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ELASTIC AND GEOMETRIC STIFFNESS MATRICES FOR THIN WALLED STIFFENER ELEMENT

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ABSTRACT

Elastic and geometric stiffness matrices for a structural stiffener member are obtained using the differential equations given by Chen and Atsuta[4]. Presence of additional elements over the conventional form is observed in the geometric Stiffness matrix. The known transformation matrix is used to express the elements displacements with respect to the middle surface of the stiffened plate.

INTRODUCTION

Elastic and geometric stiffness matrices are obtained by Akkoush[1], Barsoum[2], Tebedge[8], and Rajasekaran[5] using the approximate displacement field along the element length. Chaudhary[3], Krajcinovic[7] and Chen and Atsuta[4] obtained the stiffness matrix of the beam columns using the beam differential equations.

This work deals with the derivation of the elastic and geometric stiffness matrices using the exact displacement functions. The solution of the beam differential equations is given by Chen and Atsuta[4]. In reality, under the general conditions of loading, the differential equations of the beam columns are coupled with each other. These equations are uncoupled assuming that the axial force is applied at the shear center of the beam column or the section of the element has double symetry.

DIFFERENTIAL EQUATION SOLUTION

Figure 1 shows a beam with a unsymetric cross section subjected to an axial force of magnitude N at the shear center.

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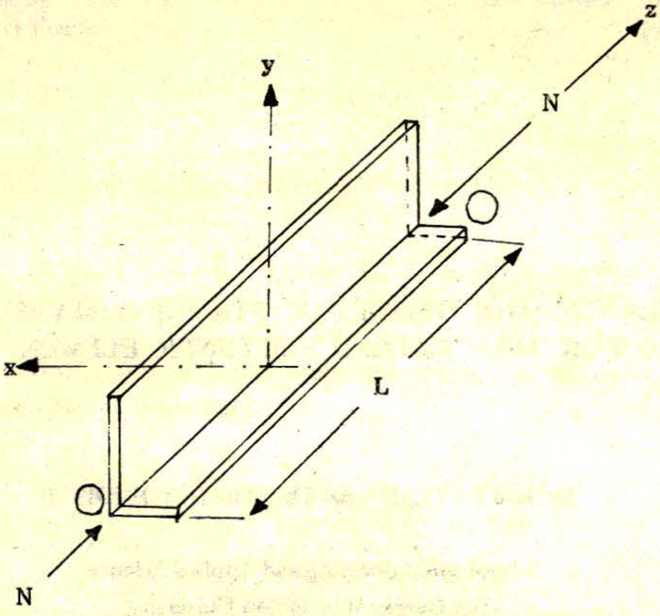

Figure 1 — Beam subjected to axial force

Then, the differential equation can be written as[4].

$$EAw'' = 0 \tag{1}$$

$$EI_y u'^{V} + Nu'' = 0 \tag{2}$$

$$EI_x v'^{V} + Nv'' = 0 \tag{3}$$

$$EI_\omega \theta'^{V} - (GK + \bar{K})\, \theta'' = 0 \tag{4}$$

In which u, v and w are the displacements of the shear center along the x, y and z axes, respectively; and θ is the counterclockwise rotation of the cross section about the z axis. Other terms in Eqs. 1,2, 3 and 4 are

$$\bar{K} = Nr^2$$

$$r^2 = \left(\frac{1}{A}\right)(I_x + I_y) + x_o^2 + y_o^2$$

$$GK = \frac{1}{3} G \sum b_i t_i^{\,3}$$

Forces and displacements at the nodes are related by the stiffness relationship. Since warping is considered only in the z direction of the element, there are seven forces and seven displacements at each node, as shown in Fig. 2.

The element stiffness matrix K (14x14) is computed by solving the uncoupled Eqs. 1, 2, 3 and 4 separately. Further, the section properties are assumed to be constant within the element. Solution of the above differential equations are as follows:

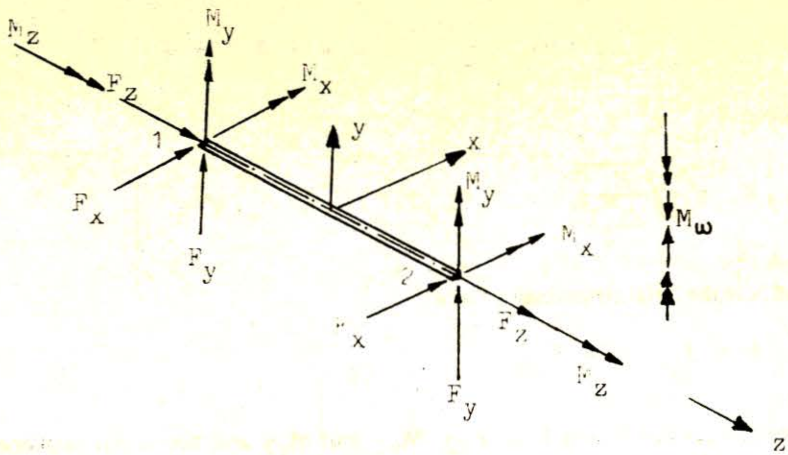

Figure 2 – Forces and displacements of an element

1. Axial Deformation

The element 1-2 is subjected to the axial and forces F_{z1} and F_{z2} and the node displacements are W_{z1} and W_{z2} as shown in Fig. 2. Equation 1 is the differential equation of the axial deformation. The general solution of Eq. 1 is

$$W_z(z) = D_1 + D_2 z \tag{5}$$

Where D_1 and D_2 are arbitrary constants. The nodal displacements and forces can be written in terms of these constants.

The displacements at the nodes

$$\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = \begin{Bmatrix} W_z(0) \\ W_z(L) \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} \tag{6}$$

and nodal forces

$$\begin{Bmatrix} F_{z1} \\ F_{z2} \end{Bmatrix} = EA \begin{Bmatrix} -W'(0) \\ W'(L) \end{Bmatrix} = EA \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} \tag{7}$$

Eliminating the arbitrary constants between Eqs. 6 and 7, the axial stiffness relationship is

$$\begin{Bmatrix} F_{z1} \\ F_{z2} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} \tag{8}$$

2. Bending and Shear Deformation

Equations 2 and 3 are the governing differential equations of the beam element for the bending and shear deformation under the applied axial load P. The general solution of the Eq. 2 is

$$u_x(x) = A_1 \text{Cos} k_x z + A_2 \text{Sin} k_x z + A_3 z + A_4 \quad (9)$$

where

$$k_y = \sqrt{\frac{N}{EI_y}} \quad (10)$$

and N is the axial compressive force,

$$N = F_{y1} = - F_{y2} \quad (11)$$

Similarly, nodal forces F_{x1}, F_{x2}, M_{y1} and M_{y2} and the nodal displacements can be written in terms of arbitrary constants as

$$\begin{Bmatrix} u_1 \\ \theta_{y1} \\ u_2 \\ \theta_{y2} \end{Bmatrix} = \begin{Bmatrix} u(0) \\ -u'(0) \\ u(L) \\ -u'(L) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -k_y & -1 & 0 \\ C_y & S_y & L & 1 \\ k_y S_y & -k_y S_y & -1 & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} \quad (12)$$

and

$$\begin{Bmatrix} F_{x1} \\ M_{y1} \\ F_{x2} \\ M_{y2} \end{Bmatrix} = EI_y \begin{Bmatrix} u'''(0) + k_y^2 u'(0) \\ u''(0) \\ -u'''(L) - k_y^2 u'(L) \\ -u''(L) \end{Bmatrix} = EI_y k_y^2 \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ Cy & S_y & 0 & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} \quad (13)$$

where

$$C_y = \text{Cos} k_y L \quad \text{and} \quad S_y = \text{Sin} k_y L \quad (14)$$

Eliminating A_1 ...... A_4 between Eqs. 12 and 13, the bending and shear stiffness relation about the y axis is

$$
\begin{Bmatrix} F_{x1} \\ M_{y1} \\ F_{x2} \\ M_{y2} \end{Bmatrix} = \frac{EI_y k_y^2}{2-2C_y-k_yLS_y} \begin{bmatrix} S_yk_y & (C_y-1) & -S_yk_y & C_y-1 \\ (C_y-1)k_y^2 & (k_yLC_y-S_y)k_y & (1-C_y)k_y^2 & (S_y-k_yL)k_y \\ -S_yk_y & (1-C_y)k_y & S_yk_y & (1-C_y) \\ (C_y-1)k_y^2 & (S_y-k_yL)k_y & (1-C_y)k_y^2 & k_y^2L(C_y-S_y) \end{bmatrix} \begin{Bmatrix} u_1 \\ \theta_{y1} \\ u_2 \\ \theta_{y2} \end{Bmatrix} \qquad (15)
$$

In a similar way the bending and shear stiffness relationship about the x axis is

$$
\begin{Bmatrix} F_{y1} \\ M_{x1} \\ F_{y2} \\ M_{x2} \end{Bmatrix} = \frac{EI_x k_x^2}{2-2C_x-k_xLS_x} \begin{bmatrix} S_xk_x & (1-C_x) & -S_xk_x & (1-C_x) \\ (1-C_x)k_x^2 & (S_x-k_xLC_x)k_x & (C_x-1)k_x^2 & (k_xL-S_x)k_x \\ -S_xk_x & (C_x-1) & S_xk_x & (C_x-1) \\ (1-C_x)k_x^2 & (k_xL-S_x)k_x & (C_x-1)k_x^2 & (S_x-k_xLC_c)k_x \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_{x1} \\ v_2 \\ \theta_{x2} \end{Bmatrix} \qquad (16)
$$

where

$$
k_x = \sqrt{\frac{N}{EI_x}} \qquad (17)
$$

and

$$
C_x = \mathrm{Cos}k_xL \quad \text{and} \quad S_x = \mathrm{Sin}k_xL \qquad (18)
$$

3. Torsion and Warping Deformation

Equation 4 is the governing differential equation of the beam element for the torsional deformation. The general solution of Eq. 4 is

$$\theta_z(z) = B_1 \mathrm{Cosh} k_x z + B_2 \mathrm{Sinh} k_z z + B_3 z + B_4 \tag{19}$$

where

$$k_z = \sqrt{\frac{GK + \bar{K}}{EI_\omega}} \tag{20}$$

and the arbitrary constants can be determined in terms of nodal displacements of the element as

$$\begin{Bmatrix} \theta_{z1} \\ \omega_1 \\ \theta_{z2} \\ \omega_2 \end{Bmatrix} = \begin{Bmatrix} \theta_z(0) \\ \theta_z'(0) \\ \theta_z(L) \\ \theta_z'(L) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & k_z & 1 & 0 \\ C_z & S_z & L & 1 \\ k_z S_z & k_z C_z & 1 & 0 \end{bmatrix} \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{Bmatrix} \tag{21}$$

and the nodal forces

$$\begin{Bmatrix} M_{z1} \\ M_{\omega 1} \\ M_{z2} \\ M_{\omega 2} \end{Bmatrix} = EI_\omega \begin{Bmatrix} \theta_z'''(0) - k_z^2 \theta_z'(0) \\ -\theta_z''(0) \\ -\theta_z'''(L) + k_z^2 \theta_z'(L) \\ \theta_z''(L) \end{Bmatrix} = k_z^2 EI_\omega \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ C_z & S_z & 0 & 0 \end{bmatrix} \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{Bmatrix} \tag{22}$$

where

$$S_z = \mathrm{Sinh} k_z L \quad \text{and} \quad C_z = \mathrm{Cosh} k_z L \tag{23}$$

Eliminating B_1 .... B_4 between Eqs. 21 and 22, the torsion and warping stiffness relationship is

$$\begin{Bmatrix} M_{z1} \\ M_{\omega 1} \\ M_{z2} \\ M_{\omega 2} \end{Bmatrix} = \frac{k_z^2 EI_\omega}{2-2C_z+k_zLS_z} \begin{bmatrix} S_zk_z & (C_z-1) & -S_zk_z & C_z-1 \\ (C_z-1)k_z^2 & (k_zLC_z-S_z)k_z & (1-C_z)k_z^2 & (S_z-k_zL)k_z \\ -S_zk_z & (1-C_z) & S_zk_z & (1-C_z)k_z \\ (C_z-1)k_z & (S_z-k_zL)k_z & (1-C_z)k_z^2 & (k_zLC_z-S_z)k_z \end{bmatrix} \begin{Bmatrix} \theta_{z1} \\ \omega_1 \\ \theta_{z2} \\ \omega_2 \end{Bmatrix} \qquad (24)$$

Summing up the above stiffness relationships together the total stiffness matrix for a thin-walled element is obtained, which is a matrix of 14th order [K(14x14)]. The nonzero elements of this matrix are

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{z1} \\ M_{x1} \\ M_{y1} \\ M_{z1} \\ M_{\omega 1} \\ F_{x2} \\ F_{y2} \\ F_{z2} \\ M_{x2} \\ M_{y2} \\ M_{z2} \\ M_{\omega 2} \end{Bmatrix} = \begin{bmatrix} k_{1,1} & & & & & & & & & & & & & \\ & k_{2,2} & & & & & & & & & & & & \\ & & k_{3,3} & & & & & & \text{SYMETRIC} & & & & & \\ & k_{4,2} & & k_{4,4} & & & & & & & & & & \\ k_{5,1} & & & & k_{5,5} & & & & & & & & & \\ & & & & & k_{6,6} & & & & & & & & \\ & & & & & & k_{7,7} & & & & & & & \\ k_{8,1} & & & & k_{8,5} & & & k_{8,8} & & & & & & \\ & k_{9,2} & & k_{9,4} & & & & & k_{9,9} & & & & & \\ & & k_{10,3} & & & & & & & k_{10,10} & & & & \\ & k_{11,2} & & k_{11,4} & & & & & k_{11,9} & & k_{11,11} & & & \\ k_{12,1} & & & & k_{12,5} & & & k_{12,8} & & & & k_{12,12} & & \\ & & & & & k_{13,6} & k_{13,7} & & & & & & k_{13,13} & \\ & & & & & k_{14,6} & k_{14,7} & & & & & & & k_{14,14} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \theta_{x1} \\ \theta_{y1} \\ \theta_{z1} \\ \omega_1 \\ u_2 \\ v_2 \\ w_2 \\ \theta_{x2} \\ \theta_{y2} \\ \theta_{z2} \\ \omega_2 \end{Bmatrix} \qquad (25)$$

where

$$k_{1,1} = k_{8,8} = \frac{EI_y k_y^3 S_y}{2-2C_y-k_y LS_y}$$

$$k_{2,2} = k_{9,9} = \frac{EI_x k_x^3 S_x}{2-2C_x-k_x LS_x}$$

$$k_{3,3} = k_{10,10} = \frac{EA}{L}$$

$$k_{4,4} = k_{11,11} = \frac{EI_x k_x (S_x-k_x LC_x)}{2-2C_x-k_x LS_x}$$

$$k_{5,5} = k_{12,12} = \frac{EI_y k_y (S_y-k_y LC_y)}{2-2C_y-k_y LS_y}$$

$$k_{6,6} = k_{13,13} = \frac{EI_\omega k_z^3 S_z}{2-2C_z+k_z LS_z}$$

$$k_{7,7} = k_{14,14} = \frac{EI_\omega k_z (k_z LC_z-S_z)}{2-2C_z+k_z LS_z}$$

$$k_{4,2} = -k_{11,9} = \frac{EI_x k_x^2 (1-C_x)}{2-2C_x-k_x LS_x}$$

$$k_{5,1} = -k_{12,8} = \frac{EI_y k_y^2 (1-C_y)}{2-2C_y-k_y LS_y}$$

$$k_{11,4} = \frac{EI_x k_x (k_x L-S_x)}{2-2C_x-k_x LS_x}$$

$$k_{12,5} = \frac{EI_y k_y (k_y L-S_y)}{2-2C_y-k_y LS_y}$$

$$k_{14,6} = \frac{-EI_\omega k_z^2 (C_z-1)}{2-2C_z+k_z LS_z}$$

$$k_{14,7} = \frac{-EI_{\omega} k_z (k_z L - S_z)}{2 - 2C_z + k_z L S_z} \tag{26}$$

Using the Taylor series expansions for the terms C_x, C_y, C_z, S_x, S_y and S_z in Eqs. 26, the nonzero elements of the elastic and geometric stiffness matrices are obtained as

$$k_E(1,1) = k_E(8,8) = \frac{12EI_y}{L^3}, \qquad k_G(1,1) = k_G(8,8) = \frac{6N}{5L}$$

$$k_E(2,2) = k_E(9,9) = \frac{12EI_x}{L^3}, \qquad k_G(2,2) = k_G(9,9) = \frac{6N}{5L}$$

$$k_E(3,3) = k_E(10,10) = \frac{AE}{L}, \qquad k_G(3,3) = k_G(10,10) = 0$$

$$k_E(4,4) = k_E(11,11) = \frac{4EI_x}{L}, \qquad k_G(4,4) = k_G(11,11) = \frac{2NL}{15}$$

$$k_E(5,5) = k_E(12,12) = \frac{4EI_y}{L}, \qquad k_G(5,5) = k_G(12,12) = \frac{2NL}{15}$$

$$k_E(6,6) = k_E(13,13) = \frac{12EI_{\omega}}{L^3} + \frac{12GK}{10L}, \qquad k_G(6,6) = k_G(13,13) = \frac{12\bar{K}}{10L}$$

$$k_E(7,7) = k_E(14,14) = \frac{4EI_{\omega}}{L} + \frac{2GK}{15L}, \qquad k_G(7,7) = k_G(14,14) = \frac{2\bar{K}}{15L}$$

$$k_E(4,2) = -k_E(11,9) = -\frac{6EI_x}{L^2}, \qquad k_G(4,2) = -k_G(11,9) = -\frac{P}{10}$$

$$k_E(5,1) = -k_E(12,8) = \frac{6EI_y}{L^2}, \qquad k_G(5,1) = -k_G(12,8) = -\frac{P}{10}$$

$$k_E(11,4) = \frac{2EI_x}{L}, \qquad k_G(11,4) = -\frac{L}{30}$$

$$k_E(12,5) = \frac{2EI_y}{L}, \qquad k_G(12,5) = -\frac{L}{30}$$

$$k_E(14,7) = \frac{2EI_\omega}{L} - \frac{6KL}{30}, \qquad k_G(14,7) = -\frac{\bar{K}L}{30}$$

$$k_E(14,6) = \frac{6EI_\omega}{L^2} + \frac{3GK}{30}, \qquad k_G(14,6) = \frac{\bar{K}}{10}$$

And the geometric stiffness matrix can be constructed as follows:

$$[k_G] = (N) \left[\begin{array}{cccccccccccccc}
\frac{6}{5L} & & & & & & & & & & & & & \\
 & \frac{6}{5L} & & & & & & & & & & & & \\
 & & 0 & & & & & & & & & & & \\
 & -\frac{1}{10} & & \frac{2L}{15} & & & & & & & \text{SYMETRIC} & & & \\
-\frac{1}{10} & & & & \frac{2L}{15} & & & & & & & & & \\
 & & & & & \frac{6r^2}{5L} & & & & & & & & \\
 & & & & & & \frac{2r^2}{15L} & & & & & & & \\
-\frac{6}{5L} & & & & -\frac{1}{10} & & & \frac{6}{5L} & & & & & & \\
 & -\frac{6}{5L} & & \frac{1}{10} & & & & & \frac{6}{5L} & & & & & \\
 & & 0 & & & & & \frac{1}{10} & & 0 & & & & \\
 & \frac{1}{10} & & -\frac{L}{30} & & & -\frac{1}{10} & & & & \frac{2L}{15} & & & \\
\frac{1}{10} & & & & -\frac{L}{30} & & & & & & & \frac{2L}{15} & & \\
 & & & & & -\frac{6r^2}{5L} & & & & & & & \frac{6r^2}{5L} & \\
 & & & & & \frac{r^2}{10} & \frac{r^2 L}{30} & & & & & & & \frac{2r^2}{15L}
\end{array} \right]$$

CONCLUSIONS

Elastic $[K_E]$ and geometric $[K_G]$ stiffness matrices are obtained using the exact displacement function of the thin walled beam elements. The elastic stiffness matrix obtained in this study is exact, and also includes the St. Venant and Wagner torsional rigidity effects. The first order terms in geometric stiffness matrix are exact to the solutions given in (1) and (5). The residual terms in geometric stiffness matrix which are not shown here are the difference between the solutions by using the exact and approximate displacement functions.

Generally, in numerical solutions of the stiffened plate using the elastic and geometric stiffness matrices are easier. For this reason, elastic and geometric stiffness matrices are obtained in this study using the exact displacement functions.

APPENDIX I.

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APPENDIX II.

NOTATION

The following symbols are used in this paper:

A = cross-sectional area

E = modulus of elasticity

G = rigidity modulus

I_ω = warping constant

I_x, I_y = moment of inertia about x and y axis, respectively

K = St. Venant torsion constant

K = bimoment

L = length of the beam

N = axial force

u,v = displacement of shear center in x and y direction, respectively

x,y,z = co-ordinate axes.