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MAPPING PROPERTIES FOR CONIC REGIONS ASSOCIATED WITH WRIGHT FUNCTIONS

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Abstract. In this paper, we are mainly interested to find sufficient conditions for the convolution operator $\mathcal{Y}_{\lambda,\mu}f(z) = z\mathsf{W}_{\lambda,\mu}(z) * f(z)$ belonging to the classes $\mathcal{UCV}(k,\alpha)$, $\mathcal{S}_p(k,\alpha)$, S_ς^* and C_ς.

1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and $\mathcal{S}$ denotes the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. Let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ denote the classes of starlike and convex functions of order α and are defined as:

$$\mathcal{S}^*(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathcal{U},\, \alpha \in [0,1) \right\}$$

and

$$\mathcal{C}(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathcal{U},\, \alpha \in [0,1) \right\}.$$

It is clear that

$$\mathcal{S}^*(0) = \mathcal{S}^* \text{ and } \mathcal{C}(0) = \mathcal{C}.$$

These classes were introduced by Robertson in 1936, for more information about these classes see [12, 17]. In 1991 Goodman [4, 5] introduced

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the class of uniformly convex functions $\mathcal{UCV}$ and uniformly starlike functions $\mathcal{S}_p$. A function $f \in \mathcal{A}$ is uniformly convex if for every circular arc τ contained in the open unit disc with the center also in the open unit disc, the image of $f(\tau)$ is convex. In 1992 Ma and Minda [7] and in 1993 Ronning [13] proved that independently:

Definition 1.1. *A function $f \in \mathcal{A}$ is uniformly convex in the open unit disc if and only if*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|. \tag{2}$$

Equivalently, we can say that a function $f \in \mathcal{A}$ is uniformly convex in the open unit if $1+\frac{zf''(z)}{f'(z)}$, is in the parabolic region.

Definition 1.2. *A function $f \in \mathcal{A}$ is in $\mathcal{S}_p$ if*

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)}-1\right|. \tag{3}$$

Next, we introduce the subclasses of $k-$uniformly convex functions of order α and a new class related to the starlike functions. These classes were defined by Bharati et al. in 1997 [2] defined as:

Definition 1.3. *A function $f \in \mathcal{A}$ is in $\mathcal{UCV}(k,\alpha)$ if and only if*

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right| + \alpha, \quad z \in \mathcal{U}, \tag{4}$$

where $0 \le k < \infty$ and $0 \le \alpha < 1$.

By using the Alexander transform, we can get the class $\mathcal{S}_p(k,\alpha)$ defined as:

Definition 1.4. *A function $f \in \mathcal{UCV}(k,\alpha)$ if and only if $zf' \in \mathcal{S}_p(k,\alpha)$.*

In 1997 Ponnusamy and Ronning [8] were introduced the classes C_ς and S^*_ς. These classes defined as follow:

Definition 1.5. *If $f \in \mathcal{A}$ and*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \varsigma, \quad (z \in \mathcal{U},\ \varsigma > 0), \tag{5}$$

then $f \in C_\varsigma$.

Definition 1.6. *If $f \in \mathcal{A}$ and*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \varsigma, \quad (z \in \mathcal{U}, \varsigma > 0), \tag{6}$$

then $f \in S^*_\varsigma$.

In 2004 Swaminathan [18], was introduced a class $P^{\tau(\eta)}_\varphi$. This class will play an very important role in our main results. The class $P^{\tau(\eta)}_\varphi$ is defined as:

Definition 1.7. *If $f \in \mathcal{A}$ and satisfies*

$$\left|\frac{(1-\varphi)\frac{f(z)}{z} + \varphi f'(z) - 1}{2\tau(1-\eta) + (1-\varphi)\frac{f(z)}{z} + \varphi f'(z) - 1}\right| < 1, \tag{7}$$

where $\varphi \in [0, 1)$, $\eta < 1$ and τ belongs to the complex numbers except 0, then $f \in P^{\tau(\eta)}_\varphi$.

Remark 1.8. *If $\tau = e^{i\xi}\cos\xi$, for $\xi \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, then the class $P^{\tau(\eta)}_\varphi$ can also be defined as:*

(8)

$$P^{\tau(\eta)}_\varphi = \left\{f \in \mathcal{A} : \Re\left\{e^{i\sigma}(1-\varphi)\frac{f(z)}{z} + \varphi f'(z) - \eta\right\} > 0, \quad \sigma \in \mathbb{R}\right\}.$$

Recently Raza et al. [14] studied some geometric properties of Wright function

$$\mathrm{W}_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \mu)}, \ \lambda > -1, \ \mu \in \mathbb{C}.$$

This series is absolutely convergent in $\mathbb{C}$, when $\lambda > -1$ and absolutely convergent in open unit disc $\mathcal{U}$ for $\lambda = -1$. Furthermore this function is entire. The Wright functions were introduced by Wright [20] and have been used in the asymptotic theory of partitions, in the theory of integral transforms of the Hankel type and in Mikusinski operational calculus. Recently, Wright functions have been found in the solution of partial differential equations of fractional order. It was found that the corresponding Green functions can be represented in terms of the Wright function [9, 15]. For positive rational number λ, the Wright function can be represented in terms of generalized hypergeometric function. For some details see [3]. In particular, the function $\mathrm{W}_{1,v+1}(-z^2/4)$ can be expressed in terms of the Bessel functions J_v, given as:

$$J_v(z) = \left(\frac{z}{2}\right)^v \mathrm{W}_{1,v+1}\left(\frac{-z^2}{4}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+v}}{n!\Gamma(n+v+1)}.$$

The Wright function generalizes various functions like Array function, Wittakar function, entire auxiliary functions, etc. For the details, we refer to [3]. Prajapat and Raza et. al [10, 14] discussed some geometric properties of the following normalization of Wright functions

$$\mathsf{W}_{\lambda,\mu}(z) = 1 + \sum_{n=1}^{\infty} \frac{\Gamma(\mu)}{n!\Gamma(\lambda n + \mu)} z^n. \ \lambda > -1, \ \mu > 0, \ z \in \mathcal{U}, \tag{9}$$

which can also be written as

$$z\mathrm{W}_{\lambda,\mu}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)} z^n. \ \lambda > -1, \ \mu > 0, \ z \in \mathcal{U}, \tag{10}$$

where $\lambda > -1, \lambda + \mu > 0$.

Let $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n \ z^n,$$

then Hadamard product (or convolution) of f and g is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n \ z^n \quad (z \in \mathcal{U}).$$

Now, we introduce the convolution operator

$$\begin{aligned} \mathcal{Y}_{\lambda,\mu} f(z) &= z\mathsf{W}_{\lambda,\mu}(z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)} a_n z^n = z + \sum_{n=2}^{\infty} A_n z^n, \end{aligned}$$

where $A_n = \frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)} a_n$. In our present work we find some sufficient conditions under which the convolution operator $\mathcal{Y}_{\lambda,\mu} f(z)$ belonging to the classes $\mathcal{UCV}(k,\alpha)$, $\mathcal{S}_p(k,\alpha)$, S_ς^* and C_ς. Here, we add some references that are closely related with this study [1, 19].

To prove our main results, we shall need the following lemmas.

Lemma 1.9. [2] *A function $f \in \mathcal{A}$ is in $\mathcal{UCV}(k,\alpha)$ if it assures*

$$\sum_{n=2}^{\infty} n \{n(1+k) - (k+\alpha)\} |a_n| \le 1 - \alpha. \tag{11}$$

Lemma 1.10. [2] *A function $f \in \mathcal{A}$ is in $\mathcal{S}_p(k,\alpha)$ if it assures*

$$\sum_{n=2}^{\infty} \{n(1+k) - (k+\alpha)\} |a_n| \le 1 - \alpha. \tag{12}$$

Lemma 1.11. [18] *If* $f \in P_{\varphi}^{\tau(\eta)}$ *defined in* (8) , *then*

$$|a_n| \leq \frac{2\,|\tau|\,(1-\eta)}{1+\varphi(n-1)}. \tag{13}$$

Lemma 1.12. [6] *If* $f \in \mathcal{A}$ *and satisfy*

$$\sum_{n=2}^{\infty} (\zeta + n - 1)\,|a_n| \leq \zeta, \quad \zeta > 0, \tag{14}$$

then $f \in S_{\varsigma}^{*}$.

Lemma 1.13. [6] *If* $f \in \mathcal{A}$ *and satisfy*

$$\sum_{n=2}^{\infty} n\,(\zeta + n - 1)\,|a_n| \leq \zeta, \quad \zeta > 0, \tag{15}$$

then $f \in C_{\varsigma}$.

Remark 1.14. *The conditions defined in* (11) , (12) , (14) *and* (15) *are also necessary if* $f \in \mathcal{A}$ *of the form of*

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0.$$

2. Main Results

These main results are the connections between the several subclasses of analytic functions by using Wright functions. For more information about that type of connections with hypergeometric funtions and Bessel functions see [6, 8, 11, 16, 18].

Theorem 2.1. *Let* $\lambda > -1$, $\mu > 0$ *and* $\alpha \in [0,1)$ *with inequality such that*

$$\frac{4(1-\eta)\cos\xi}{\varphi}\left\{\frac{(2k-\alpha+3)\,(\mu+1)}{\mu\,(2\mu+1)}\right\} \leq (1-\alpha)\,.$$

If $f \in P_{\varphi}^{\tau(\eta)}$, $\varphi \in [0,1)$ *and* $\eta < 1$, *then the convolution operator* $\mathcal{Y}_{\lambda,\mu} f(z) \in \mathcal{UCV}\,(k,\alpha)$.

Proof. Consider

$$\begin{aligned} \mathcal{Y}_{\lambda,\mu} f(z) &= z\mathsf{W}_{\lambda,\mu}(z) * f(z) \\ &= z + \sum_{n=2}^{\infty} A_n z^n, \end{aligned}$$

where $A_n = \frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)} a_n$.

To show that the convolution operator $\mathcal{Y}_{\lambda,\mu} f(z) \in \mathcal{UCV}(k,\alpha)$. From Lemma 1.9 we will prove that

$$\sum_{n=2}^{\infty} n\left\{n(1+k)-(k+\alpha)\right\}|A_n| \leq 1-\alpha.$$

Now

$$\begin{aligned} &\sum_{n=2}^{\infty} n\left\{n(1+k)-(k+\alpha)\right\} \frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)}|a_n| \\ &\qquad \leq 2(1-\eta)\cos\xi \sum_{n=2}^{\infty} n\left\{n(1+k)-(k+\alpha)\right\} \\ &\qquad \frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)} \frac{1}{1+\varphi(n-1)}. \end{aligned} \tag{16}$$

Since, $\frac{n}{1+\varphi(n-1)} \leq \frac{1}{\varphi}$, $\forall n \geq 2$, therefore (16) becomes

$$\begin{aligned} &\sum_{n=2}^{\infty} n\left\{n(1+k)-(k+\alpha)\right\} \frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)}|a_n| \\ &\leq \frac{2(1-\eta)\cos\xi}{\varphi}\left\{\begin{array}{l} (1+k)\sum_{n=2}^{\infty}\left(\frac{n\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)}\right) \\ -(k+\alpha)\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)} \end{array}\right\} \\ &= \frac{2(1-\eta)\cos\xi}{\varphi}\left\{\begin{array}{l} (1+k)\sum_{n=2}^{\infty}\left(\frac{(n-1+1)\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)}\right) \\ -(k+\alpha)\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)} \end{array}\right\} \\ &= \frac{2(1-\eta)\cos\xi}{\varphi}\left\{\begin{array}{l} (1+k)\sum_{n=2}^{\infty}\left(\frac{\Gamma(\mu)}{(n-2)!\Gamma(\lambda(n-1)+\mu)}\right) \\ +(1-\alpha)\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)} \end{array}\right\}. \end{aligned} \tag{17}$$

By using the inequaliteis

$$\begin{aligned} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} &\leq \frac{1}{(\mu)_{n-1}}, \quad \forall n \geq 2 \\ (n-1)! &\geq 2^{n-2} \text{ and } (n-2)! \geq 2^{n-3}, \quad \forall n \geq 2, \end{aligned}$$

(17) becomes

$$\begin{aligned}&\sum_{n=2}^{\infty} n\left\{n(1+k)-(k+\alpha)\right\}\frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)}|a_n|\\ \leq\ & \frac{4(1-\eta)\cos\xi}{\varphi}\left\{\frac{(2k-\alpha+3)(\mu+1)}{\mu(2\mu+1)}\right\}\\ \leq\ & 1-\alpha,\end{aligned}$$

by the theory in Lemma 1.9. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let* $\lambda > -1$, $\mu > 0$ *and* $\alpha \in [0,1)$ *with inequality such that*

$$\frac{2(1-\eta)\cos\xi}{\varphi}\left\{\frac{(\mu+1)(4\mu+2\mu k-2\mu\alpha+k+1)}{2\mu^2(2\mu+1)}\right\}\leq(1-\alpha).$$

If $f \in P_{\varphi}^{\tau(\eta)}$, $\varphi \in [0,1)$ *and* $\eta < 1$, *then the convolution operator* $\mathcal{Y}_{\lambda,\mu}f(z) \in \mathcal{S}_p(k,\alpha)$.

Proof. Consider

$$\begin{aligned}\mathcal{Y}_{\lambda,\mu}f(z) &= z\mathsf{W}_{\lambda,\mu}(z)*f(z)\\ &= z+\sum_{n=2}^{\infty}A_nz^n,\end{aligned}$$

where $A_n = \frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)}a_n$.

To show that the convolution operator $\mathcal{Y}_{\lambda,\mu}f(z) \in \mathcal{S}_p(k,\alpha)$, from Lemma 1.10 we will prove that

$$\sum_{n=2}^{\infty} n\left\{n(1+k)-(k+\alpha)\right\}|A_n|\leq 1-\alpha$$

Now

$$\begin{aligned}&\sum_{n=2}^{\infty}\left\{n(1+k)-(k+\alpha)\right\}\frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)}|a_n|\\ &\leq 2(1-\eta)\cos\xi\sum_{n=2}^{\infty}\left\{n(1+k)-(k+\alpha)\right\}\\ &\qquad\frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)}\frac{1}{1+\varphi(n-1)}.\end{aligned}\tag{18}$$

Since, $\frac{n}{1+\varphi(n-1)} \leq \frac{1}{\varphi}$, $\forall n \geq 2$, therefore (16) becomes

$$\sum_{n=2}^{\infty} \{n(1+k)-(k+\alpha)\} \frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)} |a_n|$$
$$\leq \frac{2(1-\eta)\cos\xi}{\varphi} \left\{ \begin{array}{l} (1+k)\sum_{n=2}^{\infty} \left(\frac{n\Gamma(\mu)}{n!\Gamma(\lambda(n-1)+\mu)}\right) \\ -(k+\alpha)\sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{n!\Gamma(\lambda(n-1)+\mu)} \end{array} \right\}$$
$$= \frac{2(1-\eta)\cos\xi}{\varphi} \left\{ \begin{array}{l} (1+k)\sum_{n=2}^{\infty} \left(\frac{(n-1+1)\Gamma(\mu)}{n!\Gamma(\lambda(n-1)+\mu)}\right) \\ -(k+\alpha)\sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{n!\Gamma(\lambda(n-1)+\mu)} \end{array} \right\}$$
$$\leq \frac{2(1-\eta)\cos\xi}{\varphi} \left\{ \begin{array}{l} (1+k)\sum_{n=2}^{\infty} \left(\frac{(n-1)\Gamma(\mu)}{n!\Gamma(\lambda(n-1)+\mu)}\right) \\ +(1-\alpha)\sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{n!\Gamma(\lambda(n-1)+\mu)} \end{array} \right\}. \tag{19}$$

By using the inequaliteis

$$\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \leq \frac{1}{(\mu)_{n-1}}, \qquad \forall n \geq 2$$
$$n! \geq 2(n-1) \text{ and } n! \geq 2^{n-2}, \qquad \forall n \geq 2,$$

(19) becomes

$$\sum_{n=2}^{\infty} n\{n(1+k)-(k+\alpha)\} \frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)} |a_n|$$
$$\leq \frac{2(1-\eta)\cos\xi}{\varphi} \left\{ \frac{(\mu+1)(4\mu+2\mu k-2\mu\alpha+k+1)}{2\mu^2(2\mu+1)} \right\}$$
$$\leq 1-\alpha,$$

by the theory in Lemma 1.10. This completes the proof of Theorem 2.2. □

Theorem 2.3. *Let* $\lambda > -1$, $\mu > 0$ *and* $\alpha \in [0,1)$ *with inequality such that*

$$\frac{\{2(1-\eta)\cos\xi\}}{\varphi} \left\{ \frac{(\mu+1)(\zeta+1)}{\mu(2\mu+1)} \right\} \leq \zeta.$$

If $f \in P_{\varphi}^{\tau(\eta)}$ *for* $\varphi \in [0,1)$, $\eta < 1$*and* $\zeta > 0$,*then the convolution operator* $\mathcal{Y}_{\lambda,\mu} f(z) \in S_{\zeta}^{*}$.

Proof. To prove that the convolution operator $\mathcal{Y}_{\lambda,\mu}f(z) \in S_{\varsigma}^{*}$, from Lemma 1.12, we will show that

$$\sum_{n=2}^{\infty}(\zeta+n-1)\left|A_n\right| \leq \zeta.$$

where $A_n = \frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)}a_n$, for $n \geq 2$.

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}(\zeta+n-1)\left|A_n\right| \\
= \;& \sum_{n=2}^{\infty}(\zeta+n-1)\frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)}\left|a_n\right| \\
\leq \;& \frac{2(1-\eta)\cos\xi}{\varphi}\left\{\begin{array}{c} \sum_{n=2}^{\infty}\left(\frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)}\right) \\ +(\zeta-1)\sum_{n=2}^{\infty}\left(\frac{\Gamma(\mu)}{n!\Gamma(\lambda(n-1)+\mu)}\right)\end{array}\right\} \\
\leq \;& \frac{2(1-\eta)\cos\xi}{\varphi}\left\{\begin{array}{c} \frac{1}{\mu}\sum_{n=2}^{\infty}\left(\frac{1}{2(\mu+1)}\right)^{n-2} \\ +\frac{(\zeta-1)}{2\mu}\sum_{n=2}^{\infty}\left(\frac{1}{2(\mu+1)}\right)^{n-2}\end{array}\right\} \\
= \;& \frac{2(1-\eta)\cos\xi}{\varphi}\left\{\frac{(\mu+1)(\zeta+1)}{\mu(2\mu+1)}\right\} \\
\leq \;& \zeta,
\end{aligned}$$

by the theory in Lemma 1.12. This completes the proof of Theorem 2.3. □

Theorem 2.4. *Let* $\lambda > -1$, $\mu > 0$ *and* $\alpha \in [0,1)$ *with inequality such that*

$$\{2(1-\eta)\cos\xi\}\left\{\frac{(\mu+1)(\zeta+1)}{\mu(2\mu+1)}\right\} \leq \varphi\zeta.$$

If $f \in P_{\varphi}^{\tau(\eta)}$, $\varphi \in [0,1)$, $\eta < 1$*and* $\zeta > 0$,*then the convolution operator* $\mathcal{Y}_{\lambda,\mu}f(z) \in C_{\varsigma}$.

Proof. The proof of this theorem is similar to Theorem 2.3. Therefore, we omit the details. □

Theorem 2.5. *Let $\lambda > -1$, $\mu > 0$ and $\alpha \in [0,1)$ with inequality such that*

$$\left\{\frac{(k+1)\left(4\mu^2+20\mu+20\right)}{\mu(\mu+1)(\mu+2)(2\mu+5)}\right\}+\left\{\frac{(2k+3-\alpha)\left(4\mu^2+12\mu+10\right)}{\mu(\mu+1)(2\mu+3)}\right\}$$
$$+\left\{\frac{2(1-\alpha)(1-\mu)}{\mu(2\mu+1)}\right\}\leq 1-\alpha.$$

Then $\mathcal{Y}_{\lambda,\mu}f(z)$ maps $f(z)\in\mathcal{S}$ into $\mathcal{S}_p(k,\alpha)$.

Proof. Consider

$$\begin{aligned}\mathcal{Y}_{\lambda,\mu}f(z) &= z\mathsf{W}_{\lambda,\mu}(z)*f(z)\\ &= z+\sum_{n=2}^{\infty}A_n z^n,\end{aligned}$$

where $A_n=\frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)}a_n$.

To show that the convolution operator $\mathcal{Y}_{\lambda,\mu}f(z)$ maps $f(z)\in\mathcal{S}$ of the form of (1) into $\mathcal{S}_p(k,\alpha)$, from Lemma 1.9 we will prove that

$$\sum_{n=2}^{\infty}\{n(1+k)-(k+\alpha)\}|A_n|\leq 1-\alpha.$$

Now

$$\sum_{n=2}^{\infty}\{n(1+k)-(k+\alpha)\}\frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)}|a_n|$$

$$= \left\{ \begin{array}{c} (1+k)\sum_{n=2}^{\infty}\left(\frac{n\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)}\right)|a_n| \\ -(k+\alpha)\sum_{n=2}^{\infty}\left(\frac{\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)}\right)|a_n| \end{array} \right\}$$

$$\leq \left\{ \begin{array}{c} (1+k)\sum_{n=2}^{\infty}\left(\frac{n^2\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)}\right) \\ -(k+\alpha)\sum_{n=2}^{\infty}\left(\frac{n\Gamma(\mu)}{(n-1)!\Gamma(\lambda(n-1)+\mu)}\right) \end{array} \right\}$$

$$= (1+k)\sum_{n=1}^{\infty}\left(\frac{(n+1)^2\Gamma(\mu)}{n!\Gamma(\lambda n+\mu)}\right) - (k+\alpha)\sum_{n=1}^{\infty}\left(\frac{(n+1)\Gamma(\mu)}{n!\Gamma(\lambda n+\mu)}\right)$$

$$= (1+k)\sum_{n=1}^{\infty}\left(\frac{n(n-1)+3n+1}{n!\Gamma(\lambda n+\mu)}\right)\Gamma(\mu)$$

$$- (k+\alpha)\sum_{n=1}^{\infty}\left(\frac{(n+1)}{n!\Gamma(\lambda n+\mu)}\right)\Gamma(\mu)$$

$$\leq (1+k)\sum_{n=1}^{\infty}\left(\frac{n(n-1)+3n+1}{n!(\mu)_n}\right) - (k+\alpha)\sum_{n=1}^{\infty}\left(\frac{(n+1)}{n!(\mu)_n}\right)$$

$$= (1+k)\sum_{n=1}^{\infty}\left(\frac{n(n-1)}{n!(\mu)_n}\right) + 3(1+k)\sum_{n=1}^{\infty}\left(\frac{n}{n!(\mu)_n}\right)$$

$$+ (1+k)\sum_{n=1}^{\infty}\left(\frac{1}{n!(\mu)_n}\right) - (k+\alpha)\sum_{n=1}^{\infty}\left(\frac{n}{n!(\mu)_n}\right)$$

$$- (k+\alpha)\sum_{n=1}^{\infty}\left(\frac{1}{n!(\mu)_n}\right)$$

$$= \left\{ \begin{array}{c} \frac{(1+k)}{\mu(\mu+1)}\sum_{n=2}^{\infty}\left(\frac{1}{(n-2)!(\mu+2)_{n-2}}\right) \\ +(2k+3-\alpha)\sum_{n=1}^{\infty}\left(\frac{1}{(n-1)!(\mu)_n}\right) + (1-\alpha)\sum_{n=1}^{\infty}\left(\frac{1}{n!(\mu)_n}\right) \end{array} \right\}$$

$$\leq \left[\begin{array}{c} \frac{(1+k)}{\mu(\mu+1)}\left\{2+\frac{1}{\mu+2}\sum_{n=3}^{\infty}\left(\frac{1}{2(\mu+3)}\right)^{n-3}\right\} \\ +(2k+3-\alpha)\left\{\frac{2}{\mu}+\frac{1}{\mu(\mu+1)}\sum_{n=2}^{\infty}\left(\frac{1}{2(\mu+2)}\right)^{n-2}\right\} \\ +\frac{(1-\alpha)}{\mu}\sum_{n=1}^{\infty}\left(\frac{1}{2(\mu+1)}\right)^{n-1} \end{array} \right]$$

$$
\begin{aligned}
&= \left\{\frac{(k+1)\left(4\mu^2+20\mu+20\right)}{\mu(\mu+1)(\mu+2)(2\mu+5)}\right\} \\
&\quad + \left\{\frac{(2k+3-\alpha)\left(4\mu^2+12\mu+10\right)}{\mu(\mu+1)(2\mu+3)}\right\} + \left\{\frac{2(1-\alpha)(1-\mu)}{\mu(2\mu+1)}\right\} \\
&\leq 1-\alpha,
\end{aligned}
$$

by the given hypothesis. Thus the proof of Theorem 2.5 is established. □

Theorem 2.6. *Let* $\lambda > -1$, $\mu > 0$ *and* $\alpha \in [0,1)$ *with inequality such that*

$$\frac{(5+4\zeta)(\mu+1)}{2\mu^2} \leq \zeta$$

Then $\mathcal{Y}_{\lambda,\mu} f(z)$ *maps* $f(z) \in \mathcal{S}$ *into* S^*_ς.

Theorem 2.7. *Let* $\lambda > -1$, $\mu > 0$ *and* $\alpha \in [0,1)$ *with inequality such that*

$$\frac{(28\zeta-19)(\mu+1)}{2\mu^2} \leq \zeta$$

Then $\mathcal{Y}_{\lambda,\mu} f(z)$ *maps* $f(z) \in \mathcal{S}$ *into* C_ς.

Proof. The proofs of Theorems 2.6, 2.7 are similar to the proof of Theorem 2.5. Therefore, we omit the details. □

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