Filomat 31:4 (2017), 941–952
DOI 10.2298/FIL1704941C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Projective Coordinate Spaces

Süleyman Çiftçi[a], Fatma Özen Erdoğan[a]

[a]*Department of Mathematics, Uludag University, Bursa, Turkey*

Abstract. In the present study, an $(n+1)$-dimensional module over the local ring $\boldsymbol{K} = \boldsymbol{M}_{mm}(\mathbb{R})$ is constructed. Further, an $n-$ dimensional projective coordinate space over this module is constructed with the help of equivalence classes. The points and lines of this space are determined and the points are classified. Finally, for a $3-$dimensional projective coordinate space, the incidence matrix for a line that goes through the given points and also all points of a line given with the incidence matrix are found by the use of Maple commands.

1. Introduction

The structural properties of fields have been widely studied and many simplicities are found in the operations. In the study of geometric structures constructed over fields, there are some advantages to simplify operations by using the properties of field. The algebraic structures that have less properties and geometric structures that are constructed over them have been also studied. Local rings are one of the important classes of them. In [9], F. Machala studied Klingenberg projective spaces over a local ring. Further, M. Jukl and V. Snasel adapted the concept of projective coordinate space to their study as n-dimensional coordinate projective Klingenberg space [6]. In [1], the concept of projective space over a vector space is generalized to a space over a module by the help of equivalence classes using similar methods given in [3]. Also the isomorphism between the space over a module and the $n-$dimensional coordinate projective Klingenberg space is constructed in [1].

In the present paper, one of the special types of local rings and the free dimensional modules over this special local rings are studied, and an $(n+1)-$ dimensional module over the local ring $\boldsymbol{K} = \boldsymbol{M}_{mm}(\mathbb{R})$ is constructed. Further, an $n-$dimensional projective coordinate space over this module is constructed with the help of equivalence classes. The points and lines of this space are determined and the points are classified. Finally, for a $3-$dimensional projective coordinate space, the incidence matrix for a line that goes through given points and also all points of a line given with the incidence matrix are found by the use of Maple commands.

2. Preliminaries

In this section, we recall some basic definitions, propositions and some information from [4], [5] and [2].

In many of algebra books, a local ring is defined as a ring with identity, whose non-units form an ideal. Also, a module which is constructed over a local ring $\boldsymbol{A}$ is called an $\boldsymbol{A}-$module.

2010 *Mathematics Subject Classification*. Primary 13C99; Secondary 51C05
Keywords. real plural algebra, local ring, module, projective coordinate spaces
Received: o1 December 2015; Revised: 10 March 2016; Accepted: 14 March 2016
Communicated by Bahattin Yıldız (Guest Editor)
Email addresses: sciftci@uludag.edu.tr (Süleyman Çiftçi), fatmaozen@uludag.edu.tr (Fatma Özen Erdoğan)

Definition 2.1. *[4] The real plural algebra of order m is a linear algebra $\boldsymbol{A}$ on $\mathbb{R}$ having, as a vector space over $\mathbb{R}$, a basis $\left\{1, \eta, \eta^2, \eta^3, \cdots, \eta^{m-1}\right\}$ where $\eta^m = 0$.*

Definition 2.2. *[4] Let $\boldsymbol{A}$ be a real plural algebra. By a system of projections $\boldsymbol{A} \to \mathbb{R}$, a system of mappings $p_k : \boldsymbol{A}$ onto $\mathbb{R}$ is meant which are defined for $k = 0, 1, \cdots, m-1$ as follows*

$$\forall \beta \in \boldsymbol{A}, \quad \beta = \sum_{i=0}^{m-1} b_i \eta^i \quad p_k(\beta) =: b_k.$$

Proposition 2.3. *[4] Let $\boldsymbol{A}$ be a real plural algebra. Then an element $\varepsilon \in \boldsymbol{A}$ is a unit if and only if $p_0(\varepsilon) \neq 0$.*

Proposition 2.4. *[4] The real plural algebra $\boldsymbol{A}$ is a local ring with the maximal ideal $\eta\boldsymbol{A}$. The ideals $\eta^j\boldsymbol{A}$, $1 \leq j \leq m$, are all ideals in $\boldsymbol{A}$.*

Definition 2.5. *[5] Let $\boldsymbol{A}$ be a local ring and M be a finitely generated $\boldsymbol{A}$-module. Then M is an $\boldsymbol{A}-$ space of finite dimension if there exists $E_1, E_2, \cdots, E_n$ in M with*

i) $M = \boldsymbol{A}E_1 \oplus \boldsymbol{A}E_2 \oplus \cdots \oplus \boldsymbol{A}E_n$,

ii) the map $\boldsymbol{A} \to \boldsymbol{A}E_i$ defined by $x \to xE_i$ is an isomorphism for $1 \leq i \leq n$.

Let $\boldsymbol{A}$ be a real plural algebra having a basis $\left\{1, \eta, \eta^2, \eta^3, \cdots, \eta^{m-1}\right\}$ with $\eta^m = 0$. Let $\boldsymbol{K} = \boldsymbol{M}_{mm}(\mathbb{R})$ be the linear algebra of upper triangular matrices of the form

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{m-1} \\ 0 & a_0 & a_1 & \cdots & a_{m-2} \\ 0 & 0 & a_0 & \cdots & a_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix},$$

where $a_i \in \mathbb{R}$ for $0 \leq i \leq m-1$. Then the map which is defined for every $\alpha = \sum\limits_{k=0}^{m-1} a_k \eta^k$ as

$$f(\alpha) = (a_{ij}) = \begin{cases} a_{ij} = 0, & j < i \\ a_{ij} = a_{j-i}, & j \geq i \end{cases}$$

is an isomorphism between $\boldsymbol{A}$ and $\boldsymbol{K}$ ([4]).

The set $\{\eta_0, \eta_1, \eta_2, \cdots, \eta_{m-1}\}$ is a basis of $\boldsymbol{K}$ where $\eta_k = (a_{ij})_{m \times m}$, $0 \leq k \leq m-1$ and $a_{ij} = \begin{cases} 1, & j = i+k \\ 0, & j \neq i+k \end{cases}$ for $0 \leq i, \ j \leq m-1$ ([2]).

Proposition 2.6. *[2] $M = \mathbb{R}_n^m$ is a module over the linear algebra of a matrix $\boldsymbol{K}$ and the following set is a basis of $\boldsymbol{K}-$module M;*

$$\left\{ \begin{array}{c} E_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}, \\ E_3 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \cdots & 0 \end{pmatrix}, \quad \cdots, \quad E_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \end{array} \right\}$$

Now, we would like to give a definition of being $R-$independent for a non-empty set from [9].

Definition 2.7. *[9] Let R be a local ring, R_0 be the maximal ideal of R and M be a free module with unity over R. Let S be a non-empty subset of the module M. Let M_0 be a submodule of M constructed over R_0. For $x_1, x_2, \cdots, x_k \in S$ and $\alpha_1, \alpha_2, \cdots, \alpha_k \in R$, if*

$$\sum_{i=1}^{k} \alpha_i x_i \in M_0 \Rightarrow \alpha_i \in R_0 \text{ for every } i$$

holds, then S is called $R-$independent. Otherwise, S is called an $R-$dependent subset.

Now we introduce the construction of $n-$dimensional coordinate projective Klingenberg space over a local ring R from [6]:

Structure 1: [6] Let R be a local ring with maximal ideal I. Let us denote $M = R^{n+1}$, $\overline{M} = M/IM$ and $\overline{R} = R/I$.

Let f be the natural homomorphism $M \to \overline{M}$. If the following conditions hold, the incidence structure, which is denoted by P_R, is called the *n-dimensional coordinate projective Klingenberg space over the local ring R*.

The points are just $1-$dimensional submodules $Sp\{x\}$ of M such that $f(x) \neq IM$ is an element of $\overline{M}$, namely $x \notin I^{n+1}$.

The lines are just 2-dimensional submodules $Sp\{x, y\}$ of M such that $Sp\{f(x), f(y)\}$ is a 2-dimensional subspace of $\overline{M}$.

The incidence relation is the inclusion given by

$$Sp\{x\} \circ Sp\{u, v\} \Leftrightarrow Sp\{x\} \subset Sp\{u, v\}.$$

Finally, we give the generalization of the concept of projective space over a vector space to a space over a module by the help of equivalence classes. We also give the isomorphism between the space over a module and the $n-$ dimensional coordinate projective Klingenberg space from [1].

Structure 2: [1] Let R be a local ring with maximal ideal I and $M = R^{n+1}$ be an $(n+1)$-dimensional module over the local ring R. Let us denote the submodule I^{n+1} of M by M_0. Consider the equivalence relation on the set of $M^* = M \backslash M_0$, whose equivalence classes are 1-dimensional submodules of M. Thus, if $x, y \in M^*$, then x is equivalent to y if $y = tx$ for $t \in R^* = R \backslash I$, i.e., $y_i = tx_i$ for all i. The set of equivalence classes is denoted by $P(M)$. $P(M)$ is called an *n-dimensional space* and the elements of $P(M)$ are called points. The equivalence class of the vector x is denoted by $\overline{x}$. Here x is called the coordinate vector for $\overline{x}$. In this case, tx with $t \in R \backslash I$ also represents $\overline{x}$; namely, $\overline{tx} = \overline{x}$. Here, we can denote this by $\overline{x} = \{tx \,|\, t \in R \backslash I\}$.

If $Sp\{x\}$ is a 1-dimensional submodule of M, then $\overline{x} = Sp\{x\} \backslash M_0$ is a point of $P(M)$ and $\overline{x} = \{tx \,|\, t \in R^*\}$.

The line passing through the points $\overline{x}$ and $\overline{y}$ is denoted by $[\overline{x}, \overline{y}] = Sp\{x, y\} \backslash M_0$ where x and y are R-independent vectors. Thus

$$[\overline{x}, \overline{y}] = \{ax + by \,|\, a, b \in R\} \backslash \{a'x + b'y \,|\, a', b' \in I\} = \{a''x + b''y \,|\, \exists a'', b'' \in R^*\}.$$

Let us denote the set of points and lines of the space $P(M)$, respectively, with P' and L'. We define the incidence relation as follows:

A point $\overline{u} \in P'$ is on the line $[\overline{x}, \overline{y}] \in L'$ if and only if $Sp\{u\} \subseteq \{ax + by \,|\, a, b \in R\}$.

Now, the relation between the space (P', L') which is given in [1] and $n-$dimensional coordinate projective Klingenberg space which is given in [6] is constructed with the following theorem:

Theorem 2.8. *[1] The maps $f : P \to P'$ and $f : L \to L'$ defined by respectively $f(Sp\{x\}) = \overline{x}$ for every $Sp\{x\} \in P$ and $f(Sp\{x, y\}) = [\overline{x}, \overline{y}]$ for every $Sp\{x, y\} \in L$, define an isomorphism from projective coordinate space (P, L) to the the space (P', L').*

3. Construction of the New PK-Coordinate Space

In this section, an $(n+1)-$dimensional module over the local ring $\boldsymbol{K} = \boldsymbol{M}_{mm}(\mathbb{R})$ is constructed. Then, an $n-$dimensional projective coordinate space (PK-coordinate space) over this module is constructed with the help of equivalence classes. The points and lines of this space are determined and the points are classified. Finally, a $3-$dimensional projective coordinate space is examined.

The set $M = \mathbb{R}^m_{n+1}$ is an $(n+1)-$dimensional module over the local ring $\boldsymbol{K} = \boldsymbol{M}_{mm}(\mathbb{R})$ and

$$\left\{ \begin{array}{c} E_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}, \\ E_3 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0\cdots & 0 \end{pmatrix}, \cdots, E_{n+1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \end{array} \right\}$$

is a basis of M. Furthermore a maximal ideal of $\boldsymbol{K}$ is denoted by $I = \eta_1 \boldsymbol{K}$.

Each element of a $\boldsymbol{K}$-module M can be expressed uniquely as a linear combination of $E_1, E_2, \ldots, E_{n+1}$ as follows:

$$\begin{aligned} X &= \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1(n+1)} \\ x_{21} & x_{22} & \cdots & x_{2(n+1)} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{m(n+1)} \end{pmatrix} \\ &= \begin{pmatrix} x_{m1} & x_{(m-1)1} & \cdots & x_{11} \\ 0 & x_{m1} & \cdots & x_{21} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & x_{m1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \\ &+ \cdots + \begin{pmatrix} x_{m(n+1)} & x_{(m-1)(n+1)} & \cdots & x_{1(n+1)} \\ 0 & x_{m(n+1)} & \cdots & x_{2(n+1)} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & x_{m(n+1)} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \end{aligned}$$

Let us define the set

$$M_0 = \left\{ \sum_{i=1}^{n+1} A_i E_i \,|A_i \in I,\ 1 \leq i \leq n+1 \right\}.$$

Then, we get

$$M_0 = \left\{ \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1(n+1)} \\ \vdots & \vdots & \vdots & \vdots \\ x_{(m-1)1} & x_{(m-1)2} & \cdots & x_{(m-1)(n+1)} \\ 0 & 0 & \cdots & 0 \end{pmatrix} \Bigg|\, x_{ij} \in \mathbb{R} \right\}.$$

Now, we consider equivalence relation on the elements of

$$M^* = M \setminus M_0 = \left\{ \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1(n+1)} \\ x_{21} & x_{22} & \cdots & x_{2(n+1)} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{m(n+1)} \end{pmatrix} |1 \leq i \leq n+1,\ \exists\, x_{mi} \neq 0 \right\}$$

whose equivalence classes are the one-dimensional submodules of M with the set M_0 deleted. Thus, if $X, Y \in M^*$, then X is equivalent to Y if $Y = \lambda X$ for $\lambda \in \boldsymbol{K}^* = \boldsymbol{K} \setminus I$. The set of equivalence classes is denoted by $P(M)$. Then $P(M)$ is called an $n-$dimensional projective coordinate space and the elements of $P(M)$ are called points; the equivalence class of vector X is the point $\overline{X}$. Consequently, X is called a coordinate vector for $\overline{X}$ or that X is a vector representing of $\overline{X}$. In this case, λX with $\lambda \in \boldsymbol{K}^*$ also represents $\overline{X}$; that is, by $\overline{\lambda X} = \overline{X}$. Thus, $\overline{X}$ can be expressed as follows:

$$\begin{aligned}
\overline{X} &= \begin{pmatrix} a_0 & a_1 & \cdots & a_{m-1} \\ 0 & a_0 & \cdots & a_{m-2} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & a_0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1(n+1)} \\ x_{21} & x_{22} & \cdots & x_{2(n+1)} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{m(n+1)} \end{pmatrix} \\
&= \begin{pmatrix} \sum\limits_{i=0}^{m-1} a_i x_{(i+1)1} & \sum\limits_{i=0}^{m-1} a_i x_{(i+1)2} & \cdots & \sum\limits_{i=0}^{m-1} a_i x_{(i+1)(n+1)} \\ \sum\limits_{i=0}^{m-2} a_i x_{(i+2)1} & \sum\limits_{i=0}^{m-2} a_i x_{(i+2)2} & \cdots & \sum\limits_{i=0}^{m-2} a_i x_{(i+2)(n+1)} \\ \vdots & \vdots & \vdots & \vdots \\ a_0 x_{m1} & a_0 x_{m2} & \cdots & a_0 x_{m(n+1)} \end{pmatrix}
\end{aligned}$$

where $a_0 \neq 0 \quad \wedge \quad 1 \leq i \leq n+1, \quad \exists x_{mi} \neq 0$.

Let $\overline{X}, \overline{Y}, \cdots$ be $m+1$ points such that any two of them are $\boldsymbol{K} -$ *independent*. Then the set $\Pi_m = Sp\{\overline{X}, \overline{Y}, \cdots\} \backslash M_0$ is called *a subspace of dimension m* or *m−space.*

In $P(M)$, a point is a subspace of dimension 0 and a line is a subspace of dimension 1.

For $X \in M^*$, the set $\overline{X} = \{\lambda X \,|\, \lambda \in \boldsymbol{K}^*\}$ is a $0-$dimensional subspace of $P(M)$. So, $\overline{X}$ is a point of $P(M)$.

Now, we investigate the condition of being $\boldsymbol{K}-$independent for two different points $\overline{X}$ and $\overline{Y}$ of $P(M)$.

Firstly, let us denote the coordinate vectors for the points $\overline{X}$ and $\overline{Y}$ by X and Y, respectively. We form a linear combination as

$$\begin{aligned}
&\begin{pmatrix} a_0 & a_1 & \cdots & a_{m-1} \\ 0 & a_0 & \cdots & a_{m-2} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & a_0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1(n+1)} \\ x_{21} & x_{22} & \cdots & x_{2(n+1)} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{m(n+1)} \end{pmatrix} + \\
&\begin{pmatrix} b_0 & b_1 & \cdots & b_{m-1} \\ 0 & b_0 & \cdots & b_{m-2} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & b_0 \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1(n+1)} \\ y_{21} & y_{22} & \cdots & y_{2(n+1)} \\ \vdots & \vdots & \vdots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{m(n+1)} \end{pmatrix} = \\
&= \begin{pmatrix} \sum\limits_{i=0}^{m-1} a_i x_{(i+1)1} + \sum\limits_{i=0}^{m-1} b_i y_{(i+1)1} & \cdots & \sum\limits_{i=0}^{m-1} a_i x_{(i+1)(n+1)} + \sum\limits_{i=0}^{m-1} b_i y_{(i+1)(n+1)} \\ \sum\limits_{i=0}^{m-2} a_i x_{(i+2)1} + \sum\limits_{i=0}^{m-2} b_i y_{(i+2)1} & \cdots & \sum\limits_{i=0}^{m-2} a_i x_{(i+2)(n+1)} + \sum\limits_{i=0}^{m-2} b_i y_{(i+2)(n+1)} \\ \vdots & \vdots & \vdots \\ a_0 x_{m1} + b_0 y_{m1} & \cdots & a_0 x_{m(n+1)} + b_0 y_{m(n+1)} \end{pmatrix}.
\end{aligned}$$

If this matrix is an element of M_0 then we can write

$$\begin{aligned}
a_0 x_{m1} + b_0 y_{m1} &= 0, \\
a_0 x_{m2} + b_0 y_{m2} &= 0, \\
&\vdots \\
a_0 x_{m(n+1)} + b_0 y_{m(n+1)} &= 0.
\end{aligned} \tag{1}$$

Let us denote the coefficient matrix of (1) by

$$A = \begin{pmatrix} x_{m1} & y_{m1} \\ x_{m2} & x_{m2} \\ \vdots & \vdots \\ x_{m(n+1)} & x_{m(n+1)} \end{pmatrix}.$$

If $rankA = 2$, then we get $a_0 = b_0 = 0$. So this shows that

$$\begin{pmatrix} 0 & a_1 & \cdots & a_{m-1} \\ 0 & 0 & \cdots & a_{m-2} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_1 & \cdots & b_{m-1} \\ 0 & 0 & \cdots & b_{m-2} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \in I.$$

In that case, the coordinate vectors X and Y for the points $\overline{X}$ and $\overline{Y}$, respectively, are $\boldsymbol{K}-$ *independent* if and only if the rank of the coefficient matrix is equal to 2. That is, last rows of the coordinate vectors X and Y are linearly independent vectors.

Let the set $Sp\{\overline{X}, \overline{Y}\} = \left\{\lambda X + \gamma Y \,\middle|\, \exists\, \lambda, \gamma \in \boldsymbol{K}^*\right\}$ be a 1-dimensional subspace of $P(M)$ such that $\overline{X}$ and $\overline{Y}$ are $\boldsymbol{K}-$ *independent* elements. Then $Sp\{\overline{X}, \overline{Y}\}$ is a line of $P(M)$. It is denoted by

$$Sp\{\overline{X}, \overline{Y}\} = \begin{array}{l} \begin{pmatrix} a_0 & a_1 & \cdots & a_{m-1} \\ 0 & a_0 & \cdots & a_{m-2} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & a_0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1(n+1)} \\ x_{21} & x_{22} & \cdots & x_{2(n+1)} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{m(n+1)} \end{pmatrix} \\ + \begin{pmatrix} b_0 & b_1 & \cdots & b_{m-1} \\ 0 & b_0 & \cdots & b_{m-2} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & b_0 \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1(n+1)} \\ y_{21} & y_{22} & \cdots & y_{2(n+1)} \\ \vdots & \vdots & \vdots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{m(n+1)} \end{pmatrix}, \end{array}$$

where $a_0 \neq 0 \wedge 1 \leq i \leq n+1, \exists\, x_{mi} \neq 0$ or $b_0 \neq 0 \wedge 1 \leq i \leq n+1, \exists y_{mi} \neq 0$.

We know that for every coordinate vector $X \in M^*$ of the point $\overline{X} \in P(M)$, X can be written uniquely as a linear combination of the vectors $E_1, E_2, \cdots, E_{n+1}$. So the matrix X is expressed as $X = \sum_{i=1}^{n+1} x_i E_i$ or as

$$X = (X_1, X_2, \cdots, X_n) \in \boldsymbol{K}^{n+1},$$

where

$$X_1 = \begin{pmatrix} x_{m1} & x_{(m-1)1} & \cdots & x_{11} \\ 0 & x_{m1} & \cdots & x_{21} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & x_{m1} \end{pmatrix}, X_2 = \begin{pmatrix} x_{m2} & x_{(m-1)2} & \cdots & x_{12} \\ 0 & x_{m2} & \cdots & x_{22} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & x_{m2} \end{pmatrix},$$

$$\cdots, X_{n+1} = \begin{pmatrix} x_{m(n+1)} & x_{(m-1)(n+1)} & \cdots & x_{1(n+1)} \\ 0 & x_{m(n+1)} & \cdots & x_{2(n+1)} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & x_{m(n+1)} \end{pmatrix}.$$

There are two cases:

Case 1: For the first component of the coordinate vector X of the point $\overline{X}$, if $x_{m1} \neq 0$, then $X_1 \notin I$ and

$$X_1 = \begin{pmatrix} x_{m1} & x_{(m-1)1} & \cdots & x_{11} \\ 0 & x_{m1} & \cdots & x_{21} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & x_{m1} \end{pmatrix}$$

is a unit element so there is an inverse of X_1. If we multiply both sides of the equation with the inverse matrix X_1^{-1}, we get

$$X = (I_m, X_2, \cdots, X_{n+1}) = \begin{pmatrix} 0 & x_{12} & \cdots & x_{1(n+1)} \\ 0 & x_{22} & \cdots & x_{2(n+1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{m2} & \cdots & x_{m(n+1)} \end{pmatrix}.$$

Thus, this type of points are called *proper points*.

Case 2: For the first component of the coordinate vector X of the point $\overline{X}$, if $x_{m1} = 0$, then $X_1 \in I$. So, the inverse of the matrix X_1 does not exist. Thus we call the points of $P(M)$ whose coordinate vectors are in the form

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & x_{m2} & \cdots & x_{mm} \end{pmatrix}$$

as *ideal points*.

Following expressions are valid for $s-$dimensional subspaces Π_s for $s = -1, 0, 1, \cdots, (n-1)$:

An $s-$space is the set of points whose representing vectors

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1(n+1)} \\ x_{21} & x_{22} & \cdots & x_{2(n+1)} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{m(n+1)} \end{pmatrix}$$

of the points $\overline{X}$ satisfy the equations $XA = 0$, where A is an $(n+1) \times (n-s)$ matrix of rank $n-s$ with coefficients in $\boldsymbol{K}$.

Now, we examine a $3-$dimensional projective coordinate space $P(M)$ by taking $m = 2$ and $n = 3$. For the $3-$dimensional projective coordinate space, we determine the incidence matrix of a line that goes through the given points. Also we determine all points of a line by the Maple programme whose incidence matrix is given.

Example 3.1. *In the* $3-$*dimensional projective coordinate space* $P(M)$, *any line, namely* 1-*dimensional subspace* Π_1 *is the set of points whose representing vectors* $\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{pmatrix}$ *of the points* $\overline{X}$ *satisfy the equations* $XA = 0$, *where* A *is a* 4×2 *matrix of rank* 2 *with coefficients in* $\boldsymbol{K}$. *Thus* $\Pi_1 = \left\{\overline{X} \ \middle|XA = 0, A \in \boldsymbol{K}_2^4 \diagdown I_2^4\right\}$ *is obtained. Now, we* *identify all points of a line whose incidence matrix is*

$$\begin{bmatrix} a & e \\ b & f \\ c & g \\ d & h \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix} & \begin{pmatrix} e_0 & e_1 \\ 0 & e_0 \end{pmatrix} \\ \begin{pmatrix} b_0 & b_1 \\ 0 & b_0 \end{pmatrix} & \begin{pmatrix} f_0 & f_1 \\ 0 & f_0 \end{pmatrix} \\ \begin{pmatrix} c_0 & c_1 \\ 0 & c_0 \end{pmatrix} & \begin{pmatrix} g_0 & g_1 \\ 0 & g_0 \end{pmatrix} \\ \begin{pmatrix} d_0 & d_1 \\ 0 & d_0 \end{pmatrix} & \begin{pmatrix} h_0 & h_1 \\ 0 & h_0 \end{pmatrix} \end{bmatrix} \in \boldsymbol{K}_2^4 \diagdown I_2^4.$$

As a consequence of the incidence matrix, it is trivial to see that $\exists a_0, b_0, c_0, d_0, e_0, f_0, g_0, h_0 \neq 0$.

For $XA = 0$, *we have the following cases:*

Case 1: *For the coordinate vector* X *of the point* $\overline{X}$, *if* $x_{21} \neq 0$, *then* $X = (I_2, X_2, X_3, X_4) \in \boldsymbol{K}^4$. *Thus we obtain the following equations from* $XA = 0$*:*

$$\begin{aligned} ca_0 + x_{22}b_0 + x_{23}c_0 + x_{24}d_0 &= 0, \\ a_1 + x_{22}b_1 + x_{12}b_0 + x_{23}c_1 + x_{13}c_0 + x_{24}d_1 + x_{14}d_0 &= 0, \\ e_0 + x_{22}f_0 + x_{23}g_0 + x_{24}h_0 &= 0, \\ e_1 + x_{22}f_1 + x_{12}f_0 + x_{23}g_1 + x_{13}g_0 + x_{24}h_1 + x_{14}h_0 &= 0. \end{aligned} \tag{2}$$

If we solve (2) *by using the Maple programme, we get the following solutions:*

$$\begin{aligned} x_{12} &= \frac{a'}{(b_0g_0 - c_0f_0)^2}, \\ x_{13} &= -\frac{b'}{b_0^2g_0^2 - 2c_0f_0b_0g_0 + f_0^2c_0^2}, \\ x_{22} &= -\frac{c'}{b_0g_0 - c_0f_0}, \\ x_{23} &= \frac{d'}{b_0g_0 - c_0f_0}, \\ x_{14} &= x_{14}, x_{24} = x_{24}, \end{aligned}$$

where

$$\begin{aligned} a' &= \begin{pmatrix} (c_0g_1f_0a_0 + g_0f_0c_0a_1 - g_0c_0f_1a_0 - g_0b_1c_0e_0 - g_0c_1f_0a_0 + b_0g_0c_0e_1 \\ +b_0g_0c_1e_0 - b_0c_0g_1e_0 - f_0c_0^2e_1 + f_1c_0^2e_0 + g_0^2b_1a_0 - b_0g_0^2a_1)+ \\ (-f_0c_0^2h_0 - b_0g_0^2d_0 + g_0f_0c_0d_0 + b_0g_0c_0h_0)x_{14} \\ +(c_0g_1f_0d_0 - f_0c_0^2h_1 + f_1c_0^2h_0 + g_0^2b_1d_0 - b_0g_0^2d_1 + g_0f_0c_0d_1- \\ g_0c_0f_1d_0 - g_0c_1f_0d_0 - g_0b_1c_0h_0 + b_0g_0c_1h_0 + b_0g_0c_0h_1 - b_0c_0g_1h_0)x_{24} \end{pmatrix}, \\ b' &= \begin{pmatrix} (b_0g_1f_0a_0 + f_1c_0b_0e_0 + f_0b_0c_1e_0 - f_0b_0g_0a_1 + f_0g_0b_1a_0- \\ f_0b_1c_0e_0 - b_0g_0f_1a_0 - f_0c_0b_0e_1 - b_0^2g_1e_0 + b_0^2g_0e_1 - c_1f_0^2a_0 + f_0^2c_0a_1) \\ +(f_0^2c_0d_0 + b_0^2g_0h_0 - f_0c_0b_0h_0 - f_0b_0g_0d_0)x_{14} \\ +(-b_0^2g_1h_0 - c_1f_0^2d_0 + f_0^2c_0d_1 + b_0^2g_0h_1 - b_0g_0f_1d_0 - f_0c_0b_0h_1 \\ +b_0g_1f_0d_0 + f_1c_0b_0h_0 + f_0g_0b_1d_0 + f_0b_0c_1h_0 - f_0b_0g_0d_1 - f_0b_1c_0h_0)x_{24} \end{pmatrix}, \\ c' &= (g_0a_0 - c_0e_0) + (-c_0h_0 + g_0d_0)x_{24}, d' = (-b_0e_0 + f_0a_0) + (f_0d_0 - b_0h_0)x_{24}. \end{aligned}$$

Thus, we get

$$\overline{X} = \left\{ \begin{pmatrix} 0 & \frac{a'}{(b_0g_0-c_0f_0)^2} & -\frac{b'}{b_0^2g_0^2-2c_0f_0b_0g_0+f_0^2c_0^2} & x_{14} \\ 1 & -\frac{c'}{b_0g_0-c_0f_0} & \frac{d'}{b_0g_0-c_0f_0} & x_{24} \end{pmatrix} \middle| x_{14}, x_{24} \in \mathbb{R} \right\}.$$

Case 2: *For the coordinate vector X of the point $\overline{X}$, if $x_{21} = 0$, then $\overline{X}$ is an ideal point of the form*

$$\left(\begin{array}{c} X_1 = \begin{pmatrix} 0 & x_{11} \\ 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} x_{22} & x_{12} \\ 0 & x_{22} \end{pmatrix}, \\ X_3 = \begin{pmatrix} x_{23} & x_{13} \\ 0 & x_{23} \end{pmatrix}, X_4 = \begin{pmatrix} x_{24} & x_{14} \\ 0 & x_{24} \end{pmatrix} \end{array} \right).$$

Here, we know that $\exists\ x_{22}, x_{23}, x_{24} \neq 0$. Let $x_{22} \neq 0$, then there is an inverse of X_2. If we multiply both sides of the equation with the inverse matrix X_2^{-1}, we get $X = (X_1, I_2, X_3, X_4) \in \boldsymbol{K}^4$. Thus we obtain the following equations from $XA = 0$:

$$\begin{aligned} b_0 + x_{23}c_0 + x_{24}d_0 &= 0, \\ x_{11}a_0 + b_1 + x_{23}c_1 + x_{13}c_0 + x_{24}d_1 + x_{14}d_0 &= 0, \\ f_0 + x_{23}g_0 + x_{24}h_0 &= 0, \\ x_{11}e_0 + f_1 + x_{23}g_1 + x_{13}g_0 + x_{24}h_1 + x_{14}h_0 &= 0. \end{aligned} \tag{3}$$

If we solve (3) *by using the Maple programme, we get the following solutions:*

$$\begin{aligned} x_{11} &= x_{11}, \\ x_{13} &= -\frac{a''}{(c_0h_0 - g_0d_0)^2}, \\ x_{14} &= \frac{b''}{-2g_0d_0c_0h_0 + g_0^2d_0{}^2 + c_0^2h_0^2}, \\ x_{23} &= c'', \quad x_{24} = d'', \end{aligned}$$

where

$$\begin{aligned} a'' &= \left(\begin{array}{c} (c_0h_0^2b_1 - c_0h_0d_0f_1 - c_0h_0d_1f_0 + c_0d_0h_1f_0 - c_1b_0h_0^2 + h_0d_1g_0b_0 + \\ h_0d_0g_1b_0 + h_0c_1d_0f_0 - h_0g_0d_0b_1 - d_0h_1g_0b_0 - g_1d_0^2f_0 + g_0d_0^2f_1) + \\ (c_0h_0^2a_0 - c_0h_0d_0e_0 - h_0g_0d_0a_0 + g_0d_0^2e_0)x_{11} \end{array} \right), \\ b'' &= \left(\begin{array}{c} (-c_0h_0^2f_1 + g_0c_0h_0b_1 + c_0^2h_1f_0 + c_0h_0g_1b_0 - g_0^2d_0b_1 + g_0d_0c_0f_1 - \\ c_0h_1g_0b_0 - g_1d_0c_0f_0 - g_0h_0c_1b_0 - g_0c_0d_1f_0 + d_1g_0^2b_0 + g_0c_1d_0f_0) + \\ (-c_0^2h_0e_0 + g_0d_0c_0e_0 + g_0c_0h_0a_0)x_{11} \end{array} \right), \\ c'' &= -\frac{-h_0b_0 - d_0f_0}{-c_0h_0 - g_0d_0} \text{ and } d'' = \frac{-c_0f_0 + g_0b_0}{c_0h_0 - g_0d_0}. \end{aligned}$$

Thus, we get

$$\overline{X} = \left\{ \begin{pmatrix} x_{11} & 0 & -\frac{a''}{(c_0h_0 - g_0d_0)^2} & \frac{b''}{-2g_0d_0c_0h_0 + g_0^2d_0{}^2 + c_0^2h_0^2} \\ 0 & 1 & c'' & d'' \end{pmatrix} | x_{11} \in \mathbb{R} \right\}.$$

Now conversely, we have a new situation. We determine the incidence matrix of a line whose points are given. This also has two cases:

Case 1: *Let us take the coordinate vectors*

$$X = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ 1 & x_{22} & x_{23} & x_{24} \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & y_{12} & y_{13} & y_{14} \\ 1 & y_{22} & y_{23} & y_{24} \end{pmatrix}$$

of proper points $\overline{X}$ and $\overline{Y}$, respectively. Then we search the incidence matrix of the form

$$A = \begin{bmatrix} a & e \\ b & f \\ c & g \\ d & h \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix} & \begin{pmatrix} e_0 & e_1 \\ 0 & e_0 \end{pmatrix} \\ \begin{pmatrix} b_0 & b_1 \\ 0 & b_0 \end{pmatrix} & \begin{pmatrix} f_0 & f_1 \\ 0 & f_0 \end{pmatrix} \\ \begin{pmatrix} c_0 & c_1 \\ 0 & c_0 \end{pmatrix} & \begin{pmatrix} g_0 & g_1 \\ 0 & g_0 \end{pmatrix} \\ \begin{pmatrix} d_0 & d_1 \\ 0 & d_0 \end{pmatrix} & \begin{pmatrix} h_0 & h_1 \\ 0 & h_0 \end{pmatrix} \end{bmatrix} \in \boldsymbol{K}_2^4 \diagdown I_2^4.$$

If we take coordinate vectors of these points as

$$X = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x_{22} & x_{12} \\ 0 & x_{22} \end{pmatrix}, \begin{pmatrix} x_{23} & x_{13} \\ 0 & x_{23} \end{pmatrix}, \begin{pmatrix} x_{24} & x_{14} \\ 0 & x_{24} \end{pmatrix} \right)$$

and

$$Y = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} y_{22} & y_{12} \\ 0 & y_{22} \end{pmatrix}, \begin{pmatrix} y_{23} & y_{13} \\ 0 & y_{23} \end{pmatrix}, \begin{pmatrix} y_{24} & y_{14} \\ 0 & y_{24} \end{pmatrix} \right)$$

we obtain the following equations from $XA = 0$ and $YA = 0$:

$$\begin{aligned}
a_0 + x_{22}b_0 + x_{23}c_0 + x_{24}d_0 &= 0, \\
a_1 + x_{22}b_1 + x_{12}b_0 + x_{23}c_1 + x_{13}c_0 + x_{24}d_1 + x_{14}d_0 &= 0, \\
e_0 + x_{22}f_0 + x_{23}g_0 + x_{24}h_0 &= 0, \\
e_1 + x_{22}f_1 + x_{12}f_0 + x_{23}g_1 + x_{13}g_0 + x_{24}h_1 + x_{14}h_0 &= 0, \\
a_0 + y_{22}b_0 + y_{23}c_0 + y_{24}d_0 &= 0, \\
a_1 + y_{22}b_1 + y_{12}b_0 + y_{23}c_1 + y_{13}c_0 + y_{24}d_1 + y_{14}d_0 &= 0, \\
e_0 + y_{22}f_0 + y_{23}g_0 + y_{24}h_0 &= 0, \\
e_1 + y_{22}f_1 + y_{12}f_0 + y_{23}g_1 + y_{13}g_0 + y_{24}h_1 + y_{14}h_0 &= 0.
\end{aligned} \tag{4}$$

If we solve (4) *by using the Maple programme, then we get the following solutions:*

$$\begin{aligned}
a_0 &= -\frac{a'_0}{-y_{22} + x_{22}}, \ a_1 = -\frac{a'_1}{(-y_{22} + x_{22})^2}, \\
b_0 &= -\frac{b'_0}{-y_{22} + x_{22}}, \ b_1 = \frac{b'_1}{x_{22}^2 + y_{22}^2 - 2y_{22}x_{22}}, \\
c_0 &= c_0, \ c_1 = c_1, \ d_0 = d_0, \ d_1 = d_1, \\
e_0 &= -\frac{e'_0}{-y_{22} + x_{22}}, \ e_1 = -\frac{e'_1}{(-y_{22} + x_{22})^2}, \\
f_0 &= -\frac{f'_0}{-y_{22} + x_{22}}, \ f_1 = \frac{f'_1}{x_{22}^2 + y_{22}^2 - 2y_{22}x_{22}}, \\
g_0 &= g_0, \ g_1 = g_1, \ h_0 = h_0, \ h_1 = h_1,
\end{aligned}$$

where

$$\begin{aligned}
a'_0 &= (x_{22}y_{23} - y_{22}x_{23})c_0 + (x_{22}y_{24} - y_{22}x_{24})d_0, \\
a'_1 &= \begin{pmatrix} (x_{22}^2 y_{14} + x_{22}y_{12}y_{24} - x_{22}y_{12}x_{24} - y_{22}x_{22}x_{14} - y_{22}x_{22}y_{14} + y_{22}^2 x_{14} + \\ y_{22}x_{12}x_{24} - y_{22}x_{12}y_{24})d_0 + (x_{22}^2 y_{24} - y_{22}x_{22}y_{24} - y_{22}x_{22}x_{24} + y_{22}^2 x_{24})d_1 \\ +(x_{22}^2 y_{13} - y_{22}x_{22}x_{13} - y_{22}x_{22}y_{13} - x_{22}y_{12}x_{23} + x_{22}y_{12}y_{23} + y_{22}^2 x_{13} \\ -y_{22}x_{12} + y_{22}x_{12}x_{23})c_0 + (x_{22}^2 y_{23} - y_{22}x_{22}x_{23} - y_{22}x_{22}x_{23} + y_{22}^2 x_{23})c_1 \end{pmatrix}, \\
b'_0 &= (x_{23} - y_{23})c_0 + (x_{24} - y_{24})d_0,
\end{aligned}$$

$$\begin{aligned}
b_1' &= \left(\begin{array}{c} (-x_{22}x_{13} + y_{22}x_{13} - x_{12}y_{23} + x_{12}x_{23} - y_{22}y_{13} + y_{12}y_{23} + x_{22}y_{13} - y_{12}x_{23})c_0 \\ +(-y_{22}y_{23} + y_{22}x_{23} - x_{22}x_{23} + x_{22}y_{23})c_1 \\ +(x_{12}x_{24} - x_{12}y_{24} + y_{22}x_{14} - x_{22}x_{14} + y_{12}y_{24} - y_{22}y_{14} + x_{22}y_{14})d_0 \\ +(y_{22}x_{24} - x_{22}x_{24} - y_{22}y_{24} + x_{22}x_{24})d_1 \end{array}\right), \\
e_0' &= (x_{22}y_{23} - y_{22}x_{23})g_0 + (x_{22}y_{24} - y_{22}x_{24})h_0, \\
e_1' &= \left(\begin{array}{l} (x_{22}^2y_{13} - y_{22}x_{22}y_{13} + x_{22}y_{12}y_{23} - x_{22}y_{12}x_{23} - y_{22}x_{22}x_{13} - y_{22}x_{12}y_{23} \\ +y_{22}x_{12}x_{23} + y_{22}^2x_{13})g_0 + (x_{22}^2y_{23} - y_{22}x_{22}y_{23} - y_{22}x_{22}x_{23} + y_{22}^2x_{23})g_1 \\ +(x_{22}^2y_{14} - y_{22}x_{22}x_{14} - y_{22}x_{22}y_{14} + x_{22}y_{12}y_{24} - x_{22}y_{12}x_{24} + y_{22}x_{12}x_{24} \\ -y_{22}x_{12}y_{24} + y_{22}^2x_{14})h_0 + (x_{22}^2y_{24} - y_{22}x_{22}x_{24} - y_{22}x_{22}x_{24} + y_{22}^2x_{24})h_1 \end{array}\right), \\
f_0' &= (x_{24} - y_{24})h_0 + (x_{23} - y_{23})g_0, \\
f_1' &= \left(\begin{array}{c} (x_{12}x_{24} - x_{12}y_{24} + y_{22}x_{14} - x_{22}x_{14} + y_{12}y_{24} - y_{22}y_{14} + x_{22}y_{14} - y_{12}x_{24})h_0 \\ +(y_{22}x_{24} - x_{22}x_{24} - y_{22}y_{24} + x_{22}y_{24})h_1 \\ +(y_{22}x_{13} - x_{22}x_{13} - x_{12}y_{23} + x_{12}x_{23} - y_{22}y_{13} + y_{12}y_{23} + x_{22}y_{13} - y_{12}x_{23})g_0 \\ +(y_{22}y_{23} + y_{22}x_{23} - x_{22}x_{23} + x_{22}y_{23})g_1 \end{array}\right).
\end{aligned}$$

Case 2: *Let us take the coordinate vectors*

$$X = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ 1 & x_{22} & x_{23} & x_{24} \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14} \\ 0 & y_{22} & y_{23} & y_{24} \end{pmatrix}$$

of proper and ideal points $\overline{X}$ *and* $\overline{Y}$, *respectively. Here for the point* $\overline{Y}$, *we know that* $\exists\, y_{22}, y_{23}, y_{24} \neq 0$. *So let* $y_{22} \neq 0$, *then coordinate vectors can be expressed as*

$$X = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x_{22} & x_{12} \\ 0 & x_{22} \end{pmatrix}, \begin{pmatrix} x_{23} & x_{13} \\ 0 & x_{23} \end{pmatrix}, \begin{pmatrix} x_{24} & x_{14} \\ 0 & x_{24} \end{pmatrix} \right)$$

and

$$Y = \left(\begin{pmatrix} 0 & y_{11} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} y_{23} & y_{13} \\ 0 & y_{23} \end{pmatrix}, \begin{pmatrix} y_{24} & y_{14} \\ 0 & y_{24} \end{pmatrix} \right).$$

We obtain the following equations from $XA = 0$ *and* $YA = 0$:

$$\begin{aligned}
a_0 + x_{22}b_0 + x_{23}c_0 + x_{24}d_0 &= 0, \\
a_1 + x_{22}b_1 + x_{12}b_0 + x_{23}c_1 + x_{13}c_0 + x_{24}d_1 + x_{14}d_0 &= 0, \\
e_0 + x_{22}f_0 + x_{23}g_0 + x_{24}h_0 &= 0, \\
e_1 + x_{22}f_1 + x_{12}f_0 + x_{23}g_1 + x_{13}g_0 + x_{24}h_1 + x_{14}h_0 &= 0, \\
b_0 + y_{23}c_0 + y_{24}d_0 &= 0, \\
y_{11}a_0 + b_1 + y_{23}c_1 + y_{13}c_0 + y_{24}d_1 + y_{14}d_0 &= 0, \\
f_0 + y_{23}g_0 + y_{24}h_0 &= 0, \\
y_{11}e_0 + f_1 + y_{23}g_1 + y_{13}g_0 + y_{24}h_1 + y_{14}h_0 &= 0.
\end{aligned} \tag{5}$$

If we solve (5) *by using the Maple programme, then we get the following solutions:*

$$\begin{aligned}
a_0 &= a_0'', \ a_1 = a_1'', \ b_0 = b_0'', \ b_1 = b_1'', \\
c_0 &= c_0, \ c_1 = c_1, \ d_0 = d_0, \ d_1 = d_1, \\
e_0 &= e_0'', \ e_1 = e_1'', \ f_0 = f_0'', \ f_1 = f_1'',
\end{aligned}$$

where

$$
\begin{aligned}
a_0'' &= (x_{22}y_{23} - x_{23})c_0 + (x_{22}y_{24} - x_{24})d_0, \\
a_1'' &= \begin{pmatrix} (y_{11}x_{22}^2y_{23} - x_{22}y_{11}x_{23} + x_{22}y_{13} + x_{12}y_{23} - x_{13})c_0 + (x_{22}y_{23} - x_{23})c_1 \\ +(x_{22}y_{14} - x_{22}y_{11}x_{24} + y_{11}x_{22}^2y_{24} - x_{14} + x_{12}y_{24})d_0 + (x_{22}y_{24} - x_{24})d_1 \end{pmatrix}, \\
b_0'' &= -y_{23}c_0 - y_{24}d_0, \\
b_1'' &= (-y_{11}x_{22}y_{23} + y_{11}x_{23} - y_{13})c_0 - y_{23}c_1 + (-y_{14} + y_{11}x_{24} - y_{11}x_{22}y_{24})d_0 - y_{24}d_1, \\
e_0'' &= (x_{22}y_{23} - x_{23})g_0 + (x_{22}y_{24} - x_{24})h_0, \\
e_1'' &= \begin{pmatrix} (y_{11}x_{22}^2y_{23} - x_{22}y_{11}x_{23} + x_{22}y_{13} + x_{12}y_{23} - x_{23})g_0 + (x_{22}y_{23})g_1 \\ +(x_{22}y_{14} - x_{22}y_{11}x_{24} + y_{11}x_{22}^2y_{24} - x_{14} + x_{12}y_{24})h_0 + (x_{22}y_{24} - x_{24})h_1 \end{pmatrix}, \\
f_0'' &= -y_{23}g_0 - y_{24}h_0, \\
f_1'' &= (-y_{11}x_{22}y_{23} + y_{11}x_{23} - y_{13})g_0 - y_{23}g_1 + (-y_{14} + y_{11}x_{24} - y_{11}x_{22}y_{24})h_0 - y_{24}h_1.
\end{aligned}
$$

Consequently, let $\mathbf{a} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, $\mathbf{e} = \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix}$, $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $A = \begin{bmatrix} \mathbf{a} & \mathbf{e} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, *then it is obvious that all ideal points satisfy the equation* $XA = 0$. *This equality shows that matrix representation is not necessary for an ideal line.*

4. Conclusion

In this study, PK-coordinate space is constructed for the special case $m = 2$ and $n = 3$. Similar studies can be done for different values of m and n. However it is seen that as m and n get greater values, the operations can no longer be done by hand. So the calculations must be done via computer.

Furthermore, in our study we consider only the points and lines of the PK-coordinate space which is obtained via the new method. Examination of other subspaces and finding some combinatorial results for the PK- coordinate space, are still open problems for interested researchers.

Note: The main results of the present paper are given in "AIP Conf. Proc. 1676, 020014 (2015); http://dx.doi.org/10.1063/1.4930440" without proof as an extended abstract.

Acknowledgement. This work, a part of PhD thesis of second author, is supported by The Research Fund of University of Uludag Project number KUAP(F)-2012/56.

References

[1] F. Ö. Erdoğan, Geometries Over Modules and Local Rings, PhD Thesis, Uludag University, Graduate School of Natural and Applied Sciences, Department of Mathematics, 126 p., 2014.

[2] F. Ö. Erdoğan, S. Çiftçi and A. Akpınar, On Modules Over Local Rings, An. Şt. Univ. Ovidius Constanta, Vol.:24(1), 2016.

[3] J.W.P. Hirschfeld, Projective Geometries over Finite Fields, Oxford Science Publications, New York, 1998.

[4] M. Jukl, Linear forms on free modules over certain local ring, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math., 32 (1993), 49-62.

[5] M. Jukl, Grassmann formula for certain type of modules, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math., 34 (1995), 69-74.

[6] M. Jukl and V. Snasel, Projective Equivalence of Quadrics in Klingenberg Projective Spaces Over a Special Local Ring, International Journal Of Geometry, 2 (2009), 34-38.

[7] H. Lenz, Vorlesungen über projektive Geometrie I. Leipzig, 1965.

[8] R. Lingenberg, Grundlagen der Geometrie I. Bibliographisches Institut Mannheim (Wien) Zürich, 1969.

[9] F. Machala, Fundamentalsatze der projektiven Geometrie mit Homomorphismus, Rozpravy CSAV, Rada Mat. Prirod. Ved, 90, 5, 1980.