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On integral inequalities related to the weighted and the extended Chebyshev functionals involving different fractional operators

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Abstract

The role of fractional integral operators can be found as one of the best ways to generalize classical inequalities. In this paper, we use different fractional integral operators to produce some inequalities for the weighted and the extended Chebyshev functionals. The results are more general than the available classical results in the literature.

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1 Introduction and preliminaries

Fractional calculus, which is calculus of integrals and derivatives of any arbitrary real or complex order, has gained remarkable popularity and importance during the last four decades or so, due mainly to its demonstrated applications in diverse and widespread fields ranging from natural sciences to social sciences (see, e.g., [1, 3, 17, 19, 20, 23, 24] and the references therein). Beginning with the classical Riemann–Liouville fractional integral and derivative operators, a large number of fractional integral and derivative operators and their generalizations have been presented. Also, many authors have established a variety of inequalities for those fractional integral and derivative operators, some of which have turned out to be useful in analyzing solutions of certain fractional integral and differential equations.

Definition 1.1 Let $[a,b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann–Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha \in \mathbb{C}$ $(\Re(\alpha) > 0)$ with $a \geq 0$ and $b > 0$ are defined, respectively, by

$$\bigl(J^{\alpha}_{a+}f\bigr)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt \quad \bigl(x > a; \Re(\alpha) > 0\bigr) \tag{1.1}$$

and

$$\left(J_{b-}^{\alpha}f\right)(x) := \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt \quad \left(x<b;\Re(\alpha)>0\right). \tag{1.2}$$

Here, $\Gamma(\alpha)$ is the familiar gamma function (see, e.g., [28, Sect. 1.1]). For more details and properties concerning the fractional integral operators (1.1) and (1.2), we refer the reader, for example, to the works [6, 13, 15, 18, 19, 24, 25, 27, 29, 30] and the references therein.

In [5], the Chebyshev functional for two integrable functions f and g on $[a,b]$ is defined as follows:

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\frac{1}{b-a}\left(\int_a^b g(x)\,dx\right). \tag{1.3}$$

In [3, 11, 12] the applications and several inequalities related to (1.3) are found. In [5], the weighted Chebyshev functional is defined by

$$\begin{aligned} T(f,g,p) = {}& \frac{1}{b-a}\int_a^b p(x)\,dx \int_a^b f(x)g(x)p(x)\,dx \\ & - \int_a^b f(x)p(x)\,dx \int_a^b g(x)p(x)\,dx, \end{aligned} \tag{1.4}$$

where f and g are integrable on $[a,b]$ and p is a positive and integrable function on $[a,b]$.
In [14], Elezovic *et al.* proved that

$$\begin{aligned} \left|T(f,g,p)\right| \le {}& \frac{1}{2}\left(\int_a^b\int_a^b p(x)p(y)|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\left|\int_y^x \left|f'(t)\right|^{\alpha}dt\right|^{\frac{\gamma}{\alpha}}dx\,dy\right)^{\frac{1}{\gamma}} \\ & \times\left(\int_a^b\int_a^b p(x)p(y)|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\left|\int_y^x \left|g'(t)\right|^{\beta}dt\right|^{\frac{\gamma'}{\beta}}dx\,dy\right)^{\frac{1}{\gamma'}} \\ \le {}& \frac{1}{2}\|f'\|_{\alpha}\|g'\|_{\beta}\left(\int_a^b\int_a^b p(x)p(y)|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}dx\,dy\right), \end{aligned}$$

where $f'\in L^{\alpha}([a,b])$ and $g'\in L^{\beta}([a,b])$, $\alpha,\beta,\gamma>1$, $\frac{1}{\alpha}+\frac{1}{\alpha'}=1$, $\frac{1}{\beta}+\frac{1}{\beta'}=1$, and $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$.
In [10], Dahmani *et al.* proved the following fractional integral inequality for Chebyhev functionals:

$$\begin{aligned} & 2\left|J^{\delta}p(t)J^{\delta}pfg(t) - J^{\delta}pf(t)J^{\delta}pg(t)\right| \\ & \quad \le \frac{\|f'\|_{\alpha}\|g'\|_{\beta}}{\Gamma^2(\delta)}\int_0^t\int_0^t (t-x)^{\delta-1}(t-y)^{\delta-1}|x-y|p(x)p(y)\,dx\,dy, \end{aligned}$$

where $f'\in L^{\alpha}([0,\infty))$ and $g'\in L^{\beta}([0,\infty))$, $\alpha,\beta>1$ and $\frac{1}{\alpha}+\frac{1}{\beta}=1$.
Additionally, taking a positive and integrable function q on $[a,b]$, we consider the extended Chebyshev functional [7, 22]

$$\begin{aligned} \tilde{T}(f,g,p,q) = {}& \int_a^b q(x)\,dx\int_a^b f(x)g(x)p(x)\,dx + \int_a^b p(x)\,dx\int_a^b f(x)g(x)q(x)\,dx \\ & - \int_a^b f(x)p(x)\,dx\int_a^b g(x)q(x)\,dx - \int_a^b f(x)q(x)\,dx\int_a^b g(x)p(x)\,dx. \end{aligned} \tag{1.5}$$

Many researchers have given valuable attention to functionals (1.4) and (1.5). For more details, we refer the reader to [3, 8, 21] and the references therein.

Dahmani *et al.* [9], established some inequalities for the weighted and the extended Chebyshev functionals with certain conditions via Riemann–Liouville fractional integrals, which are recalled in the following two theorems.

Theorem 1.1 *Let f and g be two differentiable functions on $[0,\infty)$ and p be a positive and integrable function on $[0,\infty)$. If $f' \in L^{\alpha}([0,\infty))$, $g' \in L^{\beta}([0,\infty))$, $\alpha,\beta,\gamma > 1$ with $\frac{1}{\alpha}+\frac{1}{\alpha'}=1$, $\frac{1}{\beta}+\frac{1}{\beta'}=1$, and $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$, then for all $t>0$, $\delta>0$, we have the inequality*

$$\begin{aligned}
&2\left|J^{\delta}p(t)J^{\delta}pfg(t)-J^{\delta}pg(t)J^{\delta}pf(t)\right| \\
&\quad\leq \left(\frac{\|f'\|_{\alpha}^{\gamma}}{\Gamma(\delta)}\int_0^t\int_0^t (t-x)^{\delta-1}(t-y)^{\delta-1}p(x)p(y)|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\,dx\,dy\right)^{\frac{1}{\gamma}} \\
&\qquad\times\left(\frac{\|g'\|_{\beta}^{\gamma'}}{\Gamma(\alpha)}\int_0^t\int_0^t (t-x)^{\delta-1}(t-y)^{\delta-1}p(x)p(y)|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\,dx\,dy\right)^{\frac{1}{\gamma'}} \\
&\quad\leq \frac{\|f'\|_{\alpha}\|g'\|_{\beta}}{\Gamma(\delta)^2}\left(\int_0^t\int_0^t (t-x)^{\delta-1}(t-y)^{\delta-1}|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}p(x)p(y)\,dx\,dy\right).
\end{aligned}$$

Theorem 1.2 *Let f and g be two differentiable functions on $[0,\infty)$ and p, q be two positive and integrable functions on $[0,\infty)$. If $f' \in L^{\alpha}([0,\infty))$, $g' \in L^{\beta}([0,\infty))$, $\alpha,\beta,\gamma>1$ with $\frac{1}{\alpha}+\frac{1}{\alpha'}=1$, $\frac{1}{\beta}+\frac{1}{\beta'}=1$, and $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$, then for all $t>0$, $\delta>0$, we have*

$$\begin{aligned}
&\left|J^{\delta}q(t)J^{\delta}pfg(t)+J^{\delta}p(t)J^{\delta}qfg(t)-J^{\delta}pf(t)J^{\delta}qg(t)-J^{\delta}qf(t)J^{\delta}pg(t)\right| \\
&\quad\leq \frac{\|f'\|_{\alpha}\|g'\|_{\beta}}{\Gamma(\delta)^2}\left(\int_0^t\int_0^t (t-x)^{\delta-1}(t-y)^{\delta-1}|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}p(x)q(y)\,dx\,dy\right).
\end{aligned}$$

2 Main results

In this section we present some inequalities for the weighted and the extended Chebyshev functionals involving the fractional integral operators, respectively, Katugampola fractional integral operator, mixed conformable fractional integral operator, and Hadamard fractional integral operator.

Definition 2.1 ([16]) Let $[a,b]\subset\mathbb{R}$ be a finite interval. Then the left- and right-hand side Katugampola fractional integrals of order $(\alpha>0)$ of $f\in X_c^p(a,b)$ are defined as follows:

$$^{\rho}\mathcal{I}_{a+}^{\alpha}f(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{t^{\rho-1}}{(x^{\rho}-t^{\rho})^{1-\alpha}}f(t)\,dt$$

and

$$^{\rho}\mathcal{I}_{b-}^{\alpha}f(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b \frac{t^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}f(t)\,dt$$

with $a<x<b$ and $\rho>0$, if the integral exists.

Theorem 2.1 *Let f and g be two differentiable functions on $[0,\infty)$ and p be a positive and integrable function on $[0,\infty)$. If $f'\in X_c^p(a^{\rho},b^{\rho})$, $g'\in X_c^p(a^{\rho},b^{\rho})$, $r,s,\gamma>1$ with $\frac{1}{r}+\frac{1}{r'}=1$,*

$\frac{1}{s}+\frac{1}{s'}=1$, *and* $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$, *then for all* $t>0$, $\alpha,\rho>0$, *we have*

$$\begin{aligned}
&2\left|{}^{\rho}\mathcal{I}^{\alpha}p(t){}^{\rho}\mathcal{I}^{\alpha}pfg(t)-{}^{\rho}\mathcal{I}^{\alpha}pg(t){}^{\rho}\mathcal{I}^{\alpha}pf(t)\right| \\
&\quad\le\left[\frac{\rho^{1-\alpha}\|f'\|_r^{\gamma}}{\Gamma(\alpha)}\int_0^t\int_0^t\frac{x^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}\frac{y^{\rho-1}}{(t^{\rho}-y^{\rho})^{1-\alpha}}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)p(y)\,dx\,dy\right]^{\frac{1}{\gamma}} \\
&\quad\quad\times\left[\frac{\rho^{1-\alpha}\|g'\|_s^{\gamma'}}{\Gamma(\alpha)}\int_0^t\int_0^t\frac{x^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}\frac{y^{\rho-1}}{(t^{\rho}-y^{\rho})^{1-\alpha}}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)p(y)\,dx\,dy\right]^{\frac{1}{\gamma'}} \\
&\quad\le\left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\right)^2\|f'\|_r\|g'\|_s\int_0^t\int_0^t\frac{x^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}\frac{y^{\rho-1}}{(t^{\rho}-y^{\rho})^{1-\alpha}}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)p(y)\,dx\,dy.
\end{aligned}\tag{2.1}$$

Proof Let us define

$$H(x,y):=\bigl(f(x)-f(y)\bigl(g(x)-g(y)\bigr)\bigr);\quad x,y\in(0,t),t>0.\tag{2.2}$$

Multiplying (2.2) by $\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{x^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}p(x)$ and integrating the resulting identity with respect to x from 0 to t, we can write

$$\begin{aligned}
&\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\frac{x^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}p(x)H(x,y)\,dx \\
&\quad={}^{\rho}\mathcal{I}^{\alpha}pfg(t)-g(y){}^{\rho}\mathcal{I}^{\alpha}pf(t)-f(y){}^{\rho}\mathcal{I}^{\alpha}pg(t)+f(y)g(y){}^{\rho}\mathcal{I}^{\alpha}p(t).
\end{aligned}\tag{2.3}$$

Again, multiplying (2.3) by $\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{y^{\rho-1}}{(t^{\rho}-y^{\rho})^{1-\alpha}}p(y)$ and integrating the resulting identity with respect to y from 0 to t, we can write

$$\begin{aligned}
&\left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\right)^2\int_0^t\int_0^t\frac{x^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}\frac{y^{\rho-1}}{(t^{\rho}-y^{\rho})^{1-\alpha}}p(x)p(y)H(x,y)\,dx\,dy \\
&\quad=2\bigl[{}^{\rho}\mathcal{I}^{\alpha}p(t){}^{\rho}\mathcal{I}^{\alpha}pfg(t)-{}^{\rho}\mathcal{I}^{\alpha}pg(t){}^{\rho}\mathcal{I}^{\alpha}pf(t)\bigr].
\end{aligned}\tag{2.4}$$

Also, on the other hand, we have

$$H(x,y):=\int_y^x\int_y^x f'(u)g'(w)\,du\,dw.\tag{2.5}$$

By employing Hölder's inequality, we have

$$\bigl|f(x)-f(y)\bigr|\le|x-y|^{\frac{1}{r'}}\left|\int_y^x\bigl|f'(u)\bigr|^r\,du\right|^{\frac{1}{r}}\tag{2.6}$$

and

$$\bigl|g(x)-g(y)\bigr|\le|x-y|^{\frac{1}{s'}}\left|\int_y^x\bigl|g'(w)\bigr|^s\,dw\right|^{\frac{1}{s}}.\tag{2.7}$$

Then, we can estimate H as follows:

$$\bigl|H(x,y)\bigr|\le|x-y|^{\frac{1}{r'}+\frac{1}{s'}}\left|\int_y^x\bigl|f'(u)\bigr|^r\,du\right|^{\frac{1}{r}}\left|\int_y^x\bigl|g'(w)\bigr|^s\,du\right|^{\frac{1}{s}}.\tag{2.8}$$

Therefore, we can write

$$\begin{aligned}
&2\left|{}^{\rho}\mathcal{I}^{\alpha}p(t){}^{\rho}\mathcal{I}^{\alpha}pfg(t)-{}^{\rho}\mathcal{I}^{\alpha}pg(t){}^{\rho}\mathcal{I}^{\alpha}pf(t)\right| \\
&\quad\leq\left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\right)^{2}\int_{0}^{t}\int_{0}^{t}\frac{x^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}\frac{y^{\rho-1}}{(t^{\rho}-y^{\rho})^{1-\alpha}}p(x)p(y) \\
&\quad\quad\times|x-y|^{\frac{1}{r'}+\frac{1}{s'}}\left|\int_{y}^{x}\left|f'(u)\right|^{r}du\right|^{\frac{1}{r}}\left|\int_{y}^{x}\left|g'(w)\right|^{s}dw\right|^{\frac{1}{s}}dx\,dy.
\end{aligned}$$

By Hölder's inequality for double integral, we obtain

$$\begin{aligned}
&2\left|{}^{\rho}\mathcal{I}^{\alpha}p(t){}^{\rho}\mathcal{I}^{\alpha}pfg(t)-{}^{\rho}\mathcal{I}^{\alpha}pg(t){}^{\rho}\mathcal{I}^{\alpha}pf(t)\right| \\
&\quad\leq\left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\right)^{2}\left[\int_{0}^{t}\int_{0}^{t}\frac{x^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}\frac{y^{\rho-1}}{(t^{\rho}-y^{\rho})^{1-\alpha}}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}\right. \\
&\quad\quad\left.\times\left|\int_{y}^{x}\left|f'(u)\right|^{r}du\right|^{\frac{\gamma}{r}}p(x)p(y)\,dx\,dy\right]^{\frac{1}{\gamma}} \\
&\quad\quad\times\left[\int_{0}^{t}\int_{0}^{t}\frac{x^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}\frac{y^{\rho-1}}{(t^{\rho}-y^{\rho})^{1-\alpha}}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}\right. \\
&\quad\quad\left.\times\left|\int_{y}^{x}\left|g'(w)\right|^{s}dw\right|^{\frac{\gamma'}{s}}p(x)p(y)\,dx\,dy\right]^{\frac{1}{\gamma'}}.
\end{aligned}\tag{2.9}$$

Using the following properties:

$$\left|\int_{x}^{y}\left|f'(u)\right|^{r}du\right|\leq\|f'\|_{r}^{r},\qquad\left|\int_{x}^{y}\left|g'(w)\right|^{s}dw\right|\leq\|g'\|_{s}^{s},$$

(2.9) can be written as

$$\begin{aligned}
&2\left|{}^{\rho}\mathcal{I}^{\alpha}p(t){}^{\rho}\mathcal{I}^{\alpha}pfg(t)-{}^{\rho}\mathcal{I}^{\alpha}pg(t){}^{\rho}\mathcal{I}^{\alpha}pf(t)\right| \\
&\quad\leq\left[\frac{\rho^{1-\alpha}\|f'\|_{r}^{\gamma}}{\Gamma(\alpha)}\int_{0}^{t}\int_{0}^{t}\frac{x^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}\frac{y^{\rho-1}}{(t^{\rho}-y^{\rho})^{1-\alpha}}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)p(y)\,dx\,dy\right]^{\frac{1}{\gamma}} \\
&\quad\quad\times\left[\frac{\rho^{1-\alpha}\|g'\|_{s}^{\gamma'}}{\Gamma(\alpha)}\int_{0}^{t}\int_{0}^{t}\frac{x^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}\frac{y^{\rho-1}}{(t^{\rho}-y^{\rho})^{1-\alpha}}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)p(y)\,dx\,dy\right]^{\frac{1}{\gamma'}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&2\left|{}^{\rho}\mathcal{I}^{\alpha}p(t){}^{\rho}\mathcal{I}^{\alpha}pfg(t)-{}^{\rho}\mathcal{I}^{\alpha}pg(t){}^{\rho}\mathcal{I}^{\alpha}pf(t)\right| \\
&\quad\leq\left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\right)^{2}\|f'\|_{r}\|g'\|_{s}\int_{0}^{t}\int_{0}^{t}\frac{x^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}\frac{y^{\rho-1}}{(t^{\rho}-y^{\rho})^{1-\alpha}}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)p(y)\,dx\,dy.
\end{aligned}$$

This completes the proof. □

Theorem 2.2 *Let f and g be two differentiable functions on* $[0,\infty)$ *and p, q be positive and integrable functions on* $[0,\infty)$. *If* $f'\in X_{c}^{p}(a^{\rho},b^{\rho})$, $g'\in X_{c}^{p}(a^{\rho},b^{\rho})$, $r,s,\gamma>1$ *with* $\frac{1}{r}+\frac{1}{r'}=1$,

$\frac{1}{s}+\frac{1}{s'}=1$, *and* $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$, *then for all* $t>0$, $\alpha,\rho>0$, *we have*

$$\begin{aligned}&\left|{}^{\rho}\mathcal{I}^{\alpha}q(t){}^{\rho}\mathcal{I}^{\alpha}pfg(t)+{}^{\rho}\mathcal{I}^{\alpha}p(t){}^{\rho}\mathcal{I}^{\alpha}qfg(t)-{}^{\rho}\mathcal{I}^{\alpha}pf(t){}^{\rho}\mathcal{I}^{\alpha}qg(t)-{}^{\rho}\mathcal{I}^{\alpha}qf(t){}^{\rho}\mathcal{I}^{\alpha}pg(t)\right|\\&\quad\le\left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\right)^{2}\|f'\|_{r}\|g'\|_{s}\\&\qquad\times\int_0^t\int_0^t\frac{x^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}\frac{y^{\rho-1}}{(t^{\rho}-y^{\rho})^{1-\alpha}}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)q(y)\,dx\,dy.\end{aligned}\tag{2.10}$$

Proof Multiplying (2.3) by $\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{y^{\rho-1}}{(t^{\rho}-y^{\rho})^{1-\alpha}}q(y)$ and integrating the resulting identity with respect to y from 0 to t, we can write

$$\begin{aligned}&\left|{}^{\rho}\mathcal{I}^{\alpha}q(t){}^{\rho}\mathcal{I}^{\alpha}pfg(t)+{}^{\rho}\mathcal{I}^{\alpha}p(t){}^{\rho}\mathcal{I}^{\alpha}qfg(t)-{}^{\rho}\mathcal{I}^{\alpha}pf(t){}^{\rho}\mathcal{I}^{\alpha}qg(t)-{}^{\rho}\mathcal{I}^{\alpha}qf(t){}^{\rho}\mathcal{I}^{\alpha}pg(t)\right|\\&\quad\le\left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\right)^{2}\int_0^t\int_0^t\frac{x^{\rho-1}}{(t^{\rho}-x^{\rho})^{1-\alpha}}\frac{y^{\rho-1}}{(t^{\rho}-y^{\rho})^{1-\alpha}}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}\\&\qquad\times\left|\int_y^x|f'(u)|^r\,du\right|^{\frac{1}{r}}\left|\int_y^x|g'(w)|^s\,dw\right|^{\frac{1}{s}}p(x)q(y)\,dx\,dy.\end{aligned}$$

Using the same arguments as in the proof of Theorem 2.1, we obtain the desired result. □

Definition 2.2 ([1]) Let f be defined on $[a,b]$ and $\alpha\in\mathbb{C}$, $\mathrm{Re}(\alpha)>0$, $\rho>0$. Then:

(i) The mixed left conformable fractional integral of f is defined by

$${}^{b}_{a}\mathfrak{J}^{\alpha,\rho}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x f(s)\left(\frac{(b-s)^{\rho}-(b-x)^{\rho}}{\rho}\right)^{\alpha-1}(b-s)^{\rho-1}\,ds;$$

and

(ii) The mixed right conformable fractional integral of f is defined by

$${}^{a}\mathfrak{J}^{\alpha,\rho}_{b}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b f(s)\left(\frac{(s-a)^{\rho}-(x-a)^{\rho}}{\rho}\right)^{\alpha-1}(s-a)^{\rho-1}\,ds.$$

For recent results related to this operators, we refer the reader to [1, 2, 4, 26].

Theorem 2.3 *Let f and g be two differentiable functions on $[0,\infty)$ and p be a positive and integrable function on $[0,\infty)$. If $f'\in L_r([0,\infty))$, $g'\in L_s([0,\infty))$, $r,s,\gamma>1$ with $\frac{1}{r}+\frac{1}{r'}=1$, $\frac{1}{s}+\frac{1}{s'}=1$, and $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$, then for all $t>0$, $\alpha,\rho>0$, we have*

$$\begin{aligned}&2\left|{}^{b}_{0}\mathfrak{J}^{\alpha,\rho}p(t){}^{b}_{0}\mathfrak{J}^{\alpha,\rho}pfg(t)-{}^{b}_{0}\mathfrak{J}^{\alpha,\rho}pg(t){}^{b}_{0}\mathfrak{J}^{\alpha,\rho}pf(t)\right|\\&\quad\le\left[\frac{\|f'\|_r^{\gamma}}{\Gamma(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}\right.\\&\qquad\left.\times\left(\frac{(b-y)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)p(y)\,dx\,dy\right]^{\frac{1}{\gamma}}\\&\qquad\times\left[\frac{\|g'\|_s^{\gamma'}}{\Gamma(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}\right.\end{aligned}$$

$$\times\left(\frac{(b-y)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)p(y)\,dx\,dy\Bigg]^{\frac{1}{\gamma'}}$$
$$\le\frac{\|f'\|_r\|g'\|_s}{\Gamma^2(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}$$
$$\times\left(\frac{(b-y)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)p(y)\,dx\,dy.$$

Proof Multiplying (2.2) by $\frac{1}{\Gamma(\alpha)}(\frac{(b-x)^{\rho}-(b-t)^{\rho}}{\rho})^{\alpha-1}(b-x)^{\rho-1}p(x)$ and integrating the resulting identity with respect to x from 0 to t, we can write

$$\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{(b-x)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}p(x)H(x,y)\,dx$$
$$={}_0^b\mathfrak{J}^{\alpha,\rho}pfg(t)-g(y){}_0^b\mathfrak{J}^{\alpha,\rho}pf(t)-f(y){}_0^b\mathfrak{J}^{\alpha,\rho}pg(t)+f(y)g(y){}_0^b\mathfrak{J}^{\alpha,\rho}p(t). \tag{2.11}$$

Now, multiplying (2.11) by $\frac{1}{\Gamma(\alpha)}(\frac{(b-y)^{\rho}-(b-t)^{\rho}}{\rho})^{\alpha-1}(b-y)^{\rho-1}p(y)$ and integrating the resulting identity with respect to y from 0 to t, we can write

$$\frac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}$$
$$\times\left(\frac{(b-y)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}p(x)p(y)H(x,y)\,dx\,dy$$
$$=2\big[{}_0^b\mathfrak{J}^{\alpha,\rho}p(t){}_0^b\mathfrak{J}^{\alpha,\rho}pfg(t)-{}_0^b\mathfrak{J}^{\alpha,\rho}pg(t){}_0^b\mathfrak{J}^{\alpha,\rho}pf(t)\big].$$

Using the same arguments as in the proof of Theorem 2.1, we obtain the desired result. □

Theorem 2.4 *Let f and g be two differentiable functions on $[0,\infty)$ and p, q be positive and integrable functions on $[0,\infty)$. If $f'\in L_r([0,\infty)$, $g'\in L_s([0,\infty)$, $r,s,\gamma>1$ with $\frac{1}{r}+\frac{1}{r'}=1$, $\frac{1}{s}+\frac{1}{s'}=1$, and $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$, then for all $t>0$, $\alpha,\rho>0$, we have*

$$\big|{}_0^b\mathfrak{J}^{\alpha,\rho}q(t){}_0^b\mathfrak{J}^{\alpha,\rho}pfg(t)+{}_0^b\mathfrak{J}^{\alpha,\rho}p(t){}_0^b\mathfrak{J}^{\alpha,\rho}qfg(t)$$
$$-{}_0^b\mathfrak{J}^{\alpha,\rho}pf(t){}_0^b\mathfrak{J}^{\alpha,\rho}qg(t)-{}_0^b\mathfrak{J}^{\alpha,\rho}qf(t){}_0^b\mathfrak{J}^{\alpha,\rho}pg(t)\big|$$
$$\le\frac{\|f'\|_r\|g'\|_s}{\Gamma^2(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}$$
$$\times\left(\frac{(b-y)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}p(x)q(y)\,dx\,dy.$$

Proof Multiplying (2.11) by $\frac{1}{\Gamma(\alpha)}(\frac{(b-y)^{\rho}-(b-t)^{\rho}}{\rho})^{\alpha-1}(b-y)^{\rho-1}q(y)$ and integrating the resulting identity with respect to y from 0 to t, we can write

$$\big|{}_0^b\mathfrak{J}^{\alpha,\rho}q(t){}_0^b\mathfrak{J}^{\alpha,\rho}pfg(t)+{}_0^b\mathfrak{J}^{\alpha,\rho}p(t){}_0^b\mathfrak{J}^{\alpha,\rho}qfg(t)$$
$$-{}_0^b\mathfrak{J}^{\alpha,\rho}pf(t){}_0^b\mathfrak{J}^{\alpha,\rho}qg(t)-{}_0^b\mathfrak{J}^{\alpha,\rho}qf(t){}_0^b\mathfrak{J}^{\alpha,\rho}pg(t)\big|$$
$$\le\frac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t\left(\frac{(b-x)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-x)^{\rho-1}$$

$$\times\left(\frac{(b-y)^{\rho}-(b-t)^{\rho}}{\rho}\right)^{\alpha-1}(b-y)^{\rho-1}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}$$
$$\times\left|\int_y^x\left|f'(u)\right|^r du\right|^{\frac{1}{r}}\left|\int_y^x\left|g'(w)\right|^s dw\right|^{\frac{1}{s}}p(x)q(y)\,dx\,dy.$$

Using the same arguments as in the proof of Theorem 2.1, we obtain the desired result. □

Definition 2.3 The Hadamard fractional integral of order $\alpha\in\mathbb{R}^+$ of a function $f(t)$, for all $t>1$, is defined as follows [19]:

$${}_HJ^{\alpha}\{f(t)\}=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau}.$$

Theorem 2.5 *Let f and g be two differentiable functions on $[1,\infty)$ and p be a positive and integrable function on $[1,\infty)$. If $f'\in L_r([1,\infty))$, $g'\in L_s([1,\infty))$, $r,s,\gamma>1$ with $\frac{1}{r}+\frac{1}{r'}=1$, $\frac{1}{s}+\frac{1}{s'}=1$, and $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$, then for all $t>1$, $\alpha>0$, we have*

$$\begin{aligned}
&2\left|{}_HJ^{\alpha}\{p(t)\}{}_HJ^{\alpha}\{pfg(t)\}-{}_HJ^{\alpha}\{pg(t)\}{}_HJ^{\alpha}\{pf(t)\}\right|\\
&\leq\left[\frac{\|f'\|_r^{\gamma}}{\Gamma(\alpha)}\int_1^t\int_1^t\left(\log\frac{t}{x}\right)^{\alpha-1}\left(\log\frac{t}{y}\right)^{\alpha-1}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}\frac{p(x)p(y)}{xy}\,dx\,dy\right]^{\frac{1}{\gamma}}\\
&\quad\times\left[\frac{\|g'\|_s^{\gamma'}}{\Gamma(\alpha)}\int_1^t\int_1^t\left(\log\frac{t}{x}\right)^{\alpha-1}\left(\log\frac{t}{y}\right)^{\alpha-1}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}\frac{p(x)p(y)}{xy}\,dx\,dy\right]^{\frac{1}{\gamma'}}\\
&\leq\frac{\|f'\|_r\|g'\|_s}{\Gamma^2(\alpha)}\int_1^t\int_1^t\left(\log\frac{t}{x}\right)^{\alpha-1}\left(\log\frac{t}{y}\right)^{\alpha-1}|x-y|^{\frac{1}{r'}+\frac{1}{s'}}\frac{p(x)p(y)}{xy}\,dx\,dy.
\end{aligned}$$

Proof Multiplying (2.2) by $\frac{(\log\frac{t}{x})^{\alpha-1}}{x\Gamma(\alpha)}p(x)$ and integrating the resulting identity with respect to x from 1 to t, we can write

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{x}\right)^{\alpha-1}\frac{p(x)}{x}H(x,y)\,dx\\
&={}_HJ^{\alpha}\{pfg(t)\}-g(y){}_HJ^{\alpha}\{pf(t)\}-f(y){}_HJ^{\alpha}\{pg(t)\}+f(y)g(y){}_HJ^{\alpha}\{p(t)\}.
\end{aligned}\tag{2.12}$$

Now, multiplying (2.12) by $\frac{(\log\frac{t}{y})^{\alpha-1}}{y\Gamma(\alpha)}p(y)$ and integrating the resulting identity with respect to y from 1 to t, we can write

$$\begin{aligned}
&\frac{1}{\Gamma^2(\alpha)}\int_1^t\int_1^t\left(\log\frac{t}{x}\right)^{\alpha-1}\left(\log\frac{t}{y}\right)^{\alpha-1}\frac{p(x)p(y)}{xy}H(x,y)\,dx\,dy\\
&=2\left[{}_HJ^{\alpha}\{p(t)\}{}_HJ^{\alpha}\{pfg(t)\}-{}_HJ^{\alpha}\{pg(t)\}{}_HJ^{\alpha}\{pf(t)\}\right].
\end{aligned}$$

Using the same arguments as in the proof of Theorem 2.1, we obtain the desired result. □

Theorem 2.6 *Let f and g be two differentiable functions on $[1,\infty)$ and p, q be positive and integrable functions on $[1,\infty)$. If $f'\in L_r([1,\infty))$, $g'\in L_s([1,\infty))$, $r,s,\gamma>1$ with $\frac{1}{r}+\frac{1}{r'}=1$,*

$\frac{1}{s} + \frac{1}{s'} = 1$, *and* $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$, *then for all* $t > 1$, $\alpha > 0$, *we have*

$$\begin{aligned}
&\big|{}_HJ^\alpha\{q(t)\}{}_HJ^\alpha\{pfg(t)\} + {}_HJ^\alpha\{p(t)\}{}_HJ^\alpha\{qfg(t)\} \\
&\quad - {}_HJ^\alpha\{pf(t)\}{}_HJ^\alpha\{qg(t)\} - {}_HJ^\alpha\{qf(t)\}{}_HJ^\alpha\{pg(t)\}\big| \\
&\quad \le \frac{\|f'\|_r\|g'\|_s}{\Gamma^2(\alpha)} \int_1^t\int_1^t \left(\log\frac{t}{x}\right)^{\alpha-1}\left(\log\frac{t}{y}\right)^{\alpha-1} |x-y|^{\frac{1}{r'}+\frac{1}{s'}} \frac{p(x)p(y)}{xy}\,dx\,dy.
\end{aligned}$$

Proof Multiplying (2.12) by $\frac{(\log\frac{t}{y})^{\alpha-1}}{y\Gamma(\alpha)}q(y)$ and integrating the resulting identity with respect to y from 1 to t, we can write

$$\begin{aligned}
&\big|{}_HJ^\alpha\{q(t)\}{}_HJ^\alpha\{pfg(t)\} + {}_HJ^\alpha\{p(t)\}{}_HJ^\alpha\{qfg(t)\} \\
&\quad - {}_HJ^\alpha\{pf(t)\}{}_HJ^\alpha\{qg(t)\} - {}_HJ^\alpha\{qf(t)\}{}_HJ^\alpha\{pg(t)\}\big| \\
&\quad \le \frac{1}{\Gamma^2(\alpha)} \int_1^t\int_1^t \left(\log\frac{t}{x}\right)^{\alpha-1}\left(\log\frac{t}{y}\right)^{\alpha-1} |x-y|^{\frac{1}{r'}+\frac{1}{s'}} \\
&\quad \times \left|\int_y^x \left|f'(u)\right|^r du\right|^{\frac{1}{r}} \left|\int_y^x \left|g'(w)\right|^s dw\right|^{\frac{1}{s}} \frac{p(x)q(y)}{xy}\,dx\,dy.
\end{aligned}$$

Using the same arguments as in the proof of Theorem 2.1, we obtain the desired result. □

3 Concluding remarks

In this paper, we established some integral inequalities related to the weighted and the extended Chebyshev functionals for different fractional integral operators. If we consider $\rho = 1$ in Theorem 2.1 and Theorem 2.2, then the obtained results will reduce to the said inequalities obtained by Dahmani *et al.* [9].

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