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SOME NEW GENERALIZATIONS OF HADAMARD–TYPE MIDPOINT INEQUALITIES INVOLVING FRACTIONAL INTEGRALS

Abstract. In this study, we formulate the identity and obtain some generalized inequalities of the Hermite–Hadamard type by using fractional Riemann–Liouville integrals for functions whose absolute values of the second derivatives are convex. The results are obtained by uniformly dividing a segment $[a,b]$ into n equal sub-intervals. Using this approach, the absolute error of a Midpoint inequality is shown to decrease approximately n^2 times. A dependency between accuracy of the absolute error (ε) of the upper limit of the Hadamard inequality and the number (n) of lower intervals is obtained.

Key words: *convexity, Hadamard inequality, Hölder's inequality, Power–mean inequality, Riemann–Liouville fractional integrals.*

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1. Introduction. The well-known definition of a convex function in literature (for example, [4] and references therein) is

Definition 1. *The function $g : [a, b] \to R$ is said to be convex, if*

$$g\left(t\xi + (1-t)\zeta\right) \leqslant tg\left(\xi\right) + (1-t)\, g\left(\zeta\right) \tag{1}$$

for all $\xi, \zeta \in [a, b]$ and $t \in [0, 1]$.

It is well known that the double Hermite-Hadamard-type inequality plays a very important role in nonlinear analysis. This inequality is stated as follows [8]:

Theorem 1. *Let $g : I \subset R \to R$ be a convex function defined on an interval I and let $a, b \in I$, with $a < b$; then the following double inequality holds:*

$$g\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{b-a}\int\limits_a^b g(\xi)d\xi \leqslant \frac{g(a)+g(b)}{2}.$$

The theory of fractional integro–differential calculus, as an extension of classical analysis, plays an important role in pure and applied science. A. M. Nakhushev describes in the monograph [14] the fundamental elements and qualitatively new properties of fractional calculus operators. Moreover, their application to solving problems of mathematical modeling of various complex systems and processes (in living and non-living systems) has a fractal structure. Butkovsky et al. [3] presented the history of the development of fractional calculus and a detailed analysis of the problems of using fractional integro-differential calculus to describe dynamics of various systems and control processes. Examples of physical systems in need of the fractional analysis theory are presented.

In the literature, there are various definitions of the fractional integral (for example, see [13], [15]), but the Riemann–Liouville definition is among the most widely used in many applications of fractional calculus.

Definition 2. *[11] Let $g \in L_1[a,b]$. The Riemann–Liouville integrals $J^{\alpha}_{a+}g$ and $J^{\alpha}_{b-}g$ of order $\alpha > 0$ are defined, respectively, by*

$$J^{\alpha}_{a+}g\ (\xi) = \frac{1}{\Gamma(\alpha)}\int\limits_a^{\xi} (\xi - t)^{\alpha-1}\, g(t)\, dt\ , \qquad x > a$$

and

$$J^{\alpha}_{b-}g\ (\xi) = \frac{1}{\Gamma(\alpha)}\int\limits_{\xi}^{b} (t - \xi)^{\alpha-1}\, g(t)\, dt\ , \qquad x < b$$

where $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-u}u^{\alpha-1}du$ is the well known Gamma function. Here, for $\alpha = 0$, $J^0_{a+}g(\xi) = J^0_{b-}g(\xi) = g(\xi)$ and for $\alpha = 1$

$$J^1_{a+}g(\xi) = J^1_{b-}g(\xi) = \int\limits_a^b g(\xi)\, d\xi.$$

The vast majority of studies on the theory of integral inequalities use two classical inequalities. This is the Hölder's inequality and its other form: the power-mean inequality:

Theorem 2. *(The Hölder inequality [12]). Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $u(\xi)$ and $v(\xi)$ are real functions, defined on $[a, b]$, and $|u|^p$, $|v|^q \in L[a, b]$, then:*

$$\int_a^b |u(\xi)\, v(\xi)|\, d\xi \leqslant \Big(\int_a^b |u(\xi)|^p\, d\xi\Big)^{\frac{1}{p}} \Big(\int_a^b |v(\xi)|^q\, d\xi\Big)^{\frac{1}{q}}. \qquad (2)$$

This inequality turns into equality if and only if $A|g(\xi)|^p = B|g(\xi)|^q$ almost everywhere; A and B are constants.

Theorem 3. *(Power mean inequality [12]). Let $q \geqslant 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $u(\xi)$ and $v(\xi)$ are real functions, defined on $[a, b]$, and if $|u|^p$, $|v|^q \in L[a, b]$, then:*

$$\int_a^b |u(\xi)\, v(\xi)|\, d\xi \leqslant \Big(\int_a^b |u(\xi)|\, d\xi\Big)^{1-\frac{1}{q}} \Big(\int_a^b |u(\xi)|\, |v(\xi)|^q\, d\xi\Big)^{\frac{1}{q}}. \qquad (3)$$

In the recent years, many authors (see [1], [5–7], [10–13], [15] and references therein) have studied Hermite–Hadamard type inequalities involving fractional integrals for improvements and generalizations. In these papers, new inequalities for functions from various convexity classes were obtained. For example, in [5], [9], [16], some new Hadamard–type integral inequalities were obtained for functions whose first derivatives are convex functions and take values at the intermediate points of the integration interval.

In this study, using the identity formulated for fractional integrals, we obtain some new generalizations of the Hadamard type inequalities for functions whose absolute values of the second derivatives are convex. In addition, the obtained results clarify the errors of the quadrature formula for the numerical integration.

2. Main Results. Let $n \in N$ and $a, b \in R$ and let $a < b$, $n \geqslant 1$. The interval $[a, b]$ with a uniform step $h = \frac{b-a}{n}$ is divided into n subintervals $[a, b] = \bigcup_{k=1}^{n} [\xi_{k-1}, \xi_k]$, where $\xi_i = a + ih$, $i = 0, 1, 2, \ldots, n$.

Lemma 1. *Let $g : I \subset R \to R$ be a twice differentiable function on I°; I° is the interior of I. If $g'' \in L[a, b]$, where $a, b \in I$, then $\forall \alpha > 1$ the following equality holds:*

$$\frac{2^{\alpha-2}\Gamma(\alpha)}{h^{\alpha-1}} \sum_{k=1}^{n} \Big[J^{\alpha-1}_{\theta_k^-} g(\xi_{k-1}) + J^{\alpha-1}_{\theta_k^+} g(\xi_k)\Big] - \sum_{k=1}^{n} g(\theta_k) =$$

$$= \frac{2^{\alpha-2}h^2}{\alpha}\sum_{k=1}^{n}\left(I_{1k}+I_{2k}\right), \quad (4)$$

where

$$h=\frac{b-a}{n}, \quad \xi_i=a+ih, \quad i=0,1,2,\dots n, \quad \theta_k=\frac{\xi_{k-1}+\xi_k}{2},$$

$$I_{1k}=\int_0^{0.5} t^{\alpha}g''\left(\left(1-t\right)\xi_{k-1}+t\xi_k\right)dt,$$

$$I_{2k}=\int_{0.5}^{1}\left(1-t\right)^{\alpha}g''\left(\left(1-t\right)\xi_{k-1}+t\xi_k\right)dt.$$

Proof. Integrate the integrals under the sum operator twice by parts and take into account that $h=\xi_k-\xi_{k-1}$ to get

$$I_{1k}=\int_0^{0.5} t^{\alpha}g''\left(\left(1-t\right)\xi_{k-1}+t\xi_k\right)dt=$$

$$=\frac{1}{2^{\alpha}h}g'\left(\theta_k\right)-\frac{\alpha}{h}\int_0^{0.5} t^{\alpha-1}g'\left(\left(1-t\right)\xi_{k-1}+t\xi_k\right)dt=$$

$$=\frac{g'\left(\theta_k\right)}{2^{\alpha}h}-\frac{\alpha g\left(\theta_k\right)}{2^{\alpha-1}h^2}+\frac{\alpha\left(\alpha-1\right)}{h^2}\int_0^{0.5} t^{\alpha-2}g\left(\left(1-t\right)\xi_{k-1}+t\xi_k\right)dt.$$

Similarly, for the second integral:

$$I_{2k}=\int_{0.5}^{1}\left(1-t\right)^{\alpha}g''\left(\left(1-t\right)\xi_{k-1}+t\xi_k\right)dt=$$

$$=-\frac{1}{2^{\alpha}h}g'\left(\theta_k\right)+\frac{\alpha}{h}\int_{0.5}^{1}\left(1-t\right)^{\alpha-1}g'\left(\left(1-t\right)\xi_{k-1}+t\xi_k\right)dt=$$

$$=-\frac{g'\left(\theta_k\right)}{2^{\alpha}h}-\frac{\alpha g\left(\theta_k\right)}{2^{\alpha-1}h^2}+\frac{\alpha(\alpha-1)}{h^2}\int_{0.5}^{1}(1-t)^{\alpha-2}g\left(\left(1-t\right)\xi_{k-1}+t\xi_k\right)dt.$$

By summing these integrals, we get:

$$I_{1k}+I_{2k}=-\frac{\alpha}{2^{\alpha-2}h^2}g\left(\theta_k\right)+\frac{\alpha\left(\alpha-1\right)}{h^2}\times$$
$$\times\Big[\int\limits_0^{0.5}t^{\alpha-2}g\left(\left(1-t\right)\xi_{k-1}+t\xi_k\right)dt+\int\limits_{0.5}^{1}\left(1-t\right)^{\alpha-2}g\left(\left(1-t\right)\xi_{k-1}+t\xi_k\right)dt\Big].$$

If we make a change of variables $\left(1-t\right)\xi_{k-1}+t\xi_k=x$ in the last equality, we get

$$I_{1k}+I_{2k}=-\frac{\alpha}{2^{\alpha-2}h^2}g\left(\theta_k\right)+\frac{\alpha\left(\alpha-1\right)}{h^2}\times$$
$$\times\Big[\int\limits_{\xi_{k-1}}^{\theta_k}\frac{\left(x-\xi_{k-1}\right)^{\alpha-2}g\left(x\right)}{h^{\alpha-1}}dx+\int\limits_{\theta_k}^{\xi_k}\frac{\left(\xi_k-x\right)^{\alpha-2}g\left(x\right)}{h^{\alpha-1}}dx\Big]=$$
$$=-\frac{\alpha}{2^{\alpha-2}h^2}g\left(\theta_k\right)+\frac{\alpha\left(\alpha-1\right)\Gamma(\alpha-1)}{h^{\alpha+1}}\left[J^{\alpha-1}_{\theta_k^-}g(\xi_{k-1})+J^{\alpha-1}_{\theta_k^+}g(\xi_k)\right].$$

Multiply both sides of the last equality by $\left(\frac{2^{\alpha-2}h^2}{\alpha}\right)$ and take into account the Gamma-function property:

$$\frac{2^{\alpha-2}h^2}{\alpha}\left(I_{1k}+I_{2k}\right)=-g\left(\theta_k\right)+\frac{2^{\alpha-2}\Gamma(\alpha)}{h^{\alpha-1}}\times$$
$$\times\left[J^{\alpha-1}_{\theta_k^-}g(\xi_{k-1})+J^{\alpha-1}_{\theta_k^+}g(\xi_k)\right]. \quad (5)$$

By summing both sides of (5) over k, we obtain (4). The proof is complete. □

Remark. *If we choose $n=1$, then from (4) we get the equality in [1] (equality (2) for $m=1$).*

Theorem 4. *Let $g:I\subset R\to R$ be twice differentiable function on I°. If $g''\in L[a,b]$, where $a,b\in I$ and $|g''|$ is a convex function; then the following inequality holds $\forall\alpha>1$:*

$$\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{h^{\alpha-1}}\sum_{k=1}^{n}\left[J^{\alpha-1}_{\theta_k^-}g(\xi_{k-1})+J^{\alpha-1}_{\theta_k^+}g(\xi_k)\right]-\sum_{k=1}^{n}g(\theta_k)\right|\leqslant$$
$$\leqslant\frac{h^2}{8\alpha(\alpha+1)}\sum_{k=1}^{n}\left(|g''(\xi_{k-1})|+|g''(\xi_k)|\right), \quad (6)$$

where $h = \frac{b-a}{n}$, $\xi_i = a + ih$, $i = 0, 1, 2, \ldots, n$ *and* $\theta_k = \frac{\xi_{k-1}+\xi_k}{2}$.

Proof. From Lemma 1, by using the triangle inequality, we obtain

$$\left| \frac{2^{\alpha-2}\Gamma(\alpha)}{h^{\alpha-1}} \sum_{k=1}^{n} \left[J^{\alpha-1}_{\theta_k^-} g(\xi_{k-1}) + J^{\alpha-1}_{\theta_k^+} g(\xi_k) \right] - \right.$$

$$\left. - \sum_{k=1}^{n} g(\theta_k) \right| \leqslant \frac{2^{\alpha-2}h^2}{\alpha} \sum_{k=1}^{n} \left(|I_{1k}| + |I_{2k}| \right). \quad (7)$$

Since the $|g''|$, the function is convex; using inequality (1) for the first integral, we get

$$|I_{1k}| = \left| \int_0^{1/2} t^\alpha g''\left((1-t)\,\xi_{k-1} + t\xi_k \right) dt \right| \leqslant$$

$$\leqslant |g''(\xi_{k-1})| \int_0^{1/2} t^\alpha (1-t) dt + |g''(\xi_k)| \int_0^{1/2} t^{\alpha+1} dt.$$

Calculating these integrals, we get

$$|I_{1k}| \leqslant \frac{\alpha+3}{(\alpha+1)(\alpha+2)\,2^{\alpha+2}} |g''(\xi_{k-1})| + \frac{1}{(\alpha+2)\,2^{\alpha+2}} |g''(\xi_k)|. \quad (8)$$

Similarly, for the second integral, we write

$$|I_{2k}| = \left| \int_{1/2}^{1} (1-t)^\alpha g''((1-t)\xi_{k-1} + t\xi_k) dt \right| \leqslant$$

$$\leqslant \frac{1}{(\alpha+2)\,2^{\alpha+2}} |g''(\xi_{k-1})| + \frac{\alpha+3}{(\alpha+1)(\alpha+2)\,2^{\alpha+2}} |g''(\xi_k)|. \quad (9)$$

By summing the relevant sides of inequalities (8) and (9), we get:

$$|I_{1k}| + |I_{2k}| \leqslant \frac{1}{(\alpha+1)\,2^{\alpha+1}} \left[|g''(\xi_{k-1})| + |g''(\xi_k)| \right]. \quad (10)$$

By multiplying both sides of the inequality (10) by the expression $\frac{2^{\alpha-2}h^2}{\alpha}$ and by summing over k, we get

$$\frac{2^{\alpha-2}h^2}{\alpha} \sum_{k=1}^{n} \left(|I_{1k}| + |I_{2k}| \right) \leqslant \frac{h^2}{8\alpha(\alpha+1)} \sum_{k=1}^{n} \left[|g''(\xi_{k-1})| + |g''(\xi_k)| \right]. \quad (11)$$

By taking (7) and (11) into account, we obtain (6). The proof is completed. □

Corollary 1. *If we choose $\alpha = 2$, then from (6), we get the inequality*

$$\left| \frac{1}{b-a} \int\limits_a^b g(x)dx - A\left(y_1, y_2, \ldots, y_n\right) \right| \leqslant \frac{(b-a)^2}{48n^2} A\left(z_1, z_2, \ldots, z_n\right), \quad (12)$$

where

$$\begin{aligned} &A(.,.) - arithmetic\ mean\ of\ n\ real\ numbers, \\ &y_k = g\left(\theta_k\right), \quad z_k = |g''(\xi_{k-1})| + |g''(\xi_k)|, \quad k = 1, 2, \ldots, n. \end{aligned}$$

Proof. For $\alpha = 2$, for the left-hand side of the inequality (6), we get

$$\left| \frac{\Gamma(2)}{h} \sum_{k=1}^{n} \left[J^1_{\theta_k^-} g(\xi_{k-1}) + J^1_{\theta_k^+} g(\xi_k) \right] - \sum_{k=1}^{n} g\left(\theta_k\right) \right| =$$

$$= \left| \frac{n}{b-a} \sum_{k=1}^{n} \left[\int\limits_{\xi_{k-1}}^{\theta_k} g(x)dx + \int\limits_{\theta_k}^{\xi_k} g(x)dx \right] - \sum_{k=0}^{n-1} g(\theta_k) \right| =$$

$$= \left| \frac{n}{b-a} \sum_{k=1}^{n} \left(\int\limits_{\xi_{k-1}}^{\xi_k} g(x)dx \right) - \sum_{k=1}^{n} g(\theta_k) \right| = \left| \frac{n}{b-a} \int\limits_a^b g(x)dx - \sum_{k=1}^{n} g(\theta_k) \right|. \quad (13)$$

For the right-hand side of the inequality (6), we write

$$\frac{h^2}{8\alpha(\alpha+1)} \sum_{k=1}^{n} \left(|g''(\xi_{k-1})| + |g''(\xi_k)| \right) =$$

$$= \frac{(b-a)^2}{48n^2} \sum_{k=1}^{n} \left[\left(|g''(\xi_{k-1})| + |g''(\xi_k)| \right) \right]. \quad (14)$$

By taking into account the equalities (13) and (14) from (6), we obtain

$$\left| \frac{n}{b-a} \int\limits_a^b g(x)dx - \sum_{k=1}^{n} g\left(\theta_k\right) \right| \leqslant \frac{(b-a)^2}{48n^2} \sum_{k=1}^{n} \left[\left(|g''(\xi_{k-1})| + |g''(\xi_k)| \right) \right].$$

By dividing both sides of the last inequality by n, we obtain (12). The proof is complete. □

Remark. *It is easy to verify that if we choose $n = 1$ and $\alpha = 2$, then, from (12), we get the inequality*

$$\left|\frac{1}{b-a}\int_a^b g(x)dx - g\left(\frac{a+b}{2}\right)\right| \leqslant \frac{(b-a)^2}{48}\left[|g''(a)| + |g''(b)|\right]. \tag{15}$$

This inequality was obtained by M. Sarikaya et al. (see [17, Proposition 1]) for convex functions.

Proposition 1. *If the absolute error ε is given, then the following inequality holds for n:*

$$n \geqslant \left\lceil \frac{b-a}{2}\sqrt{\frac{\|g''\|}{6\varepsilon}}\right\rceil,$$

where $\|g''\| = \sup_{x\in[a,b]}|g''(x)|$.

Proof. Indeed, from the right-hand side of the inequality (14), we have

$$\frac{(b-a)^2}{48n^3}\sum_{k=1}^{n}\left[|g''(\xi_{k-1})| + |g''(\xi_k)|\right] \leqslant \frac{(b-a)^2}{48n^3}(2n\|g''\|) = \frac{(b-a)^2}{24n^2}\|g''\|. \tag{16}$$

Since $\frac{(b-a)^2}{24n^2}\|g''\| \leqslant \varepsilon$, we get $n \geqslant \left\lceil \frac{b-a}{2}\sqrt{\frac{\|g''\|}{6\varepsilon}}\right\rceil$. The proof is complete. □

Remark. *If $\|g''\| = \sup_{x\in[a,b]}|g''(x)|$, then it follows from inequality (15) that*

$$\left|\frac{1}{b-a}\int_a^b g(x)dx - g\left(\frac{a+b}{2}\right)\right| \leqslant \frac{(b-a)^2}{24}\|g''\|$$

and from (12), taken into account inequality (16), we get

$$\left|\frac{1}{b-a}\int_a^b g(x)dx - \frac{1}{n}\sum_{k=1}^{n} g(\theta_k)\right| \leqslant \frac{(b-a)^2}{24n^2}\|g''\|.$$

These two inequalities show that when the interval is divided into n subintervals, the error of an Hadamard-type inequality decreases n^2 times.

Remark. *Let $R(g,h)$ be the error estimation of the Midpoint rule of numerical integration; then from (12) we obtain the well-known error estimate [2]:*

$$R(g,h)=\frac{b-a}{24}h^2\left\|g''\right\|.$$

Theorem 5. *Let $g:I\subset R\to R$ be a twice differentiable function on I°. If $g''\in L[a,b]$, where $a,b\in I$ and $|\,g''|^q$ is a convex function, then, for all $\alpha>1$, $q\geqslant 1$ and $t\in(0,1)$, the following inequality holds:*

$$\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{h^{\alpha-1}}\sum_{k=1}^{n}\left[J_{\theta_k^-}^{\alpha-1}g(\xi_{k-1})+J_{\theta_k^+}^{\alpha-1}g(\xi_k)\right]-\sum_{k=1}^{n}g\left(\theta_k\right)\right|\leqslant$$
$$\leqslant\frac{h^2}{8\alpha\left(\alpha+1\right)}\sum_{k=1}^{n}F_k,\quad(17)$$

where

$$h=\frac{b-a}{n},\ \ \xi_i=a+ih,\ i=0,1,2,\ldots,n,\ \ \theta_k=\frac{\xi_{k-1}+\xi_k}{2},$$
$$F_k=\left[\frac{(\alpha+1)\left|g''(\xi_{k-1})\right|^q+(\alpha+3)\left|g''(\xi_k)\right|^q}{2\left(\alpha+2\right)}\right]^{\frac{1}{q}}+$$
$$+\left[\frac{(\alpha+3)\left|g''(\xi_{k-1})\right|^q+(\alpha+1)\left|g''(\xi_k)\right|^q}{2\left(\alpha+2\right)}\right]^{\frac{1}{q}}.$$

Proof. From Lemma 1, we obtain, by using the triangle inequality,

$$\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{h^{\alpha-1}}\sum_{k=1}^{n}\left[J_{\theta_k^-}^{\alpha-1}g(\xi_{k-1})+J_{\theta_k^+}^{\alpha-1}g(\xi_k)\right]-\sum_{k=1}^{n}g\left(\theta_k\right)\right|\leqslant$$
$$\leqslant\frac{2^{\alpha-2}h^2}{\alpha}\sum_{k=0}^{n-1}\left(\left|I_{1k}\right|+\left|I_{2k}\right|\right).\quad(18)$$

By using the well-known Power-mean integral inequality (3) and since $|\,g''|^q$ is a convex function, we get

$$\left|I_{1k}\right|=\left|\int_0^{1/2}t^\alpha g''\left((1-t)\,\xi_{k-1}+t\xi_k\right)dt\right|\leqslant\left(\int_0^{1/2}t^\alpha dt\right)^{1-\frac{1}{q}}\times$$
$$\times\left[\left|g''(\xi_{k-1})\right|^q\int_0^{1/2}t^\alpha\left(1-t\right)dt+\left|g''(\xi_k)\right|^q\int_0^{1/2}t^{\alpha+1}dt\right]^{\frac{1}{q}}=$$

$$= \left[\frac{1}{(\alpha+1)\,2^{\alpha+1}} \right]^{1-\frac{1}{q}} \left[\frac{1}{(\alpha+1)\,(\alpha+2)\,2^{\alpha+2}} \right]^{\frac{1}{q}} \times$$
$$\times \left[(\alpha+3)\left|g''(\xi_{k-1})\right|^q + (\alpha+1)\left|g''(\xi_k)\right|^q \right]^{\frac{1}{q}}.$$

Simplifying the last statement, we get:

$$|I_{1k}| \leqslant \frac{1}{(\alpha+1)\,2^{\alpha+1}} \left[\frac{(\alpha+3)\left|g''(\xi_{k-1})\right|^q + (\alpha+1)\left|g''(\xi_k)\right|^q}{2\,(\alpha+2)} \right]^{\frac{1}{q}}. \qquad (19)$$

Note that

$$|I_{2k}| = \left| \int_{1/2}^{1} (1-t)^\alpha g''((1-t)\xi_{k-1} + t\xi_k)dt \right| =$$
$$= \left| \int_{0}^{1/2} t^\alpha g''(t\xi_{k-1} + (1-t)\xi_k)dt \right|;$$

so, similarly to the first integral, we can write down an inequality for I_{2k}:

$$|I_{2k}| \leqslant \frac{1}{(\alpha+1)\,2^{\alpha+1}} \left[\frac{(\alpha+1)\left|g''(\xi_{k-1})\right|^q + (\alpha+3)\left|g''(\xi_k)\right|^q}{2\,(\alpha+2)} \right]^{\frac{1}{q}}. \qquad (20)$$

Summing the relevant sides of the inequalities (19) and (20), we get

$$|I_{1k}| + |I_{2k}| \leqslant \frac{1}{(\alpha+1)\,2^{\alpha+1}} F_k.$$

Multiplying both sides of the last inequality by the expression $\frac{2^{\alpha-2}h^2}{\alpha}$, summing over k, and taking into account (18), we obtain (17). The proof is completed. □

Corollary 1. *Taking $\alpha = 2$, we obtain, from (17):*

$$\left| \frac{1}{b-a} \int_a^b g(x)dx - A\left(y_1, y_2, \ldots, y_n\right) \right| \leqslant \frac{(b-a)^2}{48n^2} A(F_1, F_2, \ldots, F_n), \quad (21)$$

where

$A(.,.)$ – arithmetic mean of n real numbers,

$$\begin{aligned} y_k &= g\left(\theta_k\right), \ \ \theta_k = \frac{\xi_{k-1}+\xi_k}{2}, \ \ k=1,2,\ldots,n, \\ F_k &= \left[\frac{3\left|g''(\xi_k)\right|^q + 5\left|g''(\xi_{k-1})\right|^q}{8}\right]^{\frac{1}{q}} + \left[\frac{5\left|g''(\xi_k)\right|^q + 3\left|g''(\xi_{k-1})\right|^q}{8}\right]^{\frac{1}{q}}. \end{aligned}$$

Here we get (12) for $q = 1$.

Remark. *Taking $\alpha = 2$ and $n = 1$, we get, from (17):*

$$\left|\frac{1}{b-a}\int_a^b g(x)dx - g\Big(\frac{a+b}{2}\Big)\right| \leqslant \frac{(b-a)^2}{48} \times F_1,$$

where

$$F_1 = \left[\frac{3\left|g''(a)\right|^q + 5\left|g''(b)\right|^q}{8}\right]^{1/q} + \left[\frac{5\left|g''(a)\right|^q + 3\left|g''(b)\right|^q}{8}\right]^{1/q}.$$

This inequality for convex functions has been obtained by M. Sarikaya, N. Aktan, and the author (see [17, Proposition 5]).

Theorem 6. *Let $g : I \subset R \to R$ be a twice differentiable function on I°. If $g'' \in L[a,b]$, where $a, b \in I$ and $|\ g''|^q$ is a convex function, then, for all α,q and $p > 1$, such that $\frac{1}{q} + \frac{1}{p} = 1$, the following inequality holds:*

$$\begin{aligned} &\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{h^{\alpha-1}}\sum_{k=1}^{n}\left[J^{\alpha-1}_{\theta_k^-}g(\xi_{k-1}) + J^{\alpha-1}_{\theta_k^+}g(\xi_k)\right] - \sum_{k=1}^{n} g\left(\theta_k\right)\right| \leqslant \\ &\qquad \leqslant \frac{h^2}{16\alpha}\left(\frac{1}{\alpha q - q + 3}\right)^{\frac{1}{q}} \sum_{k=1}^{n} W_k, \qquad (22) \end{aligned}$$

where

$$\begin{aligned} h &= \frac{b-a}{n}, \ \ \xi_i = a + ih, \ \ i = 0,1,2,\ldots,n, \ \ \theta_k = \frac{\xi_{k-1}+\xi_k}{2}, \\ W_k &= \left[\vartheta\left|g''(\xi_{k-1})\right|^q + \left|g''(\xi_k)\right|^q\right]^{\frac{1}{q}} + \left[\left|g''(\xi_{k-1}\ )\right|^q + \vartheta\left|g''(\xi_k)\right|^q\right]^{\frac{1}{q}}, \\ \vartheta &= \frac{q\alpha - q + 4}{q\alpha - q + 2}. \end{aligned}$$

Proof. From Lemma 1, we obtain, using the triangle inequality:

$$\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{h^{\alpha-1}}\sum_{k=1}^{n}\left[J_{\theta_k^-}^{\alpha-1}g(\xi_{k-1})+J_{\theta_k^+}^{\alpha-1}g(\xi_k)\right]-\sum_{k=1}^{n}g\left(\theta_k\right)\right|\leqslant$$

$$\leqslant\frac{2^{\alpha-2}h^2}{\alpha}\sum_{k=1}^{n}\left(|I_{1k}|+|I_{2k}|\right). \quad (23)$$

Using the well-known Hölder integral inequality (2) and since $|\,g''|^q$ is a convex function, we get

$$|I_{1k}|=\left|\int_0^{1/2}t^{\alpha}g''\left((1-t)\,\xi_{k-1}+t\xi_k\right)dt\right|=$$

$$=\left|\int_0^{1/2}t^{\frac{1}{p}}t^{\frac{1}{q}}t^{\alpha-1}g''\left((1-t)\,\xi_{k-1}+t\xi_k\right)dt\right|\leqslant$$

$$\leqslant\left(\int_0^{1/2}\left(t^{\frac{1}{p}}\right)^p dt\right)^{\frac{1}{p}}\left[|g''(\xi_{k-1})|^q\int_0^{1/2}t^{q\alpha-q+1}(1-t)dt+|g''(\xi_k)|^q\int_0^{1/2}t^{q\alpha-q+2}dt\right]^{\frac{1}{q}}=$$

$$=\left(\frac{1}{2}\right)^{\frac{3}{p}}\left[\frac{1}{(q\alpha-q+3)2^{q\alpha-q+3}}\right]^{\frac{1}{q}}\left[\frac{q\alpha-q+4}{q\alpha-q+2}\,|g''(\xi_{k-1})|^q+|g''(\xi_k)|^q\right].$$

Given the fact that

$$\left(\frac{1}{2}\right)^{\frac{3}{p}}\left[\frac{1}{(q\alpha-q+3)2^{q\alpha-q+3}}\right]^{\frac{1}{q}}=\frac{1}{2^{\alpha+2}}\left(\frac{1}{q\alpha-q+3}\right)^{\frac{1}{q}}$$

we can write an inequality for I_{1k}:

$$|I_{1k}|\leqslant\frac{1}{2^{\alpha+2}}\left(\frac{1}{q\alpha-q+3}\right)^{\frac{1}{q}}\left[\vartheta\,|g''(\xi_{k-1})|^q+|g''(\xi_k)|^q\right]^{\frac{1}{q}}. \quad (24)$$

Since

$$|I_{2k}|=\left|\int_{1/2}^{1}(1-t)^{\alpha}\,g''\left((1-t)\,\xi_{k-1}+t\xi_k\right)dt\right|=$$

$$=\left|\int_0^{1/2}t^{\alpha}g''\left(t\xi_{k-1}+(1-t)\,\xi_k\right)dt\right|,$$

we can write an inequality for I_{2k}, similarly to the first integral:

$$|I_{2k}| \leqslant \frac{1}{2^{\alpha+2}} \left(\frac{1}{q\alpha - q + 3} \right)^{\frac{1}{q}} \left[|g''(\xi_{k-1})|^q + \vartheta \, |g''(\xi_k)|^q \right]^{\frac{1}{q}}. \tag{25}$$

Summing the relevant sides of the inequalities (24) and (25), we get

$$|I_{1k}| + |I_{2k}| \leqslant \frac{1}{2^{\alpha+2}} \left(\frac{1}{q\alpha - q + 3} \right)^{\frac{1}{q}} \cdot W_k;$$

by multiplying both sides of the last inequality by the expression $\frac{2^{\alpha-2}h^2}{\alpha}$, summing over k, and taking into account (23), we obtain (22). The proof is completed. □

Corollary. *Note that $\frac{1}{p} = 1 - \frac{1}{q}$. Choose $\alpha = 2$; then we get from (22):*

$$\left| \frac{1}{b-a} \int\limits_a^b g(x)dx - A\left(y_1, y_2, \ldots, y_n\right) \right| \leqslant$$

$$\leqslant \frac{(b-a)^2}{32n^2} \psi\left(q\right) A(W_1, W_2, \ldots, W_n), \tag{26}$$

where

$A(.,.)$ – *arithmetic mean of n real numbers,*

$$y_k = g\left(\theta_k\right), \ \theta_k = \frac{\xi_{k-1} + \xi_k}{2}, \ k = 1, 2, \ldots, n, \ \ \psi\left(q\right) = \left(\frac{1}{q+3} \right)^{\frac{1}{q}},$$

$$W_k = \left[\frac{q+4}{q+2} |g''(\xi_{k-1})|^q + |g''(\xi_k)|^q \right]^{\frac{1}{q}} + \left[|g''(\xi_{k-1})|^q + \frac{q+4}{q+2} |g''(\xi_k)|^q \right]^{\frac{1}{q}}.$$

Here $\frac{1}{16} \leqslant \psi\left(q\right) < 1$ for all $q > 1$, since $\lim\limits_{q \to 1^+} \psi\left(q\right) = \frac{1}{4}$ and $\lim\limits_{q \to +\infty} \psi\left(q\right) = 1$. For $q \to 1^+$, we get (15) from (26).

3. Some Applications To The Special Means. Let us consider some special means for two real numbers α and β.

1) *Arithmetic mean* : $A(\alpha, \beta) = \frac{\alpha+\beta}{2}$;
2) *Quadratic mean* : $Q(\alpha, \beta) = \sqrt{\alpha^2 + \beta^2}$;
3) *Logarithmic mean* : $L(\alpha, \beta) = \frac{\beta - \alpha}{\ln \beta - \ln \alpha}$, $\alpha, \beta > 0$ *and* $\alpha \neq \beta$;

4) *Generalized logarithmric mean* :

$$L_n(\alpha,\beta)=\begin{cases}\alpha, & if\ \alpha=\beta,\\ \dfrac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}, & if\ \alpha\neq\beta,\ n\in Z^+.\end{cases}$$

Now, by using the results, we will give some applications to special means of positive real numbers.

Proposition 2. *Let $g(x)=\sqrt{1+x^2}$; then $\forall x\in[0,b]$ the following inequality holds:*

$$\left|A\big[Q\,(b,1)\,,\frac{1}{b}\ln\left(2A\,(b,Q\,(b,1))\right)\big]-\frac{1}{2n}A\,(y_1,y_2,\ldots,y_n)\right|\leqslant$$
$$\leqslant\frac{b^2}{16n^3}A\left[1,Q^{-3}(b,1),2n^4A\,(t_1,t_2,\ldots,t_{n-1})\right],$$

where
$$y_k=Q\left[(2k-1)\,b,2n\right],\ k=1,2,\ldots,n,$$
$$t_m=Q^{-3}\left(m^2b^2,n^2\right),\quad m=1,2,\ldots,n-1,$$
n is the number of subintervals of the interval $[0,b]$.

Proof. This inequality follows from Corollary 1.

Indeed, for the left-hand side of inequality (12) with $a=0$, we obtain:

$$\frac{1}{b-a}\int_a^b g(x)dx=\frac{1}{b}\int_0^b\sqrt{1+x^2}dx=\frac{\sqrt{1+b^2}}{2}+\frac{\ln\left(b+\sqrt{1+b^2}\right)}{2b}=$$

$$=\frac{Q(b,1)}{2}+\frac{\ln\left(2A\,(b,Q\,(b,1))\right)}{2}=A\left\{Q\,(b,1)\,,\frac{1}{b}\ln\left[2A\,(b,Q\,(b,1))\right]\right\};$$

as $\xi_i=a+i\frac{b-a}{n}=i\frac{b}{n},i=0,1,2,\ldots,n,$

$$y_k=g\left(\frac{\xi_{k-1}+\xi_k}{2}\right)=\sqrt{1+\left(\frac{\xi_{k-1}+\xi_k}{2}\right)^2}=\frac{\sqrt{(2k-1)^2b^2+4n^2}}{2n}=$$
$$=\frac{1}{2n}Q\left[(2k-1)b,2n\right].$$

As $g''(x) = \frac{1}{(1+x^2)^{3/2}}$ on the right-hand side (12), we get:

$$\frac{b^2}{48n^2} A(z_1, z_2, \ldots, z_n) = \frac{b^2}{48n^3} \sum_{k=1}^{n} \left[\frac{1}{\left(\xi_{k-1}^2 + 1\right)^{3/2}} + \frac{1}{\left(\xi_k^2 + 1\right)^{3/2}} \right] =$$
$$= \frac{b^2}{48n^3} \left[1 + 2 \sum_{k=1}^{n-1} \left(\frac{n^3}{(k^2 b^2 + n^2)^{3/2}} \right) + \frac{1}{(b^2+1)^{3/2}} \right] =$$
$$= \frac{b^2}{48n^3} \left[1 + Q^{-3}(b,1) + 2n^3 \sum_{k=1}^{n-1} Q^{-3}(kb, n) \right] =$$
$$= \frac{b^2}{16n^3} A \left[1, Q^{-3}(b,1), 2n^4 A(t_1, t_2, \ldots, t_{n-1}) \right].$$

Here $z_k = |g''(\xi_{k-1})| + |g''(\xi_k)|$ and $t_k = Q^{-3}(kb,n)$. $\square$

Proposition 3. *Let $g(x) = e^x$ and let the interval $[a, b]$ be divided into n subintervals; then $\forall x \in [a,b]$, the following inequality holds:*

$$\left| L\left(e^a, e^b\right) - e^a A\left(y_1, y_2, \ldots, y_n\right) \right| \leqslant \frac{(b-a)^2}{24n^2} EA\left(t_1, t_2, \ldots, t_n\right),$$

where

$$y_k = e^{(2k-1)\frac{h}{2}}, \quad h = \frac{b-a}{n}, \quad k = 1, 2, \ldots, n,$$
$$E = e^a \left\{ A \left[A^{\frac{1}{q}} \left(\frac{3}{4} e^{qh}, \frac{5}{4} \right), A^{\frac{1}{q}} \left(\frac{5}{4} e^{qh}, \frac{3}{4} \right) \right] \right\},$$
$$t_m = e^{(m-1)h}, \quad m = 1, 2, \ldots, n.$$

Proof. This inequality follows from Corollary 1. Indeed, for the left-hand side of the inequality (21) we get:

$$\frac{1}{b-a} \int_a^b g(x) dx = \frac{1}{b-a} \int_a^b e^x dx = \frac{e^b - e^a}{b-a} = L(e^a, e^b),$$

and, since $\xi_i = a + ih$, $h = \frac{b-a}{n}$, $i = 0, 1, 2, \ldots, n$,

$$y_k = g\left(\frac{\xi_{k-1} + \xi_k}{2} \right) = e^{a+(2k+1)\frac{h}{2}}.$$

Since $g''(x) = e^x$, we can write for the right-hand side of the inequality (21),

$$F_k = \left[\frac{3\,|g''(\xi_k)|^q + 5\,|g''(\xi_{k-1})|^q}{8}\right]^{\frac{1}{q}} + \left[\frac{5\,|g''(\xi_k)|^q + 3\,|g''(\xi_{k-1})|^q}{8}\right]^{\frac{1}{q}} =$$

$$= \left[\frac{3e^{(a+kh)q} + 5e^{(a+(k-1)h)q}}{8}\right]^{\frac{1}{q}} + \left[\frac{5e^{(a+kh)q} + 3e^{(a+(k-1)h)q}}{8}\right]^{\frac{1}{q}} =$$

$$= e^{(a+(k-1)h)}\left\{\left[\frac{3e^{hq}+5}{8}\right]^{\frac{1}{q}} + \left[\frac{5e^{hq}+3}{8}\right]^{\frac{1}{q}}\right\} =$$

$$= e^a \cdot \left\{\left[\frac{3e^{hq}+5}{8}\right]^{\frac{1}{q}} + \left[\frac{5e^{hq}+3}{8}\right]^{\frac{1}{q}}\right\} \cdot e^{(k-1)h} =$$

$$= 2e^a \cdot \left\{A\left[A^{\frac{1}{q}}\left(\frac{3}{4}e^{qh}, \frac{5}{4}\right), A^{\frac{1}{q}}\left(\frac{5}{4}e^{qh}, \frac{3}{4}\right)\right]\right\} \cdot e^{(k-1)h}.$$

Thus, $A(F_1, F_2, \ldots, F_n) = 2e^a \cdot E \cdot A(t_1, t_2, \ldots, t_n)$. □

References

[1] Bayraktar B. *Some integral inequalities Of Hermite–Hadamard type for differentiable (s, m)–convex functions via fractional integrals.*, TWMS J. App. Eng. Math,. 2020, vol. 10(3), pp. 625 – 637.

[2] Boglaev Yu. P. *Computational Mathematics and Programming.* Moscow, Higher School, 1990.

[3] Butkovskii A. G., Postnov S. S., Postnova E. A., *Fractional integro–differential calculus and its control-theoretical applications. I. Mathematical fundamentals and the problem of interpretation.* Automation and Remote Control, 2013, vol. 74, no. 4, pp. 543 – 574.
DOI: https://doi.org/10.1134/S0005117913040012

[4] Dragomir S. S., Pearce C. E. M. *Selected topics on Hermite–Hadamard inequalities and applications.* RGMIA Monographs, Victoria University, 2000. Online: www.sta.vu.edu.au/RGMIA/monographs/hermite--hadamard.html

[5] Ekinci A., Őzdemir M. E. *Some new integral inequalities via Riemann–Liuouville integral operators.* Appl. Comput. Math. 2019, vol. 18(3), pp. 288 – 295.

[6] Erdem S, Sarıkaya M. Z., Çelik N. *Some generalized inequalities involving local fractional integrals and their applications for random variables and numerical integration.* Journal of Applied Mathematics, Statistics and Informatics, 2016, vol. 12, pp. 49 – 65.

[7] Gürbüz M., Öztürk O. *Inequalities generated with Riemann – Liouville fractional integral operator.* TWMS Journal of Applied and Engineering Mathematics, 2019, vol. 9(1), pp. 91–100.

[8] Hadamard J. *Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann.* J. Math. Pures Appl, 1893, vol. 58, pp. 171 – 215.

[9] Latif M. A., Dragomir S. S. *New inequalities of Hermite–Hadamard type for functions whose derivatives in absolute value are convex with applications.* Acta Univ. M. Belii, Ser. Math., 2013, vol. 21, pp. 27 – 42.

[10] Matłoka M. *Hermite–Hadamard type inequalities for fractional integrals.* RGMIA Res. Rep. Coll., 2017, vol. 20, Art. 69. 11 p.

[11] Miller S. Ross, B. *An introduction to the fractional calculus and fractional differential equations.* USA: John Wiley & Sons, 1993.

[12] Mitrinovic D. S., Peĉariĉ J., Fink A. M. *Classical and new inequalities in analysis.* Kluwer Academic, Dordrecth, 1993.

[13] Mohammed P. O., Hamasalh F. K. *New conformable fractional integral inequalities of Hermite–Hadamard type for convex functions.* Symmetry, 2019, vol. 11(2): 263. DOI: https://doi.org/10.3390/sym11020263

[14] Nakhushev A. M. *Fractional calculus and its application.* Monographs, Moscow, Fizmatlit, 2003. (in Russian)

[15] Napoles V. J. E., Rodriguez J. M, Sigarreta J. M. *New Hermite – Hadamard type inequalities involving non – conformable integral operators.* Symmetry, 2019, vol. 11(9):1108. DOI: https://doi.org/10.3390/sym11091108

[16] Őzdemir M. E., Ekinci A., Akdemir A. O. *Some new integral inequalities for functions whose derivatives of absolute values are convex and concave.* TWMS J. Pure Appl. Math., 2019, vol. 10(2), pp. 212 – 224.

[17] Sarıkaya M.Z., Aktan N. *On the generalization of some integral inequalities and their applications.* Math. Comput. Modelling, 2011, vol. 54, pp. 2175 – 2182. DOI: https://doi.org/10.1016/j.mcm.2011.05.026

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