Free vibration of FG nanobeam using a finite-element method

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In this work, a non-local finite-element formulation is developed to analyse free vibration of functionally graded (FG) nanobeams considering power-law variation of material through thickness of the nanobeam. The Euler–Bernoulli beam theory based on Eringen's non-local elasticity theory with one length scale parameter is used to model the FG nanobeam. To this end, two types of FG nanobeams composed of two different materials are analysed by using the developed non-local finite-element formulation. First FG nanobeam is made of alumina (Al_2O_3) and steel, whereas second one is composed of silicon carbide (SiC) and stainless steel (SUS304). Numerical results are presented to show the effect of power-law exponent (k) and nanostructural length scale (e_0a/L) on the free vibration of FG nanobeams.

1. Introduction: Functionally graded materials (FGMs) which are a new emerging class of advanced materials are very attractive for an extensive range of engineering applications due to their abilities in designing of different functional performances within a part. Ceramic and metal are the most commonly used materials in the fabrication of FGM where the ceramic region stand against hot temperatures, while the metallic rich region is exposed to cold temperatures. FGMs are commonly used as thermal barrier for applications in space planes, space structures, nuclear reactors, turbine rotors, flywheels and gears, to name only a few. Moreover, they have been getting considerable attention in fabrication of nanobeams which has been very commonly used in many applications in micro- and nano-scale device such as biosensor, micro- and nano-electro-mechanical devices (MEMs and NEMs). The rapid developments in FGM are achieved by numerous theoretical and computational studies performed on understanding the mechanical behaviour of small-scale FGM structures. The most of the theoretical studies developed for FGMs are based on the non-local continuum theories since the classical continuum theories fail to account for the size effect observable at small scales.

Among them, the non-local elasticity theory pioneered by Eringen [1], especially the non-local beam theories to model micro- and nano-scale beams have received a considerable attention to model the microscale devices. For the free vibration problem, micro-/nano-beams were modelled via Timoshenko beam theory and Eringen's non-local elasticity theory by Wang *et al.* [2]. Uzun *et al.* [3] examined the free vibration behaviours of carbon nanotube (CNT) and boron nitride nanotube depending on non-local elasticity theory. They obtained non-local natural frequencies of nanobeams for various cross-section geometries and various boundary conditions via both analytical and finite-element solutions. Yayli [4] proposed a stability model to analyse the critical buckling loads of single-walled CNT embedded in an elastic medium with an attached spring using non-local elasticity theory. Also, Yayli *et al.* [5] studied the longitudinal vibration of nanorods embedded in an elastic medium with elastic restraints at both ends based on the non-local elasticity theory.

Ebrahimi *et al.* [6] presented the applicability of differential transformation method in investigations on vibrational characteristics of functionally graded (FG) size-dependent nanobeams. Anandrao *et al.* [7] investigated free vibration analysis of FG beams and they enhanced two different finite-element formulations for Euler–Bernoulli and Timoshenko beam theory. Analytical solutions of free and forced vibration of FG nanobeams were presented by Uymaz [8] using generalised beam theory. Yayli [9] presented the free axial vibration response of CNTs with arbitrary boundary conditions on the basis of the non-local elasticity theory. The nonlinear free vibration of the non-local bi-directional graded FG Euler–Bernoulli nanobeams was analysed by Nejad and Hadi [10] using the generalised differential quadrature method (GDQM). Alimoradzadeh *et al.* [11] studied nonlinear vibration analysis of axially functionally graded (AFG) beam Euler–Bernoulli beam subjected to a moving harmonic utilising Green's strain tensor. They investigated the effect of some parameters such as the power index and stiffness coefficients, among others, on the nonlinear natural frequency along with the velocity of the moving harmonic load on the nonlinear dynamic response of (AFG) Euler–Bernoulli, where it is found that these parameters have a considerable effect on both nonlinear natural frequency and response amplitude. El-Borgi *et al.* [12] investigated the non-local nonlinear free and forced vibration behaviour of FG nanobeams embedded in a nonlinear elastic foundation using the method of multiple scales and Galerkin's method.

It has seen that a considerable attention is given on the development of the non-local finite-element formulation by many researchers, recently. Eltaher *et al.* [13], on the basis of the classical beam theory, proposed a two-noded, six degrees of freedom non-local finite-element to examine free vibration of FG nanobeams. Aria and Friswell [14] presented a five noded, ten degrees of freedom non-local (strain-driven) finite-element model to examine the free vibration and buckling behaviour of FG nanobeams on the basis of first-order shear deformation theory. They predicted buckling loads and natural frequencies considering the effects of different non-local coefficients, boundary conditions, power-law indices and span-to-depth ratios. Srividhya *et al.* [15] developed a non-local nonlinear displacement finite element based on the third-order shear deformation theory of Reddy [16, 17] to analyse non-local behaviour of FG plates subjected to static loads. They performed a detailed parametric study to show the effect of side-to-thickness ratio, power-law index and non-local parameter on the load deflection characteristics of plates.

The nanobeams are used in the fabrication of micro- and nano-structures such as biosensors, MEMs and NEMs are in the scale of non-local Euler–Bernoulli beam theory in which length/diameter $\geq$10. In previous studies according to the best of our knowledge, there has been no analysis performed on the investigation of free vibration of SiC/SUS304 FG nanobeam. In this Letter, the vibration characteristics of SiC/SUS304 and Alumina/Steel FG nanobeam with simply-supported boundary condition are analysed using non-local finite-element formulation developed here and their frequency values obtained from the analyses are compared. The variation of material properties is assumed in the

thickness direction based on the power law. FG nanobeam is modelled as Euler–Bernoulli beam and investigated depending upon Eringen's non-local elasticity theory. The contribution of this Letter is the non-local elasticity theory with interpolation functions are used to compute the mass and stiffness matrix in order to construct an eigenvalue problem. The effects of small-scale parameters and power law exponent on frequencies are examined in detail for these FG nanobeams.

2. Equations of motion for FG Euler–Bernoulli beam: Two types of FG nanobeam with simply-supported boundary conditions are established with the geometric properties where L, b and h are length, width and thickness of the FG beam, respectively (Fig. 1).

The material properties of the beam are assumed to vary continuously in the thickness direction where the effective material property of FG beam is expressed by the power law as follows [8, 14]:

$$P(z) = (P_{\mathrm{U}} - P_{\mathrm{L}})\left(\frac{z}{h} + \frac{1}{2}\right)^{k} + P_{\mathrm{L}} \tag{1}$$

Here $P(z)$ is the effective material property of the beam, P_{U} and P_{L} are the material property at the upper and lower surfaces of the beam, k is the positive variable parameter called power law exponent. The effective material property refers to the properties of the beam components such as the elastic module (E), density (ρ) and so on, and can be transformed into the following FG forms:

$$E(z) = (E_{\mathrm{U}} - E_{\mathrm{L}})\left(\frac{z}{h} + \frac{1}{2}\right)^{k} + E_{\mathrm{L}} \tag{2}$$

$$\rho(z) = (\rho_{\mathrm{U}} - \rho_{\mathrm{L}})\left(\frac{z}{h} + \frac{1}{2}\right)^{k} + \rho_{\mathrm{L}} \tag{3}$$

The displacement fields for Euler–Bernoulli beam can be written as follows [18]:

$$u_1(x, z, t) = u(x, t) - z\frac{\partial w(x, t)}{\partial x} \tag{4}$$

$$u_2(x, z, t) = 0 \tag{5}$$

$$u_3(x, z, t) = w(x, t) \tag{6}$$

Here u_1, u_2 and u_3 are the displacements in the x, y, z directions, respectively. u and w denote longitudinal and transverse displacements of any point on the neutral axis, respectively. The components of strain tensor of the Euler–Bernoulli beam can be obtained using (4)–(6) as follows:

$$\varepsilon_{xx} = \frac{\partial u(x, t)}{\partial x} - z\frac{\partial^2 w(x, t)}{\partial x^2} \tag{7}$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \varepsilon_{xz} = \varepsilon_{zx} = \varepsilon_{yy} = \varepsilon_{yz} = \varepsilon_{zy} = \varepsilon_{zz} = 0 \tag{8}$$

ε_{xx} is the non-zero only strain component. Using Hook's law and equilibrium equations stress, normal force and moment expressions for the FG nanobeam can be written, respectively

$$\sigma_{xx} = E(z)\varepsilon_{xx} \tag{9}$$

$$N = A_1\frac{\partial u}{\partial x} - B_1\frac{\partial^2 w}{\partial x^2},\ M = B_1\frac{\partial u}{\partial x} - D_1\frac{\partial^2 w}{\partial x^2} \tag{10}$$

where the coefficients A_1, B_1 and D_1 are expressed as

$$A_1 = \int_A E(z)\mathrm{d}A,\ B_1 = \int_A E(z)z\mathrm{d}A,\ D_1 = \int_A E(z)z^2\mathrm{d}A \tag{11}$$

The Hamilton principle that express the actual motion minimises the difference of kinetic energy and total potential energy is

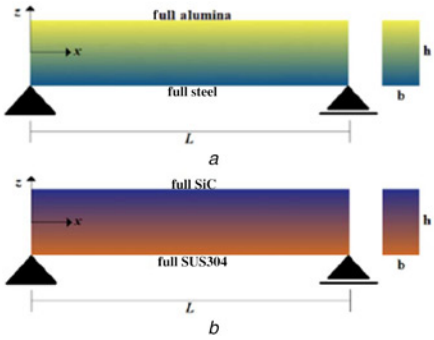

Fig. 1 *Illustration of FG nanobeams*
a Alumina/steel FG nanobeam
b SiC/SUS304 FG nanobeam

expressed as [19]

$$\int_0^T (\delta S - \delta T)\mathrm{d}t = 0 \tag{12}$$

where S and T are the strain energy and kinetic energy, respectively. S and T for an element which has volume V and length L is as below:

$$S = \frac{1}{2}\int_V \sigma_{xx}\varepsilon_{xx}\mathrm{d}V \tag{13}$$

$$T = \frac{1}{2}\int_V \rho(z)\left(\left(\frac{\partial u_1}{\partial t}\right)^2 + \left(\frac{\partial u_3}{\partial t}\right)^2\right)\mathrm{d}V \tag{14}$$

The first variation of the strain and kinetic energy are obtained as follows:

$$\begin{aligned}\delta\int_0^T S\mathrm{d}t &= \int_0^T\int_0^L \left(N\delta\left(\frac{\partial u}{\partial x}\right) - M\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\right)\mathrm{d}x\mathrm{d}t \\ &= \int_0^T\int_0^L \left(\left(A_1\frac{\partial u}{\partial x} - B_1\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial u}{\partial x}\right)\right. \\ &\quad \left.-\left(B_1\frac{\partial u}{\partial x} - D_1\frac{\partial^2 w}{\partial x^2}\right)\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\right)\mathrm{d}x\mathrm{d}t\end{aligned} \tag{15}$$

$$\delta\int_0^T T\mathrm{d}t = \int_0^T\int_0^L \left(\begin{array}{l} I_0\left(\frac{\partial u}{\partial t}\delta\left(\frac{\partial u}{\partial t}\right) + \frac{\partial w}{\partial t}\delta\left(\frac{\partial w}{\partial t}\right)\right) - \cdots \\ \ldots - I_1\left(\frac{\partial u}{\partial t}\delta\left(\frac{\partial^2 w}{\partial x\partial t}\right) + \frac{\partial^2 w}{\partial x\partial t}\delta\left(\frac{\partial u}{\partial t}\right)\right) + I_2\frac{\partial^2 w}{\partial x\partial t}\delta\left(\frac{\partial^2 w}{\partial x\partial t}\right)\end{array}\right)\mathrm{d}x\mathrm{d}t \tag{16}$$

Here I_0, I_1 and I_2 are expressed as

$$I_0 = \int_A \rho(z)\mathrm{d}A,\ I_1 = \int_A \rho(z)z\mathrm{d}A,\ I_2 = \int_A \rho(z)z^2\mathrm{d}A \tag{17}$$

Substituting (15) and (16) into (12), one can obtain the fundamental lemma of the calculus of variations within arbitrariness of δu and δw the governing equations (i.e. the Euler–Lagrange equations) of the beams as follows:

$$\delta u: \frac{\partial N}{\partial x} = I_0\frac{\partial^2 u}{\partial t^2} - I_1\frac{\partial^3 w}{\partial x\partial t^2} \tag{18}$$

$$\delta w: \frac{\partial^2 M}{\partial x^2} = I_0\frac{\partial^2 w}{\partial t^2} + I_1\frac{\partial^3 u}{\partial x\partial t^2} - I_2\frac{\partial^4 w}{\partial x^2\partial t^2} \tag{19}$$

3. Non-local elasticity theory for FG nanobeam: Unlike the local elasticity theory, the non-local constitutive equation [1] where it is

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assumed that the stress at a point is function of strain at all point in the continuum is given by

$$\left[1-\left(e_0a\right)^2\nabla^2\right]\sigma_{ij}=C_{ijkl}\varepsilon_{kl} \tag{20}$$

where σ_{ij} is the stress tensor, C_{ijkl} is the fourth-order elastic module tensor, ε_{kl} is the strain tensor, e_0 is a material constant which is determined experimentally, a is the internal characteristic length. For Euler–Bernoulli FG nanobeam, (20) can be reduced to the following form:

$$\sigma_{xx}-\left(e_0a\right)^2\frac{\partial^2\sigma_{xx}}{\partial x^2}=E(z)\varepsilon_{xx} \tag{21}$$

Integrating (21) over the cross-section area, one can obtain the axial force–strain relation as

$$N-(e_0a)^2\frac{\partial^2N}{\partial x^2}=A_1\frac{\partial u}{\partial x}-B_1\frac{\partial^2w}{\partial x^2} \tag{22}$$

Multiplying (21) by z and integrating over the cross-section area, the moment–curvature relation can be obtained as

$$M-(e_0a)^2\frac{\partial^2M}{\partial x^2}=B_1\frac{\partial u}{\partial x}-D_1\frac{\partial^2w}{\partial x^2} \tag{23}$$

Differentiating (18) with respect to x, then substituting the resulting one into (22), one can rearrange into (22) as

$$N=A_1\frac{\partial u}{\partial x}-B_1\frac{\partial^2w}{\partial x^2}+(e_0a)^2\left(I_0\frac{\partial^3u}{\partial x\partial t^2}-I_1\frac{\partial^4w}{\partial x^2\partial t^2}\right) \tag{24}$$

Substituting (19) into (23), one can obtain the following:

$$M=B_1\frac{\partial u}{\partial x}-D_1\frac{\partial^2w}{\partial x^2}+(e_0a)^2\left(I_0\frac{\partial^2w}{\partial t^2}+I_1\frac{\partial^3u}{\partial x\partial t^2}-I_2\frac{\partial^4w}{\partial x^2\partial t^2}\right) \tag{25}$$

4. Non-local finite-element formulation: The Euler–Bernoulli beam element is modelled using two nodes (at the ends of the beam element), and three degrees of freedom per node (axial displacement, transverse displacement and rotation). The mass and stiffness matrices for Euler–Bernoulli beams are obtained in the usual way from the kinetic and strain energy by applying the variational statement on FG Euler–Bernoulli nanobeam that is expressed as

$$\int_0^T\int_0^L\left(\begin{array}{l}\left(\begin{array}{l}A_1\frac{\partial u}{\partial x}\delta\left(\frac{\partial u}{\partial x}\right)-B_1\frac{\partial^2w}{\partial x^2}\delta\left(\frac{\partial u}{\partial x}\right)+\ldots\\ \cdots+(e_0a)^2\left(I_0\frac{\partial^3u}{\partial x\partial t^2}\delta\left(\frac{\partial u}{\partial x}\right)-I_1\frac{\partial^4w}{\partial x^2\partial t^2}\delta\left(\frac{\partial u}{\partial x}\right)\right)\end{array}\right)\\ -\left(\begin{array}{l}B_1\frac{\partial u}{\partial x}\delta\left(\frac{\partial^2w}{\partial x^2}\right)-D_1\frac{\partial^2w}{\partial x^2}\delta\left(\frac{\partial^2w}{\partial x^2}\right)+\ldots\\ \cdots+(e_0a)^2\left(\begin{array}{l}I_0\frac{\partial^2w}{\partial t^2}\delta\left(\frac{\partial^2w}{\partial x^2}\right)+I_1\frac{\partial^3u}{\partial x\partial t^2}\delta\left(\frac{\partial^2w}{\partial x^2}\right)-\ldots\\ \ldots-I_2\frac{\partial^4w}{\partial x^2\partial t^2}\delta\left(\frac{\partial^2w}{\partial x^2}\right)\end{array}\right)\end{array}\right)\\ -\left(\begin{array}{l}I_0\left(\frac{\partial u}{\partial t}\delta\left(\frac{\partial u}{\partial t}\right)+\frac{\partial w}{\partial t}\delta\left(\frac{\partial w}{\partial t}\right)\right)-\cdots\\ \cdots-I_1\left(\frac{\partial u}{\partial t}\delta\left(\frac{\partial^2w}{\partial x\partial t}\right)+\frac{\partial^2w}{\partial x\partial t}\delta\left(\frac{\partial u}{\partial t}\right)\right)+\cdots\\ \cdots+I_2\frac{\partial^2w}{\partial x\partial t}\delta\left(\frac{\partial^2w}{\partial x\partial t}\right)\end{array}\right)\end{array}\right)\mathrm{d}x\mathrm{d}t \tag{26}$$

ϕu and ϕw are the interpolation shape functions. The Euler–Lagrange equation and the boundary conditions can be obtained by using integration by parts. Interpolation functions can be derived from boundary conditions. They are expressed as below [19]:

$$\left[\phi_u\right]=\left[1-\frac{x}{L}\quad\frac{x}{L}\right] \tag{27}$$

$$\left[\phi_w\right]=\left[1-\frac{3x^2}{L^2}+\frac{2x^3}{L^3}x-\frac{2x^2}{L}+\frac{x^3}{L^2}\frac{3x^2}{L^2}-\frac{2x^3}{L^3}-\frac{x^2}{L}+\frac{x^3}{L^2}\right] \tag{28}$$

$\boldsymbol{K}_u$, $\boldsymbol{K}_{uw}$, $\boldsymbol{K}_w$ are the stiffness matrices, $\boldsymbol{M}_u^c$, $\boldsymbol{M}_{uw}^c$, $\boldsymbol{M}_w^c$ are the classical mass matrices and $\boldsymbol{M}_u^{nl}$, $\boldsymbol{M}_{uw}^{nl}$, $\boldsymbol{M}_w^{nl}$ are the non-local mass matrices and they expressed as

$$\boldsymbol{K}_u=\int_0^L A_1\left(\left[\phi_u\right]'\right)^{\mathrm{T}}\left[\phi_u\right]'\mathrm{d}x \tag{29}$$

$$\boldsymbol{K}_{uw}=-\left(\int_0^L B_1\left(\left[\phi_w\right]''\right)^{\mathrm{T}}\left[\phi_u\right]'\mathrm{d}x+\int_0^L B_1\left(\left[\phi_u\right]'\right)^{\mathrm{T}}\left[\phi_w\right]''\mathrm{d}x\right) \tag{30}$$

$$\boldsymbol{K}_w=\int_0^L D_1\left(\left[\phi_w\right]''\right)^{\mathrm{T}}\left[\phi_w\right]''\mathrm{d}x \tag{31}$$

$$\boldsymbol{M}_u^c=\int_0^L I_0\left(\left[\phi_u\right]\right)^{\mathrm{T}}\left[\phi_u\right]\mathrm{d}x \tag{32}$$

$$\boldsymbol{M}_{uw}^c=-\left(\int_0^L I_1\left(\left[\phi_u\right]\right)^{\mathrm{T}}\left[\phi_w\right]'\mathrm{d}x+\int_0^L I_1\left(\left[\phi_w\right]'\right)^{\mathrm{T}}\left[\phi_u\right]\mathrm{d}x\right) \tag{33}$$

$$\boldsymbol{M}_w^c=\int_0^L I_0\left(\left[\phi_w\right]\right)^{\mathrm{T}}\left[\phi_w\right]\mathrm{d}x+\int_0^L I_2\left(\left[\phi_w\right]'\right)^{\mathrm{T}}\left[\phi_w\right]'\mathrm{d}x \tag{34}$$

$$\boldsymbol{M}_u^{nl}=(e_0a)^2\int_0^L I_0\left(\left[\phi_u\right]'\right)^{\mathrm{T}}\left[\phi_u\right]'\mathrm{d}x \tag{35}$$

$$\begin{aligned}\boldsymbol{M}_{uw}^{nl}=&-\left((e_0a)^2\int_0^L I_1\left(\left[\phi_u\right]'\right)^{\mathrm{T}}\left[\phi_w\right]''\mathrm{d}x\right.\\&\left.+(e_0a)^2\int_0^L I_1\left(\left[\phi_w\right]''\right)^{\mathrm{T}}\left[\phi_u\right]'\mathrm{d}x\right)\end{aligned} \tag{36}$$

$$\begin{aligned}\boldsymbol{M}_w^{nl}=&-(e_0a)^2\int_0^L I_0\left(\left[\phi_w\right]\right)^{\mathrm{T}}\frac{\partial^2\left[\phi_w\right]}{\partial x^2}\mathrm{d}x\\&+(e_0a)^2\int_0^L I_2\left(\frac{\partial^2\left[\phi_w\right]}{\partial x^2}\right)^{\mathrm{T}}\frac{\partial^2\left[\phi_w\right]}{\partial x^2}\mathrm{d}x\end{aligned} \tag{37}$$

The classical stiffness and mass matrices have been obtained by taking the non-local parameter to zero ($e_0a/L=0$). The frequencies of FG nanobeam are found as follows:

$$\left|\boldsymbol{K}-\omega^2\boldsymbol{M}\right|=0 \tag{38}$$

Here ω is the frequency. $\boldsymbol{K}$ and $\boldsymbol{M}$ are total stiffness and mass matrices and given in the following equations:

$$\boldsymbol{K}=\boldsymbol{K}_u+\boldsymbol{K}_w+\boldsymbol{K}_{uw} \tag{39}$$

$$\boldsymbol{M}=\boldsymbol{M}_u^c+\boldsymbol{M}_{uw}^c+\boldsymbol{M}_w^c+\boldsymbol{M}_u^{nl}+\boldsymbol{M}_{uw}^{nl}+\boldsymbol{M}_w^{nl} \tag{40}$$

The present finite-element method can be extended to calculate the vibration frequencies of Timoshenko nanobeams by considering the shear effects.

5. Numerical results: In this section, several figures and tables present the vibrational frequencies for a nanobeam with different power law exponent. Several numerical examples are solved in order to point out to the possibility of enhancing or decreasing the frequencies for different (a and e_0) parameter. It also shows the effectiveness of the present method to capture significance of the power law exponent. Two types of alumina/steel and SiC/SUS304 FG nanobeams are considered here to investigate their frequencies. Ceramic and metal are the most commonly used materials to form FGM where the bottom surfaces of the beams are pure metal, whereas the top surfaces of the beams are pure ceramic. Mechanical properties of nanobeams constituents are given in Table 1. Geometrical properties of the FG nanobeams are: b (width) = 100 nm, h (thickness) = 200 nm and L (length) = 4000 nm.

The frequency values obtained from the analyses of both types of FGM with various dimensionless small-scale parameters (e_0a/L) ranging from 0 to 0.4 and various power law exponents (k) ranging from 0 to 10 are presented in Tables 2 and 3, respectively.

According to the obtained results, the frequency values of both FG nanobeams increase as the mode number increase. However, frequencies of the both FG nonobeams decrease as dimensionless small-scale parameters (e_0a/L) increases. Similarly, increasing in k values cause decreasing in frequency values of the both types of FGM nonobeams. Also, as the k increases, the properties of the beams transform from ceramic to metal (Figs. 2 and 3).

Table 1 Properties of FG nanobeams constituents

Constituents	Material	E, GPa	P, kg/cm^3
alumina (Al_2O_3) [13]	ceramic	390	3960
steel [13]	metal	210	7800
silicon carbide (SiC) [20]	ceramic	427	3210
stainless steel (SUS304) [20]	metal	207.78	8166

Table 2 Variation of first five frequencies (MHz) of FG nanobeam with k and e_0a/L (SiC/SUS304)

ω	$k=0$				
	e_0a/L				
	0	0.1	0.2	0.3	0.4
ω_1	65.3062	62.3040	55.2969	47.5251	40.6647
ω_2	260.4247	220.5102	162.1604	122.0480	96.2784
ω_3	582.9910	424.2584	273.2186	194.3911	149.4739
ω_4	1029.1840	640.8491	380.4871	263.8740	200.8142
ω_5	1593.8916	855.9662	483.4502	330.8665	250.5226
ω			$k=2$		
			e_0a/L		
	0	0.1	0.2	0.3	0.4
ω_1	38.1371	36.3838	32.2919	27.7534	23.7471
ω_2	152.1044	128.7919	94.7119	71.2837	56.2327
ω_3	340.5901	247.8567	159.6175	113.5655	87.3244
ω_4	601.4701	374.5216	222.3622	154.2118	117.3587
ω_5	931.8969	500.4558	282.6577	193.4469	146.4725
ω			$k=10$		
			e_0a/L		
	0	0.1	0.2	0.3	0.4
ω_1	32.2792	30.7953	27.3319	23.4905	20.0995
ω_2	128.7479	109.0151	80.1683	60.3377	47.5978
ω_3	288.3145	209.8143	135.1185	96.1349	73.9214
ω_4	509.2114	317.0742	188.2544	130.5575	99.3573
ω_5	789.0671	423.7520	239.3354	163.7977	124.0229

It is seen from Fig. 4 that when the e_0a/L increases, the frequency values of alumina/steel FG nanobeam and the frequency values of SiC/SUS304 FG nanobeam approach each other.

It is seen from Fig. 5 that the frequency values of SiC/SUS304 FG nanobeam are higher than alumina/steel FG nanobeam while k is small. As the k value increases, the difference between the frequencies of FG nanobeams is closed. After a certain k value, the frequency values of alumina/steel FG nanobeam become larger than the frequency values of SiC/SUS304 FG nanobeam.

Table 3 Variation of first five frequencies (MHz) of FG nanobeam with k and e_0a/L (alumina/steel)

ω	$k=0$				
	e_0a/L				
	0	0.1	0.2	0.3	0.4
ω_1	56.1924	53.6092	47.5800	40.8928	34.9897
ω_2	224.0812	189.736	139.530	105.015	82.8423
ω_3	501.6319	365.051	235.089	167.262	128.614
ω_4	885.5566	551.415	327.388	227.049	172.789
ω_5	1371.456	736.512	415.982	284.692	215.561
ω			$k=2$		
			e_0a/L		
	0	0.1	0.2	0.3	0.4
ω_1	37.2402	35.5282	31.5325	27.1007	23.1886
ω_2	148.5221	125.7585	92.4813	69.6048	54.9083
ω_3	332.5494	242.0052	155.8492	110.8844	85.2628
ω_4	587.2244	365.6511	217.0956	150.5593	114.5791
ω_5	909.7359	488.5547	275.9360	188.8467	142.9893
ω			$k=10$		
			e_0a/L		
	0	0.1	0.2	0.3	0.4
ω_1	32.4872	30.9937	27.5080	23.6418	20.2290
ω_2	129.5722	109.7130	80.6816	60.7240	47.9025
ω_3	290.1410	211.1435	135.9745	96.7439	74.3897
ω_4	512.3905	319.0537	189.4296	131.3725	99.9775
ω_5	793.9027	426.3489	240.8021	164.8015	124.7830

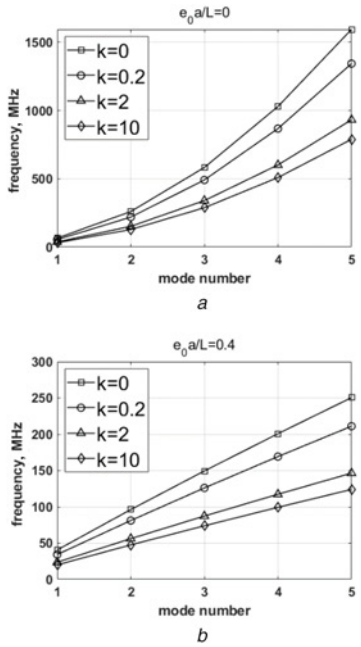

Fig. 2 *Effect of k on mode number (SiC/SUS304)*
a $e_0a/L = 0$
b $e_0a/L = 0.4$

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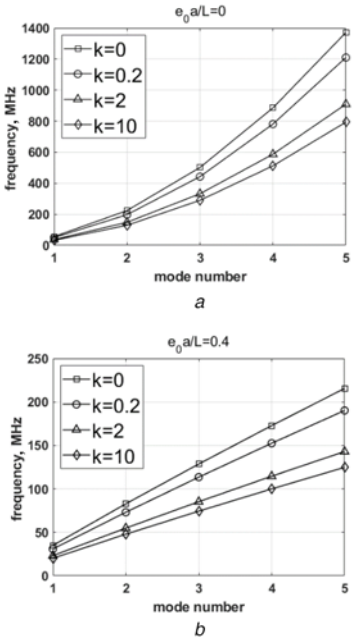

Fig. 3 *Effect of k on mode number (alumina/steel)*
a $e_0a/L = 0$
b $e_0a/L = 0.4$

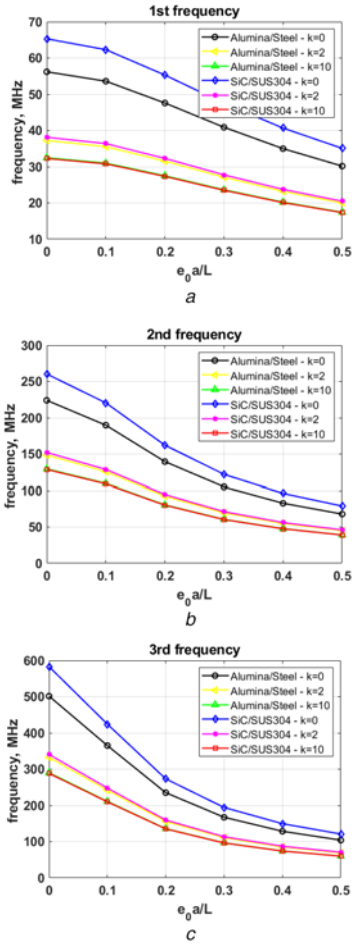

Fig. 4 *Variation of the frequencies with non-dimensional non-local parameters for different k*
a First frequency
b Second frequency
c Third frequency

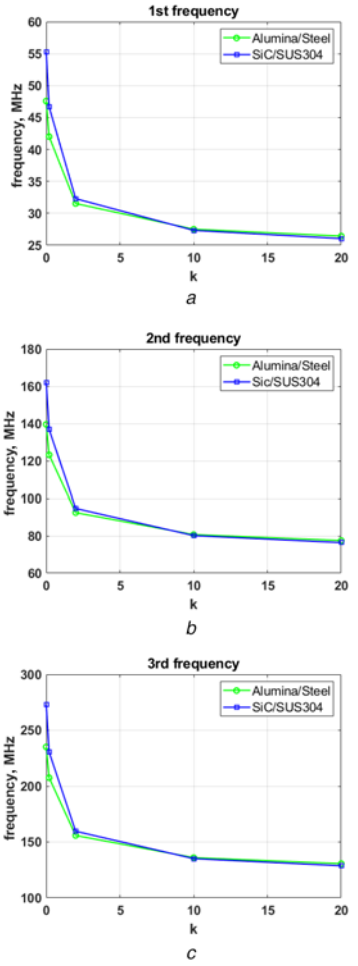

Fig. 5 *Variation of the frequencies with k ($e_0a/L = 0.2$)*
a First frequency
b Second frequency
c Third frequency

6. Conclusion: A non-local finite-element formulation is developed to analyse the non-local free vibration behaviour of FG nanobeams where the mechanical properties are assumed to vary continuously through the thickness and obey a power law distribution of the volume fraction of the constituents. Two types of FG nanobeams composed of two different materials are analysed by using the developed non-local finite-element formulation. As a result of performed analyses it has found that frequencies of the FG nonobeams decrease as dimensionless small-scale parameters (e_0a/L) increases. Similarly, increasing in k values cause decreasing in frequency values of the FG nonobeams.

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