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Boundary value problems associated with generalized Q-holomorphic functions

Sezayi Hızlıyel*

*Correspondence:
hizliyel@uludag.edu.tr
Department of Mathematics, Faculty of Art and Science, Uludağ University, Görükle, Bursa, Turkiye

Abstract

In this work, we discuss Riemann-Hilbert and its adjoint homogeneous problem associated with generalized Q-holomorphic functions and investigate the solvability of the Riemann-Hilbert problem.

Keywords: generalized Beltrami systems; Q-holomorphic functions; Riemann-Hilbert problem

Introduction

Douglis [1] and Bojarskiĭ [2] developed an analog of analytic functions for elliptic systems in the plane of the form

$$w_{\bar{z}} - qw_z = 0, \tag{1}$$

where w is an $m \times 1$ vector and q is an $m \times m$ quasi-diagonal matrix. Also, Bojarskiĭ assumed that all eigenvalues of q are less than 1. Such systems are natural ones to consider because they arise from the reduction of general elliptic systems in the plane to a standard canonical form. Subsequently Douglis and Bojarkii's theory has been used to study elliptic systems in the form

$$w_{\bar{z}} - qw_z = aw + b\overline{w} + F$$

and the solutions of such equations were called generalized (or pseudo) hyperanalytic functions. Work in this direction appears in [3–5]. These results extend the generalized (or 'pseudo') analytic function theory of Vekua [6] and Bers [7]. Also, classical boundary value problems for analytic functions were extended to generalized hyperanalytic functions. A good survey of the methods encountered in a hyperanalytic case may be found in [8, 9], also see [10].

In [11], Hile noticed that what appears to be the essential property of elliptic systems in the plane for which one can obtain a useful extension of analytic function theory is the self-commuting property of the variable matrix Q, which means

$$Q(z_1)Q(z_2) = Q(z_2)Q(z_1)$$

for any two points z_1, z_2 in the domain G_0 of Q. Further, such a Q matrix cannot be brought into a quasi-diagonal form of Bojarskiĭ by a similarity transformation. So, Hile [11] attempted to extend the results of Douglis and Bojarskiĭ to a wider class of systems in the same form with equation (1). If $Q(z)$ is self-commuting in G_0 and if $Q(z)$ has no eigenvalues of magnitude 1 for each z in G_0, then Hile called the system (1) the generalized Beltrami system and the solutions of such a system were called *Q-holomorphic functions*. Later in [12, 13], using Vekua and Bers techniques, a function theory is given for the equation

$$Dw + Aw + B\overline{w} = 0, \quad \text{where } D := \frac{\partial}{\partial \overline{z}} - Q(z)\frac{\partial}{\partial z}, \tag{2}$$

where the unknown $w(z) = \{w_{ij}(z)\}$ is an $m \times s$ complex matrix, $Q(z) = \{q_{ij}(z)\}$ is a self-commuting complex matrix with dimension $m \times m$ and $q_{k,k-1} \neq 0$ for $k = 2, \ldots, m$. $A = \{a_{ij}(z)\}$ and $B = \{b_{ij}(z)\}$ are commuting with Q. Solutions of such an equation were called *generalized Q-holomorphic functions*.

In this work, as in a complex case, following Vekua (see [6, pp.228-236]), we investigate the necessary and sufficient condition of solvability of the Riemann-Hilbert problem for equation (2).

Solvability of Riemann-Hilbert problems

In a regular domain G, we consider the problem

$$(A): \begin{cases} L[w] := Dw + Aw + B\overline{w} = F & \text{in } G, \\ \operatorname{Re}(\overline{\lambda} w) = \gamma & \text{on } \partial G. \end{cases} \tag{3}$$

We refer to this problem as boundary value problem (A). Where the unknown $w(z) = \{w_{ij}(z)\}$ is an $m \times s$ complex matrix-valued function, $Q = \{q_{ij}(z)\}$ is a Hölder-continuous function which is a self-commuting matrix with $m \times m$ and $q_{k,k-1} \neq 0$ for $k = 2, \ldots, m$. $A = \{a_{ij}(z)\}$ and $B = \{b_{ij}(z)\}$ are commuting with Q, which is

$$Q(z_1)A(z_2) = A(z_1)Q(z_1), \qquad Q(z_1)B(z_2) = B(z_1)Q(z_1).$$

It is assumed, moreover, that Q is commuting with $\overline{Q}$ and $\lambda(z) \in C^1(\Gamma)$ is commuting with Q, where $\Gamma = \partial G$, $\lambda\overline{\lambda} = I$. In respect of the data of problem (A), we also assume that A, B and $F \in L^{p,2}(\mathbb{C})$ and $\gamma \in C_\alpha(\Gamma)$. If $F \equiv 0$, $\gamma \equiv 0$, we have homogeneous problem $(\mathring{A})$.

We refer to the adjoint, homogeneous problem (A) as $(\mathring{A}')$; it is given by

$$(\mathring{A}'): \begin{cases} L'[w'] := Dw' - Aw' - B^*\overline{w'} = 0 & \text{in } G, \\ \operatorname{Re}(\frac{d\phi}{ds}\lambda w') = 0 & \text{on } \Gamma, \end{cases} \tag{4}$$

where ϕ is a generating solution for the generalized Beltrami system ([11, p.109]), $B^* = \phi_z^{-1}\overline{\phi_z B}$, $\frac{d\phi}{ds} := \frac{\partial \phi}{\partial z}\frac{dz}{ds} + \frac{\partial \phi}{\partial \overline{z}}\frac{d\overline{z}}{ds}$ and ds is the arc length differential. From the Green identity for Q-holomorphic functions (see [11, p.113]), we have

$$\operatorname{Re}\left[\frac{1}{2i}\int_\Gamma d\phi w' w - \iint_G \phi_z\left(w' L[w] - L'\left[w'\right]w\right) dx\, dy\right] = 0, \tag{5}$$

where w' is commuting with Q. For $L[w] = F$ and $L'[w'] = 0$, this becomes

$$\frac{1}{2i}\int_\Gamma d\phi(z)w'(z)\lambda(z)\gamma(z) - \operatorname{Re}\left(\iint_G \phi_z(z)w'(z)F(z)\, dx\, dy\right) = 0. \tag{6}$$

Since w' satisfies the boundary condition

$$\operatorname{Re}\left(\frac{d\phi}{ds}\lambda w'\right) = 0, \tag{7}$$

we have

$$w' = i\lambda^{-1}\left(\frac{d\phi}{ds}\right)^{-1}\varkappa, \tag{8}$$

where $\varkappa$ is a real matrix commuting with Q.

The solutions to problem $(\mathring{A}')$ may be represented by means of fundamental kernels in terms of a real, matrix density $\varkappa$ as

$$\begin{aligned} w'(z) &= P^{-1}\int_\Gamma \left(d\phi(\zeta)\Omega^{(1)}(z,\zeta)w'(\zeta) - d\overline{\phi(\zeta)}\Omega^{(2)}(z,\zeta)\overline{w'(\zeta)}\right) \\ &= iP^{-1}\int_\Gamma \left(\Omega^{(1)}(z,\zeta)\lambda^{-1}(\zeta) + \overline{\Omega^{(2)}(z,\zeta)\lambda^{-1}(\zeta)}\right)\varkappa(\zeta)\,ds, \end{aligned} \tag{9}$$

see ([14, p.543]). In (9), P is a constant matrix defined by

$$P(z) = \int_{|z|=1} (zI + \bar{z}Q)^{-1}(I\,dz + Q\,d\bar{z})$$

called P-value for the generalized Beltrami system [11]. Since $\varkappa$ is a real matrix commuting with Q, inserting the expression (9) into the boundary condition (7), we have

$$\int_\Gamma K_1(\zeta,z)\varkappa(\zeta)\,ds_\zeta = 0, \quad z,\zeta = \zeta(s) \in \Gamma, \tag{10}$$

where

$$K_1(\zeta,z) = -\operatorname{Re}\left[iP^{-1}\lambda(z)\frac{d\phi(z)}{ds}\left(\Omega^{(1)}(z,\zeta)\lambda^{-1}(z) + \overline{\Omega^{(2)}(z,\zeta)\lambda^{-1}(z)}\right)\right].$$

The integral in (10) is to be taken in the Cauchy principal value sense. If we denote this equation in an operator form by $\underset{\sim}{K}\varkappa = 0$ and its adjoint by $\underset{\sim}{K}'f = 0$, then it may be easily demonstrated that the index of (10) is $\kappa = k - k' = 0$. Here k and k' are dimensions of null spaces of $\underset{\sim}{K}$ and $\underset{\sim}{K}'$ respectively. If $\{\varkappa_1,\ldots,\varkappa_k\}$ is a complete system of solutions of (10), putting each of this into (9), we obtain the solutions of problem $(\mathring{A}')$. However, it is possible that some of these solutions may turn out to be trivial solutions, which occurs when $(\lambda\frac{d\phi}{ds})^{-1}\varkappa$ takes on the boundary values of a Q-holomorphic function ψ_j on each component of boundary contours Γ_j in $C^{1,\alpha}(\mathbb{C})$ which is, moreover, Q-holomorphic in the domain G_j bounded by the closed contour Γ_j. Let $\{\varkappa_1,\ldots,\varkappa_{\ell'}\}$ be solutions of equation (10) to which linearly independent solutions (see [15]) $w'_1,\ldots,w'_{\ell'}$ of problem $(\mathring{A}')$ correspond, then the remaining solutions $\{\varkappa_{\ell'+1},\ldots,\varkappa_k\}$ satisfy the boundary condition of the form

$$\varkappa(z) = i\lambda(z)\frac{d\phi}{ds}\Phi^-(z) \quad \text{on } \Gamma. \tag{11}$$

Here Φ^- are meant to be Q-holomorphic functions outside of $G^- := G + \Gamma$ and $\Phi(\infty) = 0$. Hence the Q-holomorphic functions satisfy the homogeneous boundary conditions

$$\overset{\circ}{A}{}'_* := \operatorname{Re}\left(\lambda \frac{d\phi}{ds}\Phi^-\right) = 0 \quad \text{on } \Gamma. \tag{12}$$

In a complex case, Vekua refers to problems of this type as being concomitant to $(\overset{\circ}{A}{}')$ and denotes them by $(\overset{\circ}{A}{}'_*)$. Let ℓ_* be a number of linearly independent solutions of this problem. Obviously, $\ell' + \ell'_* = k$.

Let us now return to the discussion of problem (A), where we assume that $\varkappa = 0$ in what follows. The solutions of this problem may be expressed in terms of the generalized Cauchy kernel as follows:

$$w(z) = w_1(z) + w_2(z) = \underset{\sim}{C}[\lambda\gamma](z) + \underset{\sim}{C}[i\lambda\mu](z),$$

where

$$\underset{\sim}{C}[\Phi] = P^{-1}\int_\Gamma d\phi(\zeta)\Omega^{(1)}(z,\zeta)\Phi(\zeta) - \overline{d\phi(\zeta)}\Omega^{(2)}(z,\zeta)\overline{\Phi(\zeta)}$$

(see [14, p.543]). From the Plemelj formulas, it is seen that the density μ must satisfy the integral equation

$$\gamma_0 = \int_\Gamma K_1(\zeta,z)\mu(z)\,ds_z, \tag{13}$$

where

$$\gamma_0(\zeta) = \gamma(\zeta) - \operatorname{Re}\left[\overline{\lambda(\zeta)}w_1^+(\zeta)\right] = -\operatorname{Re}\left[\overline{\lambda(\zeta)}w_1^-(\zeta)\right]. \tag{14}$$

Problem $(\overset{\circ}{A}_*)$ concomitant to problem $(\overset{\circ}{A})$ has the boundary condition $\operatorname{Re}[\lambda^{-1}(z)\Phi^-(z)] = 0$ on Γ, where Φ is Q-holomorphic outside $G + \Gamma$ and $\Phi(\infty) = 0$. Denoting the numbers of linearly independent solutions of $(\overset{\circ}{A})$ and $(\overset{\circ}{A}_*)$ by ℓ and ℓ_* respectively, we have $k = \ell + \ell_*$. In order that (13) is solvable, it is necessary and sufficient that the nonhomogeneous data γ_0 satisfy the auxiliary conditions

$$\int_\Gamma \varkappa_j(\zeta)\gamma_0(\zeta)\,ds_\zeta = 0 \quad (j = 1,\ldots,k), \tag{15}$$

where $\varkappa_j$ are solutions to integral equation (10). These solutions may be broken up into two groups $\{\varkappa_1,\ldots,\varkappa_{\ell'}\}$ and $\{\varkappa_{\ell'+1},\ldots,\varkappa_k\}$ such that $\varkappa_j = i\lambda(z)\frac{d\phi}{ds}w'_j(z)$ for $j = 1,\ldots,\ell'$ and $\varkappa_j = i\lambda(z)\frac{d\phi}{ds}\Phi_j$ for $j = \ell'+1,\ldots,k$, where $z \in \Gamma$. Here w'_j and Φ_j are solutions of problems $(\overset{\circ}{A}{}')$ and $(\overset{\circ}{A}{}'_*)$ respectively. The condition (15) for γ_0 given by (14) becomes

$$-\int_\Gamma \varkappa_j(\zeta)\gamma_0(\zeta)\,ds_\zeta = -i\int_\Gamma d\phi(\zeta)\lambda(\zeta)w'_j(\zeta)\gamma(\zeta) + \operatorname{Re}\left[i\int_\Gamma d\phi(\zeta)w'_j(\zeta)w_1^+(\zeta)\right]$$

for $j = 1,\ldots,\ell'$, whereas for $j = \ell'+1,\ldots,k$, we have

$$\int_\Gamma \varkappa_j(\zeta)\gamma_0(\zeta)\,ds = \operatorname{Re}\left[i\int_\Gamma d\phi(\zeta)\Phi_j^-(\zeta)w_1^-(\zeta)\right] = 0.$$

Consequently, the conditions (15) are seen to hold if (6) (with $F = 0$) holds. From the above discussion, one obtains a Fredholm-type theorem for problem (A).

Theorem 1 *Non-homogeneous boundary problem* (A) *is solvable if and only if the condition* (6) *is satisfied*, w' *being an arbitrary solution of adjoint homogeneous boundary problem* $(\overset{\circ}{A}{}')$.

This theorem immediately implies the following.

Theorem 2 *Non-homogeneous boundary problem* (A) *is solvable for an arbitrary right-hand side if and only if adjoint homogeneous problem* $(\overset{\circ}{A}{}')$ *has no solution*.

Competing interests
The author declares that they have no competing interests.

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References
1. Douglis, A: A function theoretic approach to elliptic systems of equations in two variables. Commun. Pure Appl. Math. **6**, 259-289 (1953)
2. Bojarskiĭ, BV: Theory of generalized analytic vectors. Ann. Pol. Math. **17**, 281-320 (1966) (in Russian)
3. Gilbert, RP: Constructive Methods for Elliptic Equations. Lecture Notes Math., vol. 365. Springer, Berlin (1974)
4. Gilbert, RP, Hile, GN: Generalized hypercomplex function theory. Trans. Am. Math. Soc. **195**, 1-29 (1974)
5. Gilbert, RP, Hile, GN: Hypercomplex function theory in the sense of L. Bers. Math. Nachr. **72**, 187-200 (1976)
6. Vekua, IN: Generalized Analytic Functions. Pergamon, Oxford (1962)
7. Bers, L: Theory of Pseudo-Analytic Functions. Inst. Math. Mech., New York University, New York (1953)
8. Begehr, H, Gilbert, RP: Transformations, Transmutations, and Kernel Functions. Longman, Harlow (1992)
9. Gilbert, RP, Buchanan, JL: First Order Elliptic Systems: A Function Theoretic Approach. Academic Press, Orlando (1983)
10. Begehr, H, Gilbert, RP: Boundary value problems associated with first order elliptic systems in the plane. Contemp. Math. **11**, 13-48 (1982)
11. Hile, GN: Function theory for generalized Beltrami systems. Compos. Math. **11**, 101-125 (1982)
12. Hızlıyel, S, Çağlıyan, M: Generalized *Q*-holomorphic functions. Complex Var. Theory Appl. **49**, 427-447 (2004)
13. Hızlıyel, S, Çağlıyan, M: Pseudo *Q*-holomorphic functions. Complex Var. Theory Appl. **49**, 941-955 (2004)
14. Hızlıyel, S: The Hilbert problem for generalized *Q*-homomorphic functions. Z. Anal. Anwend. **25**, 535-554 (2006)
15. Hızlıyel, S, Çağlıyan, M: The Riemann Hilbert problem for generalized *Q*-homomorphic functions. Turk. J. Math. **34**, 167-180 (2010)

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