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Tight contact structures on hyperbolic three-manifolds

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ABSTRACT

Let Σ_g denote a closed orientable surface of genus $g \geq 2$. We consider a certain infinite family of Σ_g-bundles over circle whose monodromies are taken from some collection of pseudo-Anosov diffeomorphisms. We show the existence of tight contact structure on every closed 3-manifold obtained via rational r-surgery along a section of any member of the family whenever $r \neq 2g - 1$. Combining with Thurston's hyperbolic Dehn surgery theorem, we obtain infinitely many hyperbolic closed 3-manifolds admitting tight contact structures.

1. Introduction

A *contact three-manifold* is a pair (M, ξ) where M is a smooth 3-manifold and $\xi \subset TM$ is a totally non-integrable 2-plane field distribution on M. Here we always assume that ξ is a *co-oriented positive* contact structure, that is, $\xi = \text{Ker}(\alpha)$ for a *contact* 1-form α satisfying $\alpha \wedge d\alpha > 0$ with respect to a pre-given orientation on M. A disk D in a contact 3-manifold (M, ξ) is called *overtwisted* if the boundary circle ∂D is tangent to ξ everywhere. A contact structure ξ is called overtwisted if there is an *overtwisted* disk in (M, ξ), otherwise it is called *tight*. It is known that every closed oriented 3-manifold admits an overtwisted contact structure ([7], [18]). On the other hand, there are 3-manifolds which do not admit a tight contact structure [9].

There are some classification results on tight contact structures with respect to the geometric speciality of 3-manifolds. Lisca and Stipsicz in [17] proved that a closed oriented Seifert fibered 3-manifold admits a

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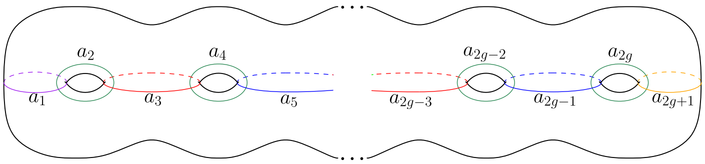

Fig. 1. Simple closed curves on the surface Σ_g.

tight contact structure if and only if it is not gotten $(2q-1)$-surgery along the $(2,2q+1)$ torus knot in S^3 for $q \geq 1$. In two independent work ([2], [15]), they showed the existence of tight contact structures on toroidal 3-manifolds. It is known that every irreducible 3-manifold that is neither toroidal nor Seifert fibered is hyperbolic. Kaloti and Tosun in [16] find infinitely many hyperbolic rational homology spheres admitting tight contact structures. Etgü in [8] also explored that infinitely many hyperbolic 3-manifolds that carry tight contact structures. His construction uses Dehn surgeries along sections of hyperbolic torus bundles over S^1. Here we'll consider Dehn surgeries along sections of surface bundles over S^1 with fiber genus at least two.

Let Σ_g be a closed connected orientable surface with genus g. In this paper assume that g is always greater than 1. We will denote $MCG(\Sigma_g)$ by the *mapping class group* of Σ_g, i.e., the group of isotopy classes of orientation preserving homeomorphisms of Σ_g. Denote by t_a the positive Dehn twist along a simple closed curve a. Let $\phi \in MCG(\Sigma_g)$ be the mapping class representing the homeomorphism

$$t_{a_1}^m t_{a_2} \cdots t_{a_{2g}} t_{a_{2g+1}}^n \tag{1}$$

where a_i's are simple closed curves on Σ_g as indicated in Fig. 1.

Denote by M_ϕ the *mapping torus* with fibers Σ_g and monodromy ϕ. Let $M_\phi(r)$ be the surgered manifold obtained by performing rational r-surgery along a section of M_ϕ. Clearly, ϕ has a fixed point, so such a section exists. The following theorems give examples required:

Theorem 1.1. *Suppose $g \geq 2$, $m, n \in \mathbb{Z}$, $r \in \mathbb{Q}$ and ϕ as indicated in (1). Then $M_\phi(r)$ is hyperbolic for all but finitely many m and r.*

Theorem 1.2. *Suppose $g \geq 1$, $r \in \mathbb{Q}$ and ϕ as indicated in (1). Then $M_\phi(r)$ admits a tight contact structure ξ for any $m, n \in \mathbb{Z}^+$ and for all $r \neq 2g-1$.*

Corollary 1.3. *Suppose $g \geq 2$, $m, n \in \mathbb{Z}^+$, $r \in \mathbb{Q}$ and ϕ as indicated in (1). Then $M_\phi(r)$ is a hyperbolic manifold admitting a tight contact structure for all but finitely many $m \in \mathbb{Z}^+$ and for all but finitely many $r \neq 2g-1$.* □

The proof of Theorem 1.1 and Theorem 1.2 will be given in Section 2 and Section 3.

2. Proof of Theorem 1.1

In order to prove the theorem, we'll focus on pseudo-Anosov homeomorphisms and construct infinitely many hyperbolic 3-manifolds via pseudo-Anosov monodromies. A hyperbolic 3-manifold is a 3-manifold which admits a complete finite-volume hyperbolic structure. Thurston [21] demonstrated that an orientable surface bundle over circle whose fiber is a compact surface of negative Euler characteristic is hyperbolic if and only if the monodromy of the surface bundle is a pseudo-Anosov homeomorphism. Another deep result of Thurston is hyperbolic Dehn surgery theorem which states that a hyperbolic 3-manifold remains hyperbolic after Dehn filling along a link for all slopes except finitely many of them (for details see [22]). In order to apply these results, we need a lemma where we construct infinitely many pseudo-Anosov diffeomorphisms as products of certain Dehn twists:

Lemma 2.1. *Let ϕ be the class in $MCG(\Sigma_g)$ as described in (1) above. Then ϕ is pseudo-Anosov for any integer n and for all but at most 7 consecutive values of m.*

Denote by $\iota(\alpha, \beta)$ *geometric intersection number* of the curves α and β. We say a set of simple closed curves $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$ *fills* Σ_g if $\Sigma_g \setminus \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ is a disjoint union of topological disks. In order to prove Lemma 2.1, we use the following theorem of Fathi:

Theorem 2.2 *([11]). Let f be the class in $MCG(\Sigma_g)$ and let γ be a simple closed curve in Σ_g. If the orbit of γ under f fills Σ_g, then $t_\gamma^m f$ is a pseudo-Anosov class except for at most 7 consecutive values of m.*

Proof of Lemma 2.1. Let γ represents the curve a_1 and let f be the product of Dehn twists $t_{a_1} t_{a_2} \cdots t_{a_{2g}} t^n_{a_{2g+1}}$ where a_i's are simple closed curves in Fig. 1. Since $i(a_i, a_{i+1}) = 1$ for all $i \in \{1, 2, \dots, 2g\}$ and $i(a_i, a_{i+2}) = 0$ for all $i \in \{1, 2, \dots, 2g-1\}$, conclude that

$$f(\gamma) = t_{a_1} t_{a_2}(a_1) = a_2, \quad f^2(\gamma) = t_{a_1} t_{a_2} t_{a_3}(a_2) = t_{a_1}(a_3) = a_3,$$

(see Proposition 3.12, [10]) and inductively,

$$f^i(\gamma) = f(a_i) = t_{a_1} \cdots t_{a_i} t_{a_{i+1}}(a_i) = t_{a_1} \cdots t_{a_{i-1}}(a_{i+1}) = a_{i+1} \text{ for all } i \in \{1, 2, \dots, 2g-1\}.$$

Clearly the complement of the orbit of γ under f, i.e. $\Sigma_g \setminus \{f^i(\gamma)\}_{i \in \mathbb{N}}$, is a subset of

$$\Sigma_g \setminus \{\gamma, f(\gamma), \cdots, f^{2g-1}(\gamma)\} = \Sigma_g \setminus \{a_1, \dots, a_{2g}\}$$

which is a topological disk. Hence the orbit set fills the surface. As a result of Theorem 2.2, ϕ is pseudo-Anosov except for at most 7 consecutive m values. □

Now we have a family of pseudo-Anosov monodromies. It is known that a surface bundle over circle with two or greater fiber genus is hyperbolic if and only if the monodromy of the surface bundle is pseudo-Anosov by Thurston's work [21]. So the surface bundles M_ϕ are all hyperbolic but at most 7 consecutive values of m. By hyperbolic Dehn surgery theorem the surgered manifolds $M_\phi(r)$ are hyperbolic for all $m, n \in \mathbb{Z}$ and $r \in \mathbb{Q}$ except 7 values of m and finitely many "bad" slopes r. This finishes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

We will analyze the proof with respect to the parity of the genus g of the fiber Σ_g. First assume $g \geq 3$ odd. Note that conjugation of the monodromy by any class of $MCG(\Sigma_g)$ does not change the mapping torus up to diffeomorphism. Since

$$t_{a_2} \cdots t_{a_{2g}} t^n_{a_{2g+1}} t^m_{a_1} = t^{-m}_{a_1} \phi \, t^m_{a_1}$$

we may replace ϕ in (1) with the mapping class $t_{a_2} \cdots t_{a_{2g}} t^n_{a_{2g+1}} t^m_{a_1}$. Also observe that $M_\phi(r)$ can be also obtained from a Dehn surgery on the binding of an open book decomposition whose page is Σ^1_g (punctured Σ_g) and monodromy can be still assumed to be $\phi \in MCG(\Sigma^1_g)$. We will construct the required contact structure ξ on $M_\phi(r)$ via Dehn surgery on the open book decomposition (Σ^1_g, ϕ) along its binding.

It is known (see [1], [12], [13], [19]) that the contact structure supported by an open book decomposition is Stein fillable if and only if the monodromy is the product of positive Dehn twists. Hence the contact

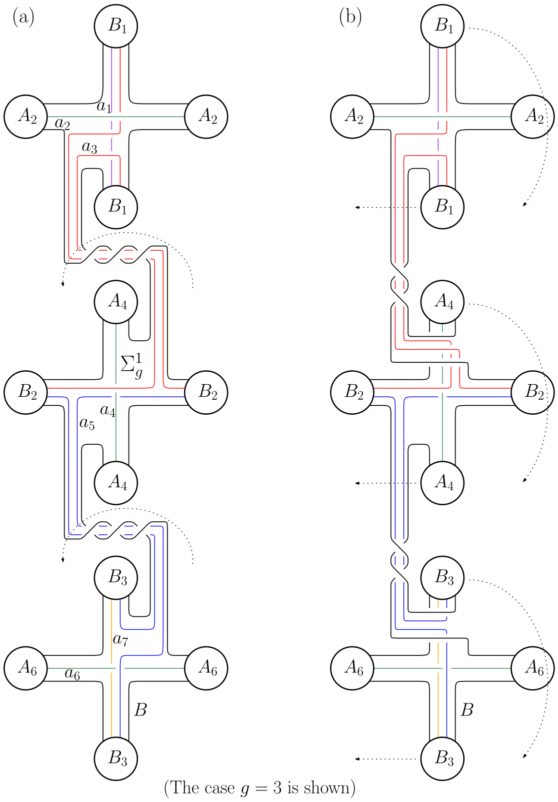

Fig. 2. (a) The handlebody diagram of X_ϕ, (b) another handle description of X_ϕ.

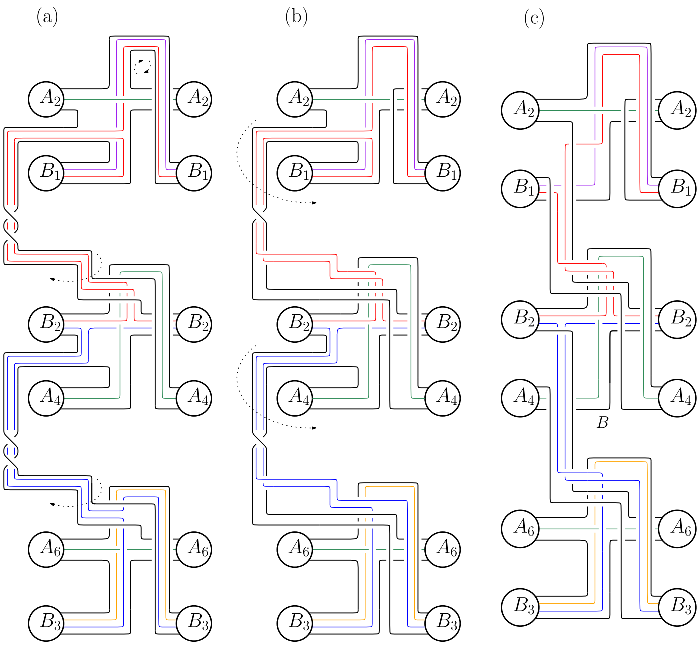

Fig. 3. Other handle descriptions of X_ϕ.

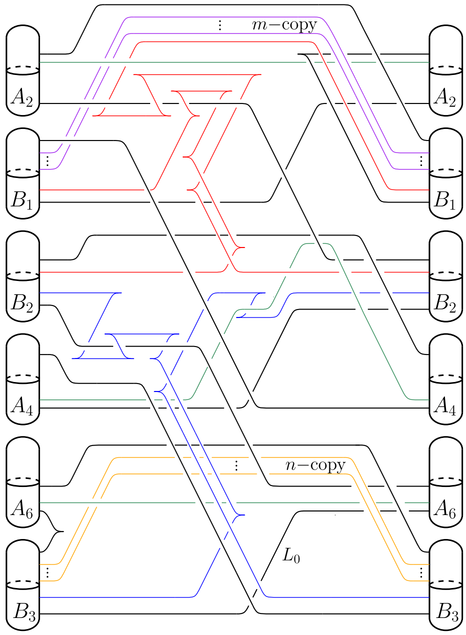

Fig. 4. A Stein structure on X_ϕ.

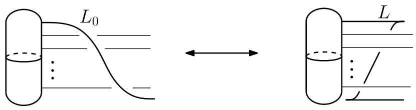

Fig. 5. Passing around a foot of a handle.

structure, say ξ_0, (before the surgery along binding) supported by (Σ_g^1, ϕ) is Stein fillable. More precisely, consider the handlebody diagram of the smooth 4-manifold X_ϕ given in Fig. 2-(a) (in the case of genus 3) with "$2g$" 1-handles and "$m + n + 2g - 1$" 2-handles. Note that Fig. 2-(a) describes a Lefschetz fibration structure on X_ϕ with a regular fiber Σ_g^1 and the vanishing cycles $a_1, a_2, ..., a_{2g+1}$. There are n copies for a_{2g+1} and m copies for a_1 (not drawn for simplicity). All coefficients (except on B) are -1 with respect to the framing given by the page Σ_g^1.

We remark that no handle is attached along the binding of the induced open book (Σ_g^1, ϕ) on the boundary ∂X_ϕ which is realized as B in the figure.

Next starting from the topological description in Fig. 2-(a) of X_ϕ, we'll get a diagram describing a Stein structure on X_ϕ inducing ξ_0 as follows: First we flip the twisted bands over the 1-handles as pointed out in Fig. 2-(a) and get Fig. 2-(b). Fig. 3-(a) gives another handle description of X_ϕ obtained by moving the feet of 1-handles as indicated by the dotted arrows in Fig. 2-(b). Then flip the bands as shown in Fig. 3-(a) to get rid of one more left half twist for each band (see Fig. 3-(b)), and obtain Fig. 3-(c) by flipping the connecting bands over the feet of 1-handles suggested by the dotted arrows in Fig. 3-(b). Fig. 4 defines a Stein structure on X_ϕ obtained by putting the attaching circles in Fig. 3-(c) into Legendrian positions, where a Legendrian realization L_0 of B in the tight contact boundary ∂X_ϕ is also provided. All coefficients (except on L_0) are -1 with respect to Thurston–Bennequin (contact) framing, "tb", in ∂X_ϕ and no handle is attached along L_0. Note that $tb(L_0) = 2$ (the case $g = 3$ is shown in Fig. 4). In the general case, $tb(L_0) = g - 1$.

Finally, we pass g strands of L_0 around the left feet of the corresponding 1-handles as in Fig. 5 to get a Legendrian representation L of B depicted in Fig. 6 with $tb(L) = 2g - 1$ (see [14] for details). Note that Fig. 6 describes the same Stein structure on X_ϕ as in Fig. 4.

Now if $g \geq 2$ is even, we replace the monodromy ϕ with $t^n_{a_{2g+1}} t_{a_2} \cdots t_{a_{2g}} t^m_{a_1}$ since

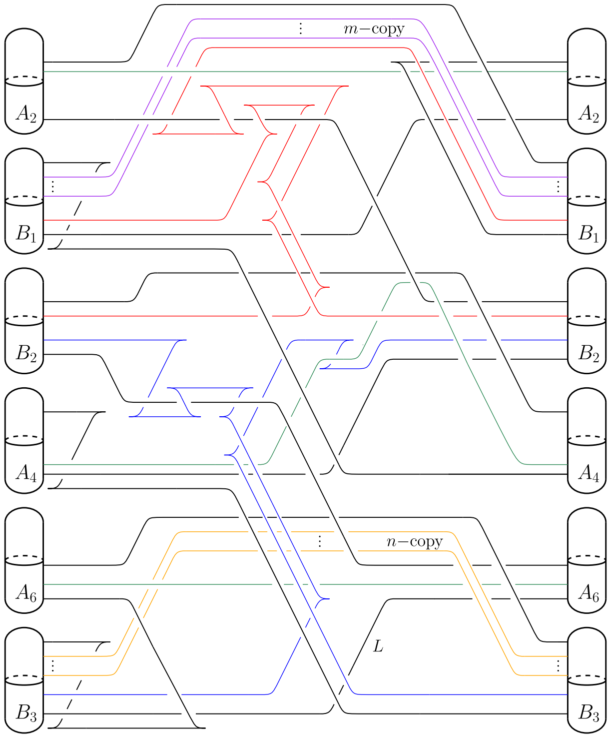

Fig. 6. The same Stein structure on X_ϕ as in Fig. 4-(b), and another Legendrian realization L of the binding B in the tight contact boundary ∂X_ϕ. L is obtained from L_0 by applying the move in Fig. 5 (smooth but non-Legendrian isotopy of L_0) g times using the left feet of the corresponding 1-handles (when $g = 3$, handles are B_1, A_4, B_3). All coefficients (except on L) are -1 with respect to Thurston–Bennequin (contact) framing in ∂X_ϕ. No handle attached along L. Note that $tb(L) = 5$ (the case $g = 3$ is shown). In the general case, $tb(L) = 2g - 1$.

$$t^n_{a_{2g+1}} t^{-m}_{a_1} \phi\, t^{-n}_{a_{2g+1}} t^m_{a_1} = t^n_{a_{2g+1}} t_{a_2} \cdots t_{a_{2g}} t^m_{a_1}.$$

Then starting from the handlebody diagram given in Fig. 7-(a) (where the case $g = 4$ is shown) and following the moves as in the case of odd genus, one can get Fig. 7-(b) describing a Stein structure realizing a Legendrian representation L with $tb(L) = 2g - 1$ as in Fig. 6.

One should note that we need to consider different monodromies (but still giving the same mapping torus) depending on the parity of g to make the contact and the page framing on any attaching circle coincide.

Now (in any case of g) in the final Stein diagram (Fig. 6/Fig. 7-(b)), we first (Legendrian) slide (Stein) 2-handle corresponding a_3 over the ones represented by the curves $a_1, a_5, a_7, ..., a_{2g+1}$, and then cancel the 2-handles represented by $a_5, a_7, ..., a_{2g-1}$ with the corresponding 1-handles. Second, we (Legendrian) slide 2-handles represented by the curves a_1 and a_{2g+1} over a fixed one chosen from each family, and then cancel 1-handles B_1 and B_g with the chosen 2-handles corresponding a_1 and a_{2g+1} respectively. Also we cancel each 1-handle A_i with the 2-handle corresponding a_i for each i even (see [4]). As a result, we obtain another picture for the same Stein structure on X_ϕ which can be also seen as a surgery diagram for ξ_0 on ∂X_ϕ. Finally, we set $r' = r - 2g + 1$ and perform r'-contact surgery along $L \subset (\partial X_\phi, \xi_0)$ to get a contact structure ξ on $M_\phi(r)$ given in Fig. 8 where we use continued fractions for a_1, a_{2g+1} families, and "Move 4" in Figure 9 of [14] for the slid a_3.

First suppose $r' = r - 2g + 1 < 0$. We know any contact surgery with negative contact framing can be converted to a sequence of contact (-1)-surgeries and (-1)-surgeries preserve Stein fillability ([5], [6]). Thus $(M_\phi(r), \xi)$ is Stein fillable (hence tight).

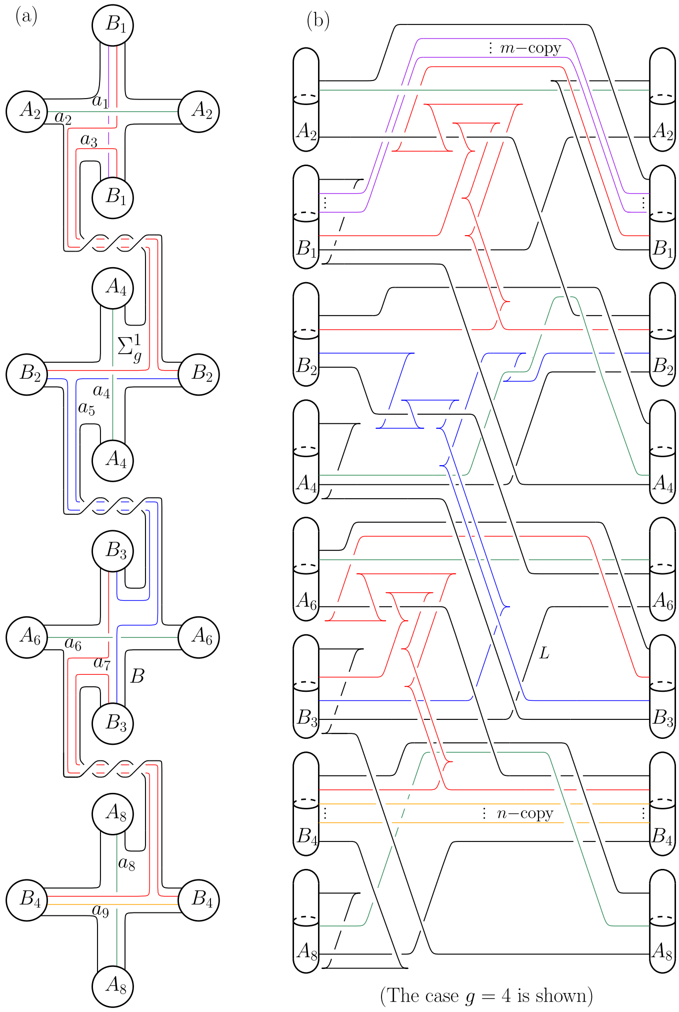

Fig. 7. X_ϕ and its Stein structure (when g is even).

Now let $r' = r - 2g + 1 > 0$. By Thurston–Winkelnkemper construction ([23]), it is known that the binding B is transverse to the contact structure supported by the open book decomposition. Also since ∂X_ϕ is Stein fillable, ξ_0 has nonzero contact invariant [20]. In Theorem 1.6 of [3], Conway states that if K is a (integrally) fibered transverse knot in a contact 3-manifold (M, η) where η is tight (resp. has nonvanishing contact class), then the surgered manifold obtained via inadmissible transverse r-surgery along K is also tight (resp. has nonvanishing contact class) for $r > 2g - 1$ where g is the genus of K. Since $r' > 0$, contact r'-surgery can be converted to sequences of contact (± 1)-surgeries [5]. Note that contact $(+1)$-surgery along L is equivalent to inadmissible $(tb(L) + 1)$-surgery (see [3] for details). Hence we conclude that $(M_\phi(r), \xi)$ has nonzero contact invariant (hence tight) through Conway's result because of the fact $tb(L) + 1 > 2g - 1$. This finishes the proof of Theorem 1.2. □

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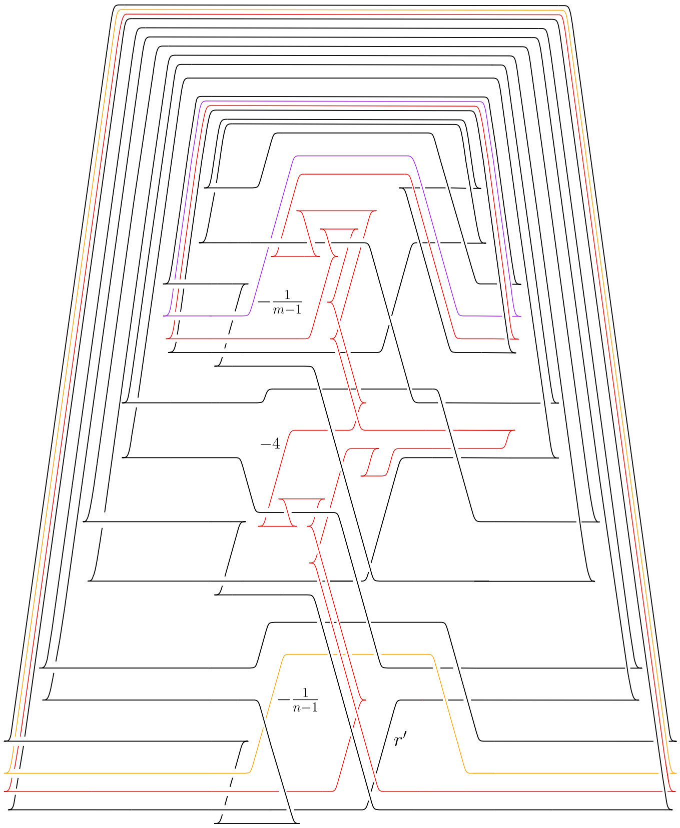

Fig. 8. The contact 3-manifold $(M_\phi(r), \xi)$. (The case $g = 3$ is shown.)

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