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On a New Subclass of Harmonic Univalent Functions

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ABSTRACT

In the acquaint article, we scrutinize some fundamental attribute of a subclass of harmonic univalent functions defined by a new alteration. Like these, coefficient disparities, distortion bounds, convolutions, convex combinations and extreme points.

Keywords: Harmonic, univalent, a new linear operator, multiplier transformation, distortion bounds.

1. Introduction

Let $\mathbb{D} = \{\mathsf{z} : |\mathsf{z}| < 1\}$ indicates the open unit disk and let $\mathcal{H}$ denotes the family of continuous complex valued harmonic functions in $\mathbb{D}$. Let $\mathcal{A}$ denotes the class of functions which are analytic in $\mathbb{D}$. It is clear that $\mathcal{A}$ is a subclass of $\mathcal{H}$. If $\mathfrak{f}$ and $\mathfrak{g}$ are selected from $\mathcal{A}$, harmonic $\mathfrak{f}$ function in $\mathbb{D}$ can be expressed as $\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}}$. It is usually called the analytic part of $\mathfrak{f}$ for the $\mathfrak{h}$ function and the co-analytic part for the $\mathfrak{g}$ function. We know that $\mathcal{S}$ denotes the class of normalized analytic univalent functions in $\mathbb{D}$. Attention that if the co-analytic part's members are zero, then $\mathcal{H}$ degrades to the class of $\mathcal{S}$. A sufficient and necessary condition for $\mathfrak{f}$ to be sense-preserving and locally univalent in $\mathbb{D}$ is that $|\mathfrak{h}'(\mathsf{z})| > |\mathfrak{g}'(\mathsf{z})|$ (see Clunie and Sheil-Small (1984)). $\mathcal{SH}$ denotes the class of functions $\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}}$ which are harmonic univalent and sense-preserving in the unit disk $\mathbb{D}$ for which $\mathfrak{f}(0) = \mathfrak{f}_{\mathsf{z}}(0) - 1 = 0$. Also, attention that if the co-analytic part of $\mathfrak{f}$ function is zero, then $\mathcal{SH}$ reduces to $\mathcal{S}$. Then we can state $\mathfrak{h}$ and $\mathfrak{g}$ analytic functions as for $\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}}$ as follows

$$\mathfrak{h}(\mathsf{z}) = \mathsf{z} + \sum_{j=2}^{\infty} a_j \mathsf{z}^j \quad \text{and} \quad \mathfrak{g}(\mathsf{z}) = \sum_{j=1}^{\infty} b_j \mathsf{z}^j. \tag{1}$$

One demonstrates clearly that the sense-preserving feature alludes to $|b_1| < 1$. The subclass $\mathcal{SH}^0$ of $\mathcal{SH}$ contains entire functions in $\mathcal{SH}$ which have the extra feature $\mathfrak{f}_{\overline{\mathsf{z}}}(0) = 0$.

Geometric functions theory has been studied a lot in recent years (For example; Olatunji and Dutta (2019) , Kumar and Ravichandran (2017)).

Clunie and Sheil-Small (1984) researched $\mathcal{SH}$ class's geometric subclasses as well as some coefficient bounds. Since then, there have been many articles about $\mathcal{SH}$ and related subclasses.

For $\mathfrak{f} \in \mathcal{S}$, the differential operator D^n $(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ of $\mathfrak{f}$ was acquainted by Salagean (1983). This operator was developed and modified by many researchers over time. As a simple example for $\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}}$ given by (1), Jahangiri et al. (2002) defined the modified Salagean operator of $\mathfrak{f}$ as

$$D^n \mathfrak{f}(\mathsf{z}) = D^n \mathfrak{h}(\mathsf{z}) + (-1)^n \overline{D^n \mathfrak{g}(\mathsf{z})},$$

where

$$D^n \mathfrak{h}(\mathsf{z}) = \mathsf{z} + \sum_{j=2}^{\infty} j^n a_j \mathsf{z}^j \quad \text{and} \quad D^n \mathfrak{g}(\mathsf{z}) = \sum_{j=1}^{\infty} j^n b_j \mathsf{z}^j.$$

Now, for $\mathfrak{f} \in \mathcal{A}$ functions, let $\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}}$ like (1), we define the modified multiplier transformation of $\mathfrak{f}$

$$I_{\vartheta}^{0,\zeta}(\varrho,\xi)\mathfrak{f}(\mathsf{z})=\mathfrak{f}(\mathsf{z}),$$

$$I_{\vartheta}^{1,\zeta}(\varrho,\xi)\mathfrak{f}(\mathsf{z})=\frac{\zeta-\xi+\varrho-\vartheta}{\zeta+\varrho}\mathfrak{f}(\mathsf{z})+\frac{\xi+\vartheta}{\zeta+\varrho}(\mathsf{z}\mathfrak{f}_{\mathsf{z}}(\mathsf{z})-\overline{\mathsf{z}}\mathfrak{f}_{\overline{\mathsf{z}}}(\mathsf{z})) \tag{2}$$

$$I_{\vartheta}^{n,\zeta}(\varrho,\xi)\mathfrak{f}(\mathsf{z})=I_{\vartheta}^{1,\zeta}(\varrho,\xi)\left(I_{\vartheta}^{n-1,\zeta}(\varrho,\xi)\mathfrak{f}(\mathsf{z})\right).\ (n\in\mathbb{N}_0) \tag{3}$$

Where $\zeta,\xi,\vartheta,\varrho>0$. If $\mathfrak{f}$ is given by (1), then from (2) and (3) we see that

$$\begin{aligned} I_{\vartheta}^{n,\zeta}(\varrho,\xi)\mathfrak{f}(\mathsf{z}) &= \mathsf{z}+\sum_{j=2}^{\infty}\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}\right]^n a_j\mathsf{z}^j \\ &+ (-1)^n\sum_{j=1}^{\infty}\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}\right]^n\overline{b_j\mathsf{z}^j}. \end{aligned} \tag{4}$$

Let $\mathfrak{f}$ is given by (1). Thus we obtain that

$$\begin{aligned} I_{\vartheta}^{n,\zeta}(\varrho,\xi)\mathfrak{f}(\mathsf{z}) &:= \mathfrak{f}(\mathsf{z})*\underbrace{\phi_{\varrho,\vartheta}^{\xi,\zeta}(\mathsf{z})*...*\phi_{\varrho,\vartheta}^{\xi,\zeta}(\mathsf{z})}_{n\ times} \\ &= \mathfrak{h}*\underbrace{\phi_{1,\varrho,\vartheta}^{\xi,\zeta}(\mathsf{z})*...*\phi_{1,\varrho,\vartheta}^{\xi,\zeta}(\mathsf{z})}_{n\ times}+\overline{\mathfrak{g}*\underbrace{\phi_{2,\varrho,\vartheta}^{\xi,\zeta}*...*\phi_{2,\varrho,\vartheta}^{\xi,\zeta}}_{n\ times}}, \end{aligned} \tag{5}$$

where "$*$" shows convolution of power series or the usual Hadamard product and

$$\begin{aligned} \phi_{\varrho,\vartheta}^{\xi,\zeta}(\mathsf{z}) &= \frac{\mathsf{z}-\left(\frac{\zeta-\xi+\varrho-\vartheta}{\zeta+\varrho}\right)\mathsf{z}^2}{(1-\mathsf{z})^2}+\frac{\left[1-\frac{2(\xi+\vartheta)}{\zeta+\varrho}\right]\overline{\mathsf{z}}-\left[1-\frac{\xi+\vartheta}{\zeta+\varrho}\right]\overline{\mathsf{z}}^2}{(1-\overline{\mathsf{z}})^2} \\ &= \phi_{1,\varrho,\vartheta}^{\xi,\zeta}(\mathsf{z})+\overline{\phi_{2,\varrho,\vartheta}^{\xi,\zeta}(\mathsf{z})} \end{aligned}$$

If special numbers are selected for the parameters n,ζ,ϑ,ϱ and ξ The following operators, which are examined by various authors, are obtained:

for $\mathfrak{f}\in\mathcal{A}$,

(i) $I_{1}^{n,1}(0,0)\mathfrak{f}(\mathsf{z})=D^n\mathfrak{f}(\mathsf{z})$ (Salagean (1983)),

(ii) $I_{\vartheta}^{n,1}(\lambda,0)\mathfrak{f}(\mathsf{z})=I_{\vartheta}^{n}\mathfrak{f}(\mathsf{z})$ (Cho and Srivastava (2003), Cho and Kim (2003), Flett (1972)),

(iii) $I_{1}^{n,1}(1,0)\mathfrak{f}(\mathsf{z})=I^n\mathfrak{f}(\mathsf{z})$ (Uralegaddi and Somanatha (1992)),

(iv) $I_{\vartheta}^{n,1}(0,0)\mathfrak{f}(\mathsf{z}) = D_{\vartheta}^{n}\mathfrak{f}(\mathsf{z})$ (Al-Oboudi (2004)),

(v) $I_{\vartheta}^{n,1}(l,0)\mathfrak{f}(\mathsf{z}) = D^{n}(\vartheta,l)\mathfrak{f}(\mathsf{z}); l > 0$ (Catas (2009))

for $\mathfrak{f} \in \mathcal{H}$,

(iv) $I_{1}^{n,1}(0,0)\mathfrak{f}(\mathsf{z}) = D^{n}\mathfrak{f}(\mathsf{z})$ (Jahangiri et al. (2002)),

(v) $I_{1}^{n,1}(\gamma,0)\mathfrak{f}(\mathsf{z}) = I_{\gamma}^{n}\mathfrak{f}(\mathsf{z}); \gamma > 0$ (Yasar and Yalcin (2012)),

(vi) $I_{\vartheta}^{n,1}(0,0)\mathfrak{f}(\mathsf{z}) = D_{\vartheta}^{n}\mathfrak{f}(\mathsf{z})$ (Yasar and Yalcin (2013)),

(vii) $I_{\varrho}^{n,\gamma}(\varrho,0)\mathfrak{f}(\mathsf{z}) = I_{\gamma,\varrho}^{n}\mathfrak{f}(\mathsf{z})$ (Bayram and Yalcin (2017)).

$\mathcal{SH}(\zeta,\vartheta,\varrho,\xi,n,\delta)$ represents the subclass of $\mathcal{SH}$ comprising of functions $\mathfrak{f}$ in type (1) which provide below the circumstance

$$\operatorname{Re}\left(\frac{I_{\vartheta}^{n+1,\zeta}(\varrho,\xi)\mathfrak{f}(\mathsf{z})}{I_{\vartheta}^{n,\zeta}(\varrho,\xi)\mathfrak{f}(\mathsf{z})}\right) \geq \delta, \quad 0 \leq \delta < 1 \tag{6}$$

where $I_{\vartheta}^{n,\zeta}\mathfrak{f}(\mathsf{z})$ is described by (4).

We allow to the subclass $\overline{\mathcal{SH}}(\zeta,\vartheta,\varrho,\xi,n,\delta)$ occurring of harmonic functions $\mathfrak{f}_n = \mathfrak{h} + \overline{\mathfrak{g}}_n$ in $\mathcal{SH}$, therefore, $\mathfrak{h}$ and $\mathfrak{g}_n$ are in type

$$\mathfrak{h}(\mathsf{z}) = \mathsf{z} - \sum_{j=2}^{\infty} a_j \mathsf{z}^j, \ \mathfrak{g}_n(\mathsf{z}) = (-1)^n \sum_{j=1}^{\infty} b_j \mathsf{z}^j, \quad a_j, \ b_j \geq 0. \tag{7}$$

If the parameters are chosen appropriately, $\mathcal{SH}(\zeta,\vartheta,\varrho,\xi,n,\delta)$ classes are reduced to different subclasses of harmonic univalent functions. Like,

(i) $\mathcal{SH}(1,1,0,0,0,0) = \mathcal{SH}^*(0)$ (Avci and Zlotkiewicz (1990), Silverman (1998), Silverman and Silvia (1999)),

(ii) $\mathcal{SH}(1,1,0,0,0,\delta) = \mathcal{SH}^*(\delta)$ (Jahangiri (1999)),

(iii) $\mathcal{SH}(1,1,0,0,1,0) = \mathcal{KH}(0)$ (Avci and Zlotkiewicz (1990), Silverman (1998), Silverman and Silvia (1999)),

(iv) $\mathcal{SH}(1,1,0,0,1,\delta) = \mathcal{KH}(\delta)$ (Jahangiri (1999)),

(v) $\mathcal{SH}(1,1,0,0,n,\delta) = \mathcal{H}(n,\delta)$ (Jahangiri et al. (2002)),

(vi) $\mathcal{SH}(1,1,\varrho,0,n,\delta)=\mathcal{SH}(\gamma,n,\delta)$ (Yasar and Yalcin (2012)),

(vii) $\mathcal{SH}(1,\vartheta,0,0,n,\delta)=\mathcal{SH}(\vartheta,n,\delta)$ (Yasar and Yalcin (2013)),

(viii) $\mathcal{SH}(\gamma,\varrho,\varrho,0,n,\delta)=\mathcal{SH}(\gamma,\varrho,n,\delta)$ (Bayram and Yalcin (2017)),

Define $\mathcal{SH}^0(\zeta,\vartheta,\varrho,\xi,n,\delta):=\mathcal{SH}(\zeta,\vartheta,\varrho,\xi,n,\delta)\cap\mathcal{SH}^0$ and

$$\overline{\mathcal{SH}}^0(\zeta,\vartheta,\varrho,\xi,n,\delta):=\overline{\mathcal{SH}}(\zeta,\vartheta,\varrho,\xi,n,\delta)\cap SH^0.$$

2. Primary Conclusions

Theorem 2.1. *Let $\mathfrak{f}=\mathfrak{h}+\overline{\mathfrak{g}}$. Let $\mathfrak{h}$ and $\mathfrak{g}$ are given by (1) with $b_1=0$. Let*

$$\sum_{j=2}^{\infty}\left[\tfrac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}\right]^n\left[\tfrac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}-\delta\right]|a_j|+$$
$$\sum_{j=2}^{\infty}\left[\tfrac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}\right]^n\left[\tfrac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}+\delta\right]|b_j|\leq 1-\delta, \qquad (8)$$

where $\xi+\vartheta\geq 2(\zeta+\varrho)$, $n\in\mathbb{N}_0$, $0\leq\delta<1$. In that case $\mathfrak{f}$ is harmonic univalent, sense-preserving in $\mathbb{D}$ and $\mathfrak{f}\in\mathcal{SH}^0(\zeta,\vartheta,\varrho,\xi,n,\delta)$.

As a special notation for convenience, we make

$$L_n=\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}\right]^n$$

and

$$M_n=\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}\right]^n$$

in this article.

Proof. If $\mathsf{z}_1 \neq \mathsf{z}_2$,

$$\begin{aligned}
\left|\frac{\mathfrak{f}(\mathsf{z}_1)-\mathfrak{f}(\mathsf{z}_2)}{\mathfrak{h}(\mathsf{z}_1)-\mathfrak{h}(\mathsf{z}_2)}\right| &\geq 1-\left|\frac{\mathfrak{g}(\mathsf{z}_1)-\mathfrak{g}(\mathsf{z}_2)}{\mathfrak{h}(\mathsf{z}_1)-\mathfrak{h}(\mathsf{z}_2)}\right| = 1-\left|\frac{\sum\limits_{j=2}^{\infty} b_j\left(\mathsf{z}_1^j-\mathsf{z}_2^j\right)}{(\mathsf{z}_1-\mathsf{z}_2)+\sum\limits_{j=2}^{\infty} a_j\left(\mathsf{z}_1^j-\mathsf{z}_2^j\right)}\right| \\
&> 1-\frac{\sum\limits_{j=2}^{\infty} j\left|b_j\right|}{1-\sum\limits_{j=2}^{\infty} j\left|a_j\right|} \\
&\geq 1-\frac{\sum\limits_{j=2}^{\infty}\frac{M_n\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}\right]}{1-\delta}\left|b_j\right|}{1-\sum\limits_{j=2}^{\infty}\frac{L_n\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}\right]}{1-\delta}\left|a_j\right|} \geq 0,
\end{aligned}$$

that demonstrates univalence. The attention that $\mathfrak{f}$ is sense-preserving in $\mathbb{D}$. Therefore

$$\begin{aligned}
\left|\mathfrak{h}'(\mathsf{z})\right| &\geq 1-\sum_{j=2}^{\infty} j\left|a_j\right|\left|\mathsf{z}\right|^{j-1} \\
&> 1-\sum_{j=2}^{\infty}\frac{L_n\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}-\delta\right]}{1-\delta}\left|a_j\right| \\
&\geq \sum_{j=2}^{\infty}\frac{M_n\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}+\delta\right]}{1-\delta}\left|b_j\right| \\
&> \sum_{j=2}^{\infty} j\left|b_j\right|\left|\mathsf{z}\right|^{j-1} \\
&\geq \left|\mathfrak{g}'(\mathsf{z})\right|.
\end{aligned}$$

If we use the consubstantiality that $\mathrm{Re}\,\omega \geq \delta \Leftrightarrow |1-\delta+\omega| \geq |1+\delta-\omega|$, it suffices to prove that

$$\left|(1-\delta)I_{\vartheta}^{n,\zeta}\mathfrak{f}(\mathsf{z})+I_{\vartheta}^{n+1,\zeta}\mathfrak{f}(\mathsf{z})\right|-\left|(1+\delta)I_{\vartheta}^{n,\zeta}\mathfrak{f}(\mathsf{z})-I_{\vartheta}^{n+1,\zeta}\mathfrak{f}(\mathsf{z})\right| \geq 0. \qquad (9)$$

Substituting for $I_\vartheta^{n,\zeta}\mathfrak{f}(\mathsf{z})$ and $I_\vartheta^{n+1,\zeta}\mathfrak{f}(\mathsf{z})$ in (9), we have

$$\begin{aligned}
&\left|(1-\delta)I_\vartheta^{n,\zeta}\mathfrak{f}(\mathsf{z})+I_\vartheta^{n+1,\zeta}\mathfrak{f}(\mathsf{z})\right|-\left|(1+\delta)I_\vartheta^{n,\zeta}\mathfrak{f}(\mathsf{z})-I_\vartheta^{n+1,\zeta}\mathfrak{f}(\mathsf{z})\right|\\
\geq\ & 2(1-\delta)\,|\mathsf{z}|-\sum_{j=2}^{\infty}L_n\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}+1-\delta\right]|a_j|\,|\mathsf{z}|^j\\
&-\sum_{j=2}^{\infty}M_n\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}-1+\delta\right]|b_j|\,|\mathsf{z}|^j\\
&-\sum_{j=2}^{\infty}L_n\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}-1-\delta\right]|a_j|\,|\mathsf{z}|^j\\
&-\sum_{j=2}^{\infty}M_n\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}+1+\delta\right]|b_j|\,|\mathsf{z}|^j\\
>\ & 2(1-\delta)\,|\mathsf{z}|\left\{1-\sum_{j=2}^{\infty}L_n\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}-\delta\right]|a_j|\right.\\
&\left.-\sum_{j=2}^{\infty}M_n\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}+\delta\right]|b_j|\right\}.
\end{aligned}$$

Then the last statement is not negative by (8). □

Theorem 2.2. *Let $\mathfrak{f}_n=\mathfrak{h}+\overline{\mathfrak{g}}_n$ be given by (7) with $b_1=0$. Then $\mathfrak{f}_n\in\overline{\mathcal{SH}}^0(\zeta,\vartheta,\varrho,\xi,n,\delta)$ if and only if*

$$\sum_{j=2}^{\infty}L_n\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}-\delta\right]a_j+\sum_{j=2}^{\infty}M_n\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}+\delta\right]b_j \leq 1-\delta, \tag{10}$$

where $\xi+\vartheta\geq 2(\zeta+\varrho)$, $n\in\mathbb{N}_0$, $0\leq\delta<1$.

Proof. The "if" part of the proof is obtained by Theorem 1 $\overline{\mathcal{SH}}^0(\zeta,\vartheta,\varrho,\xi,n,\delta)\subset\mathcal{SH}^0(\zeta,\vartheta,\varrho,\xi,n,\delta)$. To show the "only if" part, we need to show $\mathfrak{f}_n\notin\overline{\mathcal{SH}}^0(\zeta,\vartheta,\varrho,\xi,n,\delta)$ if the stipulation (10) doesn't hold. Attention that a sufficient and necessary condition for $\mathfrak{f}_n=\mathfrak{h}+\overline{\mathfrak{g}}_n$ given by (7), to be in $\overline{\mathcal{SH}}^0(\zeta,\vartheta,\varrho,\xi,n,\delta)$ is that (6) to be satisfied. This is same with

$$\operatorname{Re}\left\{\frac{(1-\delta)\mathsf{z}-\sum_{j=2}^{\infty}L_n\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}-\delta\right]a_j\mathsf{z}^j-\sum_{j=2}^{\infty}M_n\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}+\delta\right]b_j\overline{\mathsf{z}}^j}{\mathsf{z}-\sum_{j=2}^{\infty}L_na_j\mathsf{z}^j+\sum_{j=2}^{\infty}M_nb_j\overline{\mathsf{z}}^j}\right\}\geq 0.$$

The above stipulation must hold for all values of $|\mathsf{z}| = r < 1$. With selecting these values of z on the positive real axis where $0 \leq \mathsf{z} = r < 1$. We ought to have

$$\frac{1-\delta-\sum_{j=2}^{\infty}\left(L_n\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}-\delta\right]a_j-M_n\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}+\delta\right]b_j\right)r^{j-1}}{1-\sum_{j=2}^{\infty}L_na_jr^{j-1}+\sum_{j=2}^{\infty}M_nb_jr^{j-1}}\geq 0 \tag{11}$$

If the stipulation (10) is not valid, then the expression in (11) is negative for r values approaching to 1. Therefore there exist $\mathsf{z}_0 = r_0$ in $(0, 1)$ for which the quotient in (11) is negative.
This shows the required stipulation for $f_n \in \overline{\mathcal{SH}}^0(\zeta, \vartheta, \varrho, \xi, n, \delta)$. □

Theorem 2.3. *Let $\mathfrak{f}_n$ be given by (7). For the $\mathfrak{f}_n$ functions to be in the $\overline{\mathcal{SH}}^0(\zeta, \vartheta, \varrho, \xi, n, \delta)$ class, a necessary and sufficient condition is*

$$\mathfrak{f}_n(\mathsf{z}) = \sum_{j=1}^{\infty}\left(X_j\mathfrak{h}_j(\mathsf{z}) + Y_j\mathfrak{g}_{n_j}(\mathsf{z})\right),$$

where

$$\mathfrak{h}_1(\mathsf{z}) = \mathsf{z}, \quad \mathfrak{h}_j(\mathsf{z}) = \mathsf{z} - \frac{1-\delta}{L_n\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}-\delta\right]}\mathsf{z}^j \quad (j = 2, 3, ...),$$

and for $j = 2, 3, ...$

$$\mathfrak{g}_{n_1}(\mathsf{z}) = \mathsf{z}, \quad \mathfrak{g}_{n_j}(\mathsf{z}) = \mathsf{z} + (-1)^n\frac{1-\delta}{M_n\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}+\delta\right]}\overline{\mathsf{z}}^j$$

$$X_j \geq 0,\ Y_j \geq 0,\ \sum_{j=1}^{\infty}(X_j + Y_j) = 1,\ \ \xi + \vartheta \geq 2(\zeta + \varrho),\ n \in \mathbb{N}_0,\ 0 \leq \delta < 1.$$

Especially, the extreme points of $\overline{\mathcal{SH}}^0(\zeta,\vartheta,\varrho,\xi,n,\delta)$ *are* $\{\mathfrak{h}_j\}$ *and* $\{\mathfrak{g}_{n_j}\}$.

Proof. For $\mathfrak{f}_n$ functions in type (7) we have

$$\begin{aligned}\mathfrak{f}_n(\mathsf{z}) &= \sum_{j=1}^{\infty}\left(X_j\mathfrak{h}_j(\mathsf{z})+Y_j\mathfrak{g}_{n_j}(\mathsf{z})\right)\\ &= \sum_{j=1}^{\infty}(X_j+Y_j)\,\mathsf{z}-\sum_{j=2}^{\infty}\frac{1-\delta}{L_n\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}-\delta\right]}X_j\mathsf{z}^j\\ &\quad+(-1)^n\sum_{j=2}^{\infty}\frac{1-\delta}{M_n\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}+\delta\right]}Y_j\overline{\mathsf{z}}^j.\end{aligned}$$

Then

$$\begin{aligned}&\sum_{j=2}^{\infty}\frac{L_n\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}-\delta\right]}{1-\delta}\left(\frac{1-\alpha}{L_n\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}-\delta\right]}X_j\right)\\ &+\sum_{j=2}^{\infty}\frac{M_n\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}+\delta\right]}{1-\delta}\left(\frac{1-\delta}{M_n\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}+\delta\right]}Y_j\right)\\ =\ &\sum_{j=2}^{\infty}X_j+\sum_{j=2}^{\infty}Y_j=1-X_1-Y_1\le 1\end{aligned}$$

and so $\mathfrak{f}_n\in\overline{\mathcal{SH}}^0(\zeta,\vartheta,\varrho,\xi,n,\delta)$. Conversely, if $\mathfrak{f}_n\in\overline{\mathcal{SH}}^0(\zeta,\vartheta,\varrho,\xi,n,\delta)$, then

$$a_j\le\frac{1-\delta}{L_n\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}-\delta\right]}$$

and

$$b_j\le\frac{1-\delta}{M_n\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}+\delta\right]}.$$

Set

$$X_j=\frac{L_n\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}-\delta\right]}{1-\delta}a_j,\ (j=2,3,...)$$

$$Y_j=\frac{M_n\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}+\delta\right]}{1-\delta}b_j,\ (j=2,3,...)$$

and

$$X_1+Y_1=1-\left(\sum_{j=2}^{\infty}X_j+Y_j\right)$$

where $X_j,\ Y_j\geq 0$. If so, as necessary, we have

$$\mathfrak{f}_n(\mathsf{z})=(X_1+Y_1)\mathsf{z}+\sum_{j=2}^{\infty}X_j\mathfrak{h}_j(\mathsf{z})+\sum_{j=2}^{\infty}Y_j\mathfrak{g}_{n_j}(\mathsf{z})=\sum_{j=1}^{\infty}\left(X_j\mathfrak{h}_j(\mathsf{z})+Y_j\mathfrak{g}_{n_j}(\mathsf{z})\right).$$

□

Theorem 2.4. *Let $\mathfrak{f}_n\in\overline{\mathcal{SH}}^0(\zeta,\vartheta,\varrho,\xi,n,\delta)$. Then for $|\mathsf{z}|=r<1$ and $\xi+\vartheta\geq 2(\zeta+\varrho)$, $n\in\mathbb{N}_0$, $0\leq\delta<1$. we have*

$$|\mathfrak{f}_n(\mathsf{z})|\leq r+\frac{1-\delta}{\left[\frac{\zeta+\xi+\varrho+\vartheta}{\zeta+\varrho}\right]^n\left[\frac{\zeta+\xi+\varrho+\vartheta}{\zeta+\varrho}-\delta\right]}r^2,$$

and

$$|\mathfrak{f}_n(\mathsf{z})|\geq r-\frac{1-\delta}{\left[\frac{\zeta+\xi+\varrho+\vartheta}{\zeta+\varrho}\right]^n\left[\frac{\zeta+\xi+\varrho+\vartheta}{\zeta+\varrho}-\delta\right]}r^2.$$

Proof. Here we only will prove the rightside of the inequality. The leftside of the inequality might be shown like this way. Let $\mathfrak{f}_n\in\overline{\mathcal{SH}}^0(\zeta,\vartheta,\varrho,\xi,n,\delta)$. If we take absolute value of $\mathfrak{f}_n$, then we obtain

$$\begin{aligned}|\mathfrak{f}_n(\mathsf{z})|&\leq r+\sum_{j=2}^{\infty}(a_j+b_j)\,r^2\\&\leq r+\frac{(1-\delta)\,r^2}{\left[\frac{\zeta+\xi+\varrho+\vartheta}{\zeta+\varrho}\right]^n\left[\frac{\zeta+\xi+\varrho+\vartheta}{\zeta+\varrho}-\delta\right]}\sum_{j=2}^{\infty}\frac{L_n\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}-\delta\right]}{1-\delta}a_j\\&+\frac{(1-\delta)\,r^2}{\left[\frac{\zeta+\xi+\varrho+\vartheta}{\zeta+\varrho}\right]^n\left[\frac{\zeta+\xi+\varrho+\vartheta}{\zeta+\varrho}-\delta\right]}\sum_{j=2}^{\infty}\frac{M_n\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}+\delta\right]}{1-\delta}b_j\\&\leq r+\frac{(1-\delta)}{\left[\frac{\zeta+\xi+\varrho+\vartheta}{\zeta+\varrho}\right]^n\left[\frac{\zeta+\xi+\varrho+\vartheta}{\zeta+\varrho}-\delta\right]}r^2.\end{aligned}$$

We can obtain covering result in following corollary with the left-hand side inequality in Theorem 2.4. □

Corollary 2.1. *Let $\mathfrak{f}_n$ of the form (7) be so that $\mathfrak{f}_n \in \overline{\mathcal{SH}}^0(\zeta, \vartheta, \varrho, \xi, n, \delta)$, where $\xi + \vartheta \geq 2(\zeta + \varrho)$, $n \in \mathbb{N}_0$, $0 \leq \delta < 1$. Then*

$$\left\{ \mathsf{w} : |\mathsf{w}| < 1 - \frac{(1-\delta)}{\left[\frac{\zeta+\xi+\varrho+\vartheta}{\zeta+\varrho}\right]^n \left[\frac{\zeta+\xi+\varrho+\vartheta}{\zeta+\varrho} - \delta\right]} \right\} \subset \mathfrak{f}_n(\mathbb{D}).$$

Theorem 2.5. *Under convex combinations $\overline{\mathcal{SH}}^0(\zeta, \vartheta, \varrho, \xi, n, \delta)$ is closed.*

Proof. Let $\mathfrak{f}_{n_i} \in \overline{\mathcal{SH}}^0(\zeta, \vartheta, \varrho, \xi, n, \delta)$ for $i = 1, 2, ...$, where $\mathfrak{f}_{n_i}$ is given by

$$\mathfrak{f}_{n_i}(\mathsf{z}) = \mathsf{z} - \sum_{j=2}^{\infty} a_{j_i} \mathsf{z}^j + (-1)^n \sum_{j=2}^{\infty} b_{j_i} \overline{\mathsf{z}}^j.$$

Then by (10),

$$\sum_{j=2}^{\infty} \frac{L_n \left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho} - \delta\right]}{1-\delta} a_{j_i} + \sum_{j=2}^{\infty} \frac{M_n \left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho} + \delta\right]}{1-\delta} b_{j_i} \leq 1. \quad (12)$$

For $\sum_{i=1}^{\infty} \mathfrak{t}_i = 1$, $0 < \mathfrak{t}_i < 1$, the convex combination of $\mathfrak{f}_{n_i}$ can be expressed as

$$\sum_{i=1}^{\infty} \mathfrak{t}_i \mathfrak{f}_{n_i}(z) = z - \sum_{j=2}^{\infty} \left(\sum_{i=1}^{\infty} \mathfrak{t}_i a_{j_i} \right) \mathsf{z}^j + (-1)^n \sum_{j=2}^{\infty} \left(\sum_{i=1}^{\infty} \mathfrak{t}_i b_{j_i} \right) \overline{\mathsf{z}}^j.$$

Then by (12),

$$\sum_{j=2}^{\infty} \frac{L_n\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}-\delta\right]}{1-\delta}\left(\sum_{i=1}^{\infty} \mathfrak{t}_i a_{j_i}\right)$$
$$+\sum_{j=2}^{\infty} \frac{M_n\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}+\delta\right]}{1-\delta}\left(\sum_{i=1}^{\infty} \mathfrak{t}_i b_{j_i}\right)$$
$$=\sum_{i=1}^{\infty} \mathfrak{t}_i \sum_{j=2}^{\infty} \frac{L_n\left[\frac{\zeta+(\xi+\vartheta)(j-1)+\varrho}{\zeta+\varrho}-\delta\right]}{1-\delta} a_{j_i}$$
$$+\sum_{i=1}^{\infty} \mathfrak{t}_i \sum_{j=2}^{\infty} \frac{M_n\left[\frac{-\zeta+(\xi+\vartheta)(j+1)-\varrho}{\zeta+\varrho}+\delta\right]}{1-\delta} b_{j_i}$$
$$\leq \sum_{i=1}^{\infty} \mathfrak{t}_i = 1.$$

This is the condition required by (10) and so $\sum_{i=1}^{\infty} \mathfrak{t}_i \mathfrak{f}_{n_i}(\mathsf{z}) \in \overline{\mathcal{SH}}^0(\zeta,\vartheta,\varrho,\xi,n,\delta)$. □

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