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Some properties on the lexicographic product of graphs obtained by monogenic semigroups

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Abstract

In (Das *et al.* in J. Inequal. Appl. 2013:44, 2013), a new graph $\Gamma(\mathcal{S}_M)$ on monogenic semigroups $\mathcal{S}_M$ (with zero) having elements $\{0, x, x^2, x^3, \ldots, x^n\}$ was recently defined. The vertices are the non-zero elements $x, x^2, x^3, \ldots, x^n$ and, for $1 \leq i, j \leq n$, any two distinct vertices x^i and x^j are adjacent if $x^i x^j = 0$ in $\mathcal{S}_M$. As a continuing study, in an unpublished work, some well-known indices (first Zagreb index, second Zagreb index, Randić index, geometric-arithmetic index, atom-bond connectivity index, Wiener index, Harary index, first and second Zagreb eccentricity indices, eccentric connectivity index, the degree distance) over $\Gamma(\mathcal{S}_M)$ were investigated by the same authors of this paper.

In the light of the above references, our main aim in this paper is to extend these studies to the lexicographic product over $\Gamma(\mathcal{S}_M)$. In detail, we investigate the diameter, radius, girth, maximum and minimum degree, chromatic number, clique number and domination number for the lexicographic product of any two (not necessarily different) graphs $\Gamma(\mathcal{S}_M^1)$ and $\Gamma(\mathcal{S}_M^2)$.

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1 Introduction and preliminaries

The base of the graph $\Gamma(\mathcal{S}_M)$ is actually zero-divisor graphs (*cf.* [1]). In fact, the history of studying zero-divisor graphs began over commutative rings by the paper [2], and then it was followed over commutative and noncommutative rings by some of the joint papers [3–5]. After that the same terminology has been converted to commutative and noncommutative semigroups [6, 7].

In a recent study [1], the graph $\Gamma(\mathcal{S}_M)$ is defined by changing the adjacency rule of vertices and not destroying the main idea. Detailed, the authors considered a finite multiplicative monogenic semigroup with zero as the set

$$\mathcal{S}_M = \{0, x, x^2, x^3, \ldots, x^n\}. \tag{1}$$

Then, by following the definition given in [7], an undirected (zero-divisor) graph $\Gamma(\mathcal{S}_M)$ associated to $\mathcal{S}_M$ was obtained as in the following. The vertices of the graph are labeled by the nonzero zero-divisors (in other words, all nonzero element) of $\mathcal{S}_M$, and any two distinct vertices x^i and x^j, where $(1 \leq i,j \leq n)$ are connected by an edge in the case $x^ix^j = 0$ with the rule $x^ix^j = x^{i+j} = 0$ if and only if $i + j \geq n + 1$. The fundamental spectral properties such as the diameter, girth, maximum and minimum degree, chromatic number, clique number, degree sequence, irregularity index and dominating number for this new graph are presented in [1]. Furthermore, in an unpublished work, the same authors of this paper studied the first and second Zagreb indices, Randić index, geometric-arithmetic index and atom-bond connectivity index, Wiener index, Harary index, the first and second Zagreb eccentricity indices, eccentric connectivity index and the degree distance to indicate the importance of the graph $\Gamma(\mathcal{S}_M)$.

It is known that studying the *extension* of graphs is also an important tool (see, for instance, [8, 9]) since there are so many applications in science. With this idea, the *lexicographic product* $G[H]$ of any two simple graphs G and H (in some references, it is also called *composition product* [10]) is defined which has the vertex set $V(G) \times V(H)$ such that any two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are connected to each other by an edge if and only if $u_1v_1 \in E(G)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$ (see, for instance, [11–13]). In here, we replace G_1 by $\Gamma(\mathcal{S}_M^1)$ and G_2 by $\Gamma(\mathcal{S}_M^2)$ (as defined in (1)), where $\mathcal{S}_M^1 = \{x, x^2, x^3, \dots, x^n\}$ with 0 and $\mathcal{S}_M^2 = \{y, y^2, y^3, \dots, y^m\}$ with 0 such that $n \geq m$. Hence, the lexicographic product $\Gamma(\mathcal{S}_M^1)[\Gamma(\mathcal{S}_M^2)]$ has a vertex set $V(\Gamma(\mathcal{S}_M^1)[\Gamma(\mathcal{S}_M^2)]) = V(\Gamma(\mathcal{S}_M^1)) \times V(\Gamma(\mathcal{S}_M^2))$ which is given by

$$\left.\begin{array}{ll} \{(x,y),(x^2,y),\dots,(x^n,y), & (x,y^2),(x^2,y^2),\dots,(x^n,y^2), \\ \vdots & \vdots \\ (x,y^{m-1}),(x^2,y^{m-1}),\dots,(x^n,y^{m-1}), & (x,y^m),(x^2,y^m),\dots,(x^n,y^m)\} \end{array}\right\}. \tag{2}$$

Here, any two vertices (x^i, y^j) and (x^a, y^b) are connected to each other if and only if

$$\left.\begin{array}{l} x^ix^a \in E(\Gamma(\mathcal{S}_M^1)) \iff x^ix^a = 0 \iff i + a \geq n + 1 \quad \text{or} \\ x^i = x^a \quad \text{and} \quad y^jy^b \in E(\Gamma(\mathcal{S}_M^2)) \iff x^i = x^a \quad \text{and} \quad j + b \geq m + 1 \end{array}\right\}. \tag{3}$$

In this paper, by considering $\Gamma(\mathcal{S}_M^1)[\Gamma(\mathcal{S}_M^2)]$, we present some certain results for the diameter, radius, girth, maximum and minimum degrees, and finally chromatic, clique and domination numbers.

2 Main results

It is known that the *girth* of a simple graph G is the length of the shortest cycle contained in that graph. However, if G does not contain any cycle, then the girth of it is assumed to be infinity. Thus the first theorem of this paper is the following.

Theorem 1

$$\operatorname{girth}\bigl(\Gamma(\mathcal{S}_M^1)\bigl[\Gamma(\mathcal{S}_M^2)\bigr]\bigr) = 3.$$

Proof By considering (3), we easily conclude that

(i) $x^nx^{n-1} = 0$ implies $(x^n, y^m) \sim (x^{n-1}, y^m)$,
(ii) $x^{n-1}x^2 = 0$ implies $(x^{n-1}, y^m) \sim (x^2, y^m)$,
(iii) $x^2x^n = 0$ implies $(x^2, y^m) \sim (x^n, y^m)$.

Then, thinking the above steps at the same time, we get

$$(x^n, y^m) \sim (x^{n-1}, y^m) \sim (x^2, y^m) \sim (x^n, y^m),$$

as desired. □

The degree $\deg_G(v)$ of a vertex v of G is the number of vertices adjacent to v. Among all degrees, the *maximum* $\Delta(G)$ (or the *minimum* $\delta(G)$) degrees of G is the number of the largest (or the smallest) degree in G [14].

Theorem 2 *The maximum and minimum degrees of* $\Gamma(\mathcal{S}_M^1)[\Gamma(\mathcal{S}_M^2)]$ *are*

$$\Delta\bigl(\Gamma(\mathcal{S}_M^1)\bigl[\Gamma(\mathcal{S}_M^2)\bigr]\bigr) = nm - 1 \quad \textit{and} \quad \delta\bigl(\Gamma(\mathcal{S}_M^1)\bigl[\Gamma(\mathcal{S}_M^2)\bigr]\bigr) = m + 1,$$

respectively.

Proof It is obvious that the vertex set $V(\Gamma(\mathcal{S}_M^1)[\Gamma(\mathcal{S}_M^2)])$ in (2) has a total of nm vertices. Among these vertices, let us take the vertex (x^n, y^m). So, the maximum degree Δ of the graph $\Gamma(\mathcal{S}_M^1)[\Gamma(\mathcal{S}_M^2)]$ is equal to

$$nm - 1$$

since the vertex (x^n, y^m) is adjacent to all the other vertices.

On the other hand, let us take the vertex (x, y). Then, again by (3), the adjacency of (x, y) with a vertex (x^i, y^j) holds only if we have $i = n$ or $i = 1$ and $j = m$. That means the vertex (x, y) is connected to $(x^n, y), (x^n, y^2), \ldots, (x^n, y^m)$ and (x, y^m). Thus $\delta(\Gamma(\mathcal{S}_M^1)[\Gamma(\mathcal{S}_M^2)]) = m + 1$, as required. □

We recall that the *distance* (length of the shortest path) between two vertices u and v of G is denoted by $d_G(u, v)$. Moreover, the *diameter* of a simple graph G is defined by

$$\operatorname{diam}(G) = \max\bigl\{d_G(u, v) : u \text{ and } v \text{ are vertices of } G\bigr\}.$$

We then have the next result.

Theorem 3

$$\operatorname{diam}\bigl(\Gamma(\mathcal{S}_M^1)\bigl[\Gamma(\mathcal{S}_M^2)\bigr]\bigr) = 2.$$

Proof Obviously, the vertex (x, y) in (2) has at least one neighborhood, and so the diameter can be figured out by considering the distance between (x, y) and one of the other vertices in the vertex set. Therefore, by (3), the vertex (x, y) is adjacent only to the vertices $(x^n, y), (x^n, y^2), \ldots, (x^n, y^m)$ and (x, y^m). However (x^n, y^m) is adjacent to all the vertices defined in (2). Therefore the diameter should be obtained by considering the distance between (x, y) and (x^i, y^j), where $1 \leq i \leq n - 1$, $1 \leq j \leq m$. In here, we must assume that the case $i = 1$ and $j = m$ does not hold at the same time since there exists an adjacency

$$(x, y) \sim (x^n, y^m) \sim (x^i, y^j).$$

Hence the result. □

The *eccentricity* of a vertex v, denoted by $\varepsilon(v)$, in a connected graph G is the maximum distance between v and any other vertex u of G. (For a disconnected graph, all vertices are defined to have infinite eccentricity.) It is clear that $\operatorname{diam}(G)$ is equal to the *maximum eccentricity* among all the vertices of G. On the other hand, the *minimum eccentricity* is called the *radius* [15, 16] of G and denoted by

$$\operatorname{rad}(G) = \min_{u}\Bigl\{\max_{v}\bigl\{d_G(u,v)\bigr\}\Bigr\}.$$

Theorem 4

$$\operatorname{rad}\bigl(\Gamma\bigl(\mathcal{S}_M^1\bigr)\bigl[\Gamma\bigl(\mathcal{S}_M^2\bigr)\bigr]\bigr) = 1.$$

Proof We know that the vertex (x^n, y^m) is adjacent to all the vertices in (2). Thus the radius can be figured out by considering the distance between (x^n, y^m) and one of the other vertices in the set (2). So,

$$\bigl(x^n, y^m\bigr) \sim \bigl(x^i, y^j\bigr), \quad 1 \le i \le n, 1 \le j \le m.$$

Hence, the eccentricity $\varepsilon[(x^n, y^m)]$ is equal to 1, which implies the required result. □

A subset D of the vertex set $V(G)$ of a graph G is called a *dominating set* if every vertex $V(G)\backslash D$ is joined to at least one vertex of D by an edge. Additionally, the *domination number* $\gamma(G)$ is the number of vertices in the smallest dominating set for G. (We may refer to [14] for the fundamentals of a domination number.)

In our case, by (2), the dominating set is defined by $\{(x^n, y^m)\}$ since the vertices (x^n, y^m) are adjacent to all the other vertices. Hence we obtain the next result.

Theorem 5 $\gamma(\Gamma(\mathcal{S}_M^1)[\Gamma(\mathcal{S}_M^2)]) = 1$.

Basically, the coloring of G is to be an assignment of colors (elements of some set) to the vertices of G, one color to each vertex, so that adjacent vertices are assigned distinct colors. If n colors are used, then the coloring is referred to as an *n-coloring*. If there exists an n-coloring of G, then G is called *n-colorable*. The minimum number n for which G is n-colorable is called the *chromatic number* of G and is denoted by $\chi(G)$.

In addition, there exists another graph parameter, namely the *clique* of a graph G. In fact, depending on the vertices, each of the maximal complete subgraphs of G is called a clique. Moreover, the largest number of vertices in any clique of G is called the *clique number* and denoted by $\omega(G)$. In general, by [14], it is well known that $\chi(G) \ge \omega(G)$ for any graph G. For every induced subgraph H of G, if $\chi(H) = \omega(H)$ holds, then G is called a *perfect graph* [17].

By constructing the next result (see Theorem 6 below) for the chromatic number over the lexicographic product of the graphs $\Gamma(\mathcal{S}_M^1)$ and $\Gamma(\mathcal{S}_M^2)$, we shall present a negative answer of a result given in [18] (see Remark 1).

We recall that for a real number r, the notation $\lceil r\rceil$ denotes the least integer $\ge r$. This fact will be needed for some of our results below.

The proof of the following lemma can be found in [1, Theorem 6].

Lemma 1 ([1]) *For a monogenic semigroup $\mathcal{S}_M$ as in* (1), *the chromatic number of the graph $\Gamma(\mathcal{S}_M)$ is given by*

$$1+\left\lceil \frac{n-1}{2} \right\rceil.$$

The next result is an extension of the above lemma to the lexicographic product.

Theorem 6 *The chromatic number of $\Gamma(\mathcal{S}_M^1)[\Gamma(\mathcal{S}_M^2)]$ is equal to*

$$\left(1+\left\lceil \frac{n-1}{2} \right\rceil\right)\left(1+\left\lceil \frac{m-1}{2} \right\rceil\right).$$

In other words, $\chi(\Gamma(\mathcal{S}_M^1)[\Gamma(\mathcal{S}_M^2)]) = \chi(\Gamma(\mathcal{S}_M^1))\chi(\Gamma(\mathcal{S}_M^2))$.

Proof *First step*: The list of vertices that the vertex (x^n, y^m) is adjacent to all the other vertices was given in (2). That means the color that was used for (x^n, y^m) cannot be used for any other vertices. So, let us suppose that the color used for the vertex (x_1^n, x_2^m) is labeled by $\mathcal{C}_1^1$. Secondly, if we consider the vertex (x^n, y^{m-1}), then it is easy to see that (x^n, y^{m-1}) is adjacent to all the vertices except the vertex (x^n, y). Thus, the color for (x^n, y^{m-1}), say $\mathcal{C}_1^2$, can be also used only for (x^n, y). As a similar idea, the vertex (x^n, y^{m-2}) is adjacent to all the vertices except the vertices (x^n, y) and (x^n, y^2). Thus the color, say $\mathcal{C}_1^3$, for (x^n, y^{m-2}) can be also used only for the vertex (x^n, y^2). (Notice that the color $\mathcal{C}_1^2$ has been already used for (x^n, y) in the previous step.) After that, following the same progress, we see that the total of $1+\lceil\frac{m-1}{2}\rceil$ different colors is needed to handle the coloring of all vertices in the set $\{(x^n, y^j); 1 \leq j \leq m\}$.

Second step: Except the set of vertices $\{(x,y),(x,y^2),\ldots,(x,y^m)\}$, the vertex (x^{n-1}, y^m) is adjacent to all the other vertices defined in (2). On the other hand, while each element in the sets

$$\left\{(x,y),\left(x,y^2\right),\ldots,\left(x,y^{\lceil\frac{m}{2}\rceil}\right)\right\} \quad \text{and} \quad \left\{\left(x,y^{\lceil\frac{m}{2}\rceil+1}\right),\left(x,y^{\lceil\frac{m}{2}\rceil+2}\right),\ldots,\left(x,y^m\right)\right\}$$

is adjacent to each other, there also exists an adjacency among the vertices

$$\left(x,y^{\lceil\frac{m}{2}\rceil+1}\right),\left(x,y^{\lceil\frac{m}{2}\rceil+2}\right),\ldots,\left(x,y^m\right).$$

That means the color used for (x^{n-1}, y^m) can be also used for the vertices

$$\left\{(x,y),\left(x,y^2\right),\ldots,\left(x,y^{\lceil\frac{m}{2}\rceil}\right)\right\}.$$

So, let us suppose that the color used for (x^{n-1}, y^m) and the vertices

$$(x,y),\left(x,y^2\right),\ldots,\left(x,y^{\lceil\frac{m}{2}\rceil}\right)$$

is labeled by $\mathcal{C}_2^1$.

Now let us secondly consider the vertex (x^{n-1}, y^{m-1}). Since (x^{n-1}, y^{m-1}) is not adjacent to vertices $(x,y),(x,y^2),\ldots,(x,y^m)$ and (x^{n-1}, y), the color, say $\mathcal{C}_2^2$, for (x^{n-1}, y^{m-1}) can be also used only for

$$(x, y^{\lceil \frac{m}{2} \rceil+1}) \quad \text{and} \quad (x^{n-1}, y).$$

(The color $\mathcal{C}_2^1$ has been already used for the vertices $(x, y), (x, y^2), \ldots, (x, y^{\lceil \frac{m}{2} \rceil})$ in the previous step.) Similarly, it is easy to see that the vertex (x^{n-1}, y^{m-2}) is not adjacent to the vertices

$$(x, y), (x, y^2), \ldots, (x, y^m), (x^{n-1}, y) \quad \text{and} \quad (x^{n-1}, y^2).$$

Thus the color, say $\mathcal{C}_2^3$, for (x^{n-1}, y^{m-2}) can be also used only for the vertices $(x, y^{\lceil \frac{m}{2} \rceil+2})$ and (x^{n-1}, y^2). (Again, notice that the color $\mathcal{C}_2^2$ has been already used for (x^{n-1}, y) previously.)

Finally, we need the total of $1 + \lceil \frac{m-1}{2} \rceil$ different colors for the coloring of vertices in the set

$$\left\{(x^{n-1}, y^j), (x, y^i); 1 \le j \le m, \left\lceil \frac{m}{2} \right\rceil + 1 \le i \le m\right\}.$$

Third step: The vertex (x^{n-2}, y^m) cannot be adjacent to the vertices

$$\{(x, y), (x, y^2), \ldots, (x, y^m), (x^2, y), (x^2, y^2), \ldots, (x^2, y^m)\}$$

in the set (2). In the second step, we have already colored the vertices $(x, y), (x, y^2), \ldots, (x, y^m)$. Furthermore, again similarly as in the second step, the vertices

$$\{(x^2, y), (x^2, y^2), \ldots, (x^2, y^{\lceil \frac{m}{2} \rceil})\} \quad \text{and} \quad \{(x^2, y^{\lceil \frac{m}{2} \rceil+1}), (x^2, y^{\lceil \frac{m}{2} \rceil+2}), \ldots, (x^2, y^m)\}$$

are adjacent to each other, and also there exists an adjacency among the vertices

$$(x^2, y^{\lceil \frac{m}{2} \rceil+1}), (x^2, y^{\lceil \frac{m}{2} \rceil+2}), \ldots, (x^2, y^m).$$

That means the color that used for (x^{n-2}, y^m) can be also used for the vertices

$$\{(x^2, y), (x^2, y^2), \ldots, (x^2, y^{\lceil \frac{m}{2} \rceil})\}.$$

So, let us suppose that the color used for (x^{n-2}, y^m) and $(x^2, y), (x^2, y^2), \ldots, (x^2, y^{\lceil \frac{m}{2} \rceil})$ is labeled by $\mathcal{C}_3^1$. Moreover, if we consider the vertex (x^{n-2}, y^{m-1}), then it is clear that it is not adjacent to the vertices

$$(x^2, y), (x^2, y^2), \ldots, (x^2, y^m) \quad \text{and} \quad (x^{n-1}, y).$$

Hence, the color, say $\mathcal{C}_3^2$, for the vertex (x^{n-2}, y^{m-1}) can be also used only for

$$(x^2, y^{\lceil \frac{m}{2} \rceil+1}) \quad \text{and} \quad (x^{n-2}, y).$$

(We note that the color $\mathcal{C}_3^1$ has been already used for

$$(x^2, y), (x^2, y^2), \ldots, (x^2, y^{\lceil \frac{m}{2} \rceil})$$

previously.) Finally, the vertex (x^{n-2}, y^{m-2}) is not adjacent to the vertices

$$(x^2,y),(x^2,y^2),\dots,(x^2,y^m),(x^{n-2},y) \quad \text{and} \quad (x^{n-2},y^2).$$

Thus the color, say $\mathcal{C}_3^3$, for (x^{n-2}, y^{m-2}) can be also used only for the vertices

$$\left(x^2,y^{\lceil \frac{m}{2}\rceil+2}\right) \quad \text{and} \quad \left(x^{n-2},y^2\right).$$

(Note that the color $\mathcal{C}_3^2$ has been already used for (x^{n-2}, y).) Following a similar process as in the third step, one can see that the total of $1 + \lceil \frac{m-1}{2} \rceil$ different colors is needed to handle the coloring of all the vertices in the set

$$\left\{(x^{n-2},y^j),(x^2,y^i);1\leq j\leq m,\left\lceil\frac{m}{2}\right\rceil+1\leq i\leq m\right\}.$$

By applying the same procedure as in the above steps, one can see that to handle the coloring of all the vertices in the set (2), we need the total of $1 + \lceil \frac{n-1}{2} \rceil$ steps. In fact, each step has $1 + \lceil \frac{m-1}{2} \rceil$ different colors. Therefore we obtain

$$\chi\left(\Gamma\left(\mathcal{S}_M^1\right)\left[\Gamma\left(\mathcal{S}_M^2\right)\right]\right)=\left(1+\left\lceil\frac{n-1}{2}\right\rceil\right)\left(1+\left\lceil\frac{m-1}{2}\right\rceil\right),$$

as desired. □

Remark 1 It is clear that $\chi(G[H]) \leq \chi(G)\chi(H)$. This trivial upper bound is attained for any G and H with $\chi(G) = \omega(G)$ and $\chi(H) = \omega(H)$. However, in Theorem 6, we obtained an equality between $\chi(\Gamma(\mathcal{S}_M^1)[\Gamma(\mathcal{S}_M^2)])$ and $\chi(\Gamma(\mathcal{S}_M^1))\chi(\Gamma(\mathcal{S}_M^2))$. But it was shown in [18], that there is not any product $*$ of graphs for which the equality $\chi(G*H) = \chi(G)\chi(H)$ holds for all graphs G and H.

In [10, Theorem 3.1], the authors proved that the clique number is preserved under the lexicographic product for any graphs G and H. In the following, we deal with this result by considering our special graphs. Before that, we need to present the following lemma, the truthfulness of which is quite clear.

Lemma 2 *For any $m \in \mathbb{N}^+$, there always exists*

$$m-\left\lceil\frac{m}{2}\right\rceil=\left\lceil\frac{m-1}{2}\right\rceil.$$

Theorem 7 *The clique number of $\Gamma(\mathcal{S}_M^1)[\Gamma(\mathcal{S}_M^2)]$ is equal to*

$$\omega\left(\Gamma\left(\mathcal{S}_M^1\right)\left[\Gamma\left(\mathcal{S}_M^2\right)\right]\right)=\left(1+\left\lceil\frac{n-1}{2}\right\rceil\right)\left(1+\left\lceil\frac{m-1}{2}\right\rceil\right).$$

Proof In the proof, we must first check whether the subgraph is complete or not (which means any two distinct vertices in the vertex set of this subgraph are adjacent). Now let us consider the graph $\Gamma(\mathcal{S}_M^1)[\Gamma(\mathcal{S}_M^2)]$. According to the definition, a subgraph will be complete if, for all distinct vertices (x^i, y^j) and (x^a, y^b),

$$x^i \sim x^a \quad (i.e., x^i \cdot x^a = 0) \quad \text{or} \quad x^i = x^a \quad \text{and} \quad y^j \sim y^b \quad (i.e., y^j \cdot y^b = 0) \tag{4}$$

i.e., $(x^i, y^j) \sim (x^a, y^b)$ for all i, j, a, b.

On the other hand, the equality in (4) will hold only in case the sum $i + a$ would be at least equal to the $n + 1$ or the sum $j + b$ would be at least equal to the $m + 1$ and $i = a$. Therefore, for any two vertices (x^i, y^j) and (x^a, y^b), we must have at least

$$i = \left\lceil \frac{n}{2} \right\rceil, \qquad a = \left\lceil \frac{n}{2} \right\rceil + 1 \quad \text{or} \quad i = a, \qquad j = \left\lceil \frac{m}{2} \right\rceil, \qquad b = \left\lceil \frac{m}{2} \right\rceil + 1$$

since $\lceil \frac{n}{2} \rceil + \lceil \frac{n}{2} \rceil + 1 = n + 1$ and $\lceil \frac{m}{2} \rceil + \lceil \frac{m}{2} \rceil + 1 = m + 1$. This process will be given a maximal complete subgraph, say A, with the vertex set

$$V(A) = \left\{ \begin{array}{c} (x^{\lceil \frac{n}{2} \rceil}, y^{\lceil \frac{m}{2} \rceil}), (x^{\lceil \frac{n}{2} \rceil}, y^{\lceil \frac{m}{2} \rceil + 1}), \ldots, (x^{\lceil \frac{n}{2} \rceil}, y^m), \\ (x^{\lceil \frac{n}{2} \rceil + 1}, y^{\lceil \frac{m}{2} \rceil}), (x^{\lceil \frac{n}{2} \rceil + 1}, y^{\lceil \frac{m}{2} \rceil + 1}), \ldots, (x^{\lceil \frac{n}{2} \rceil + 1}, y^m), \\ \vdots \\ (x^n, y^{\lceil \frac{m}{2} \rceil}), (x^n, y^{\lceil \frac{m}{2} \rceil + 1}), \ldots, (x^n, y^m) \end{array} \right\}.$$

Note that the number of elements in the set $V(A)$ is given by

$$\left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right)\left(m - \left\lceil \frac{m}{2} \right\rceil + 1\right) = \left(1 + \left\lceil \frac{n-1}{2} \right\rceil\right)\left(1 + \left\lceil \frac{m-1}{2} \right\rceil\right) \quad \text{(by Lemma 2)}.$$

Hence we obtain $\omega(\Gamma(\mathcal{S}_M^1)[\Gamma(\mathcal{S}_M^2)]) = (1 + \lceil \frac{n-1}{2} \rceil)(1 + \lceil \frac{m-1}{2} \rceil)$, as required. □

Remark 2 By Theorems 6 and 7,

$$\chi(\Gamma(\mathcal{S}_M^1)[\Gamma(\mathcal{S}_M^2)]) = \omega(\Gamma(\mathcal{S}_M^1)[\Gamma(\mathcal{S}_M^2)]) = \left(1 + \left\lceil \frac{n-1}{2} \right\rceil\right)\left(1 + \left\lceil \frac{m-1}{2} \right\rceil\right), \tag{5}$$

which implies that the lexicographic product preserves the perfectness property for the special graphs $\Gamma(\mathcal{S}_M^1)$ and $\Gamma(\mathcal{S}_M^2)$. We note that each graph in here is perfect by [1]. Actually, Eq. (5) implies a special case of the result in [19] since in this reference the authors proved that the lexicographic product $G[H]$ is perfect iff G and H are perfect.

Example 1 For the semigroups

$$\mathcal{S}_M^5 = \{x, x^2, x^3, x^4, x^5\} \quad \text{and} \quad \mathcal{S}_M^4 = \{y, y^2, y^3, y^4\}$$

as in (1), let us consider the graph $\Gamma(\mathcal{S}_M^5) \otimes \Gamma(\mathcal{S}_M^4)$. Depending on the results presented in this paper, we can state the following equalities:

(i) $\mathrm{girth}(\Gamma(\mathcal{S}_M^5)[\Gamma(\mathcal{S}_M^4)]) = 3$ (by Theorem 1).
(ii) $\Delta(\Gamma(\mathcal{S}_M^5)[\Gamma(\mathcal{S}_M^4)]) = 19$ and $\delta(\Gamma(\mathcal{S}_M^5) \otimes \Gamma(\mathcal{S}_M^4)) = 5$ (by Theorem 2).
(iii) $\mathrm{diam}(\Gamma(\mathcal{S}_M^5)[\Gamma(\mathcal{S}_M^4)]) = 2$ (by Theorem 3).
(iv) $\mathrm{rad}(\Gamma(\mathcal{S}_M^5)[\Gamma(\mathcal{S}_M^4)]) = 1$ (by Theorem 4).

(v) $\gamma(\Gamma(\mathcal{S}_M^5)[\Gamma(\mathcal{S}_M^4)]) = 1$ (by Theorem 5).

(vi) $\chi(\Gamma(\mathcal{S}_M^5)[\Gamma(\mathcal{S}_M^4)]) = 9$ (by Theorem 6).

(vii) $\omega(\Gamma(\mathcal{S}_M^5)[\Gamma(\mathcal{S}_M^4)]) = 9$ (by Theorem 7).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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References
1. Das, KC, Akgunes, N, Cevik, AS: On a graph of monogenic semigroup. J. Inequal. Appl. **2013**, 44 (2013). doi:10.1186/1029-242X-2013-44
2. Beck, I: Coloring of commutating ring. J. Algebra **116**, 208-226 (1988)
3. Anderson, DF, Livingston, PS: The zero-divisor graph of commutative ring. J. Algebra **217**, 434-447 (1999)
4. Anderson, DF, Badawi, A: On the zero-divisor graph of a ring. Commun. Algebra **36**(8), 3073-3092 (2008)
5. Anderson, DD, Naseer, M: Beck's coloring of a commutative ring. J. Algebra **159**, 500-514 (1991)
6. DeMeyer, FR, DeMeyer, L: Zero-divisor graphs of semigroups. J. Algebra **283**, 190-198 (2005)
7. DeMeyer, FR, McKenzie, T, Schneider, K: The zero-divisor graph of a commutative semigroup. Semigroup Forum **65**, 206-214 (2002)
8. Imrich, W: Factoring cardinal product graphs in polynomial time. Discrete metric spaces (Villeurbanne, 1996). Discrete Math. **192**(1-3), 119-144 (1998)
9. Imrich, W, Klavžar, S: Product Graphs. Structure and Recognition. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York (2000)
10. Došlić, T, Ghorbani, M, Hosseinzadeh, MA: The relationships between Wiener index, stability number and clique number of composite graphs. Bull. Malays. Math. Sci. Soc. **36**(1), 165-172 (2013)
11. Čižek, N, Klavžar, S: On the chromatic number of the lexicographic product and the Cartesian sum of graphs. Discrete Math. **134**, 17-24 (1994)
12. Li, F, Wang, W, Xu, Z, Zhao, H: Some results on the lexicographic product of vertex-transitive graphs. Appl. Math. Lett. **24**, 1924-1926 (2011)
13. Klavžar, S: Coloring graph products - a survey. Discrete Math. **155**, 135-145 (1996)
14. Gross, JL, Yellen, J (eds.): Handbook of Graph Theory. CRC Press, Boca Raton (2004)
15. Akgunes, N, Cevik, AS: A new bound of radius of irregularity index. Appl. Math. Comput. **219**, 5750-5753 (2013)
16. Erdös, P, Pach, J, Pollack, R, Tuza, Z: Radius, diameter, and minimum degree. J. Comb. Theory, Ser. B **47**, 73-79 (1989)
17. Lovász, L: Normal hypergraphs and the weak perfect graph conjecture. Discrete Math. **2**, 253-267 (1972)
18. Puš, V: Chromatic number of products of graphs. Comment. Math. Univ. Carol. **29**, 457-463 (1988)
19. Ravindra, G, Parthasarathy, KR: Perfect product graphs. Discrete Math. **20**, 177-186 (1977)

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