \tag{36}$$

Solving b_0 from Equation (36), we have

$$b_0 = \pm 2b_1. \tag{37}$$

Substituting Equation (37) into Equation (34) results in a compact-like solution, which reads.

$$u(x,t) = \frac{1}{2K^2}\left(K^4 - K^2 + W^2\right) \pm \frac{3K^2}{\cos(Kx+Wt) \mp 1}. \quad (38)$$

Remark 9. *Comparing our results and Jafari* et al. *results [38] then it can be seen that the results are same.*

Remark 10. *We have verified obtained the solutions of Equation (33), Equation (34) and Equation (38) with the aid of Maple.*

4 Conclusion

In this study we considered the ill posed Boussinesq equation. We first discussed the exact travelling wave solutions with the exp function method. We have constructed generalized solitonary, periodic and compact-like solutions.The obtained exact solutions should be very useful in various areas of applied mathematics and they can interpret some physical phenomena. The Exp-function method has more advantages: it is direct and concise. In addition, this method clearly avoids some linearization processes, unrealistic assumptions and consequently it provides exact solutions efficiently.Then we considered a nonlocal conservation method. We constructed an adjoint equation by applying the formal Lagrangian to the variational derivative. We showed that the ill posed Boussinesq equation is not self-adjoint. Using the notion of nonlinear self-adjoint we obtained numerous local conservation laws. The conserved vectors obtained here can be used in reductions and solutions of the underlying equation [45].

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