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Research Article

Properties of a Class of p-Harmonic Functions

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A p times continuously differentiable complex-valued function $F = u + iv$ in a domain $D \subseteq \mathbb{C}$ is p-harmonic if F satisfies the p-harmonic equation $\Delta \cdots \Delta F = 0$, where p is a positive integer. By using the generalized Salagean differential operator, we introduce a class of p-harmonic functions and investigate necessary and sufficient coefficient conditions, distortion bounds, extreme points, and convex combination of the class.

1. Introduction

A continuous complex-valued function $f = u + iv$ in a domain $D \subseteq \mathbb{C}$ is harmonic if both u and v are real harmonic in D; that is, $\Delta u = 0$ and $\Delta v = 0$. Here Δ represents the complex Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{1}$$

In any simply connected domain D we can write $f = h+\overline{g}$, where h and g are analytic in D. We call h the analytic part and g the coanalytic part of f. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $J_f = |f_z|^2 - |f_{\overline{z}}|^2 > 0$ in D. See [1, 2].

Denote by SH the class of functions $f = h + \overline{g}$ that are harmonic, univalent, and sense preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in SH$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{j=2}^{\infty} a_j z^j, \qquad g(z) = \sum_{j=1}^{\infty} b_j z^j, \quad |b_1| < 1. \tag{2}$$

The properties of the class SH and its geometric subclasses have been investigated by many authors; see ([1–6]). Note that SH reduces to the class S of normalized analytic univalent functions in U if the coanalytic part of f is identically zero.

A p times continuously differentiable complex-valued function $F = u + iv$ in a domain $D \subseteq \mathbb{C}$ is p-harmonic if F satisfies the p-harmonic equation $\Delta \cdots \Delta F = 0$, where p is a positive integer.

A function F is p-harmonic in a simply connected domain D if and only if F has the following representation:

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} f_{p-k+1}(z), \tag{3}$$

where $\Delta f_{p-k+1}(z) = 0$ in D for each $k \in \{1, \ldots, p\}$. f_{p-k+1} has the form

$$f_{p-k+1} = h_{p-k+1} + \overline{g}_{p-k+1}, \tag{4}$$

where

$$h_p(z) = z + \sum_{j=2}^{\infty} a_{j,p} z^j,$$
$$h_{p-k+1}(z) = \sum_{j=1}^{\infty} a_{j,p-k+1} z^j, \quad (k \geq 2), \tag{5}$$
$$g_{p-k+1}(z) = \sum_{j=1}^{\infty} b_{j,p-k+1} z^j, \quad (k \geq 1).$$

Denote by SH_p the class of functions F of the form (3) that are harmonic, univalent, and sense-preserving in the unit disk. Apparently, if $p = 1$ and $p = 2$, F is harmonic and biharmonic, respectively.

Biharmonic functions have been studied by several authors, such as, [7–9]. Also, biharmonic functions arise in many physical situations, particularly, in fluid dynamics and elasticity problems. They have many important applications in engineering, biology, and medicine, such as in [10, 11].

For a function f in S, differential operator D^n $(n \in \mathbb{N}_0)$ was introduced by Sălăgean [12]. Al-Oboudi [13] generalized D^n as follows:

$$D_\lambda^0 f(z) = f(z),$$
$$D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z), \quad \lambda \geq 0, \tag{6}$$
$$D_\lambda^n f(z) = D_\lambda^1 \left(D_\lambda^{n-1} f(z)\right).$$

When $\lambda = 1$, we get the Salagean differential operator.

For $f(z) = h(z) + \overline{g(z)}$ given by (2), Li and Liu [14] defined the following generalized Salagean operator D_λ^n in SH:

$$D_\lambda^n f(z) = D_\lambda^n h(z) + \overline{D_\lambda^n g(z)}, \quad \lambda \geq 0, \tag{7}$$

where

$$D_\lambda^n h(z) = z + \sum_{j=2}^{\infty} [1 + (k-1)\lambda]^n a_j z^j,$$
$$D_\lambda^n g(z) = \sum_{j=1}^{\infty} [1 + (k-1)\lambda]^n b_j z^j. \tag{8}$$

For a p-harmonic function F given by (3), we define the following operator:

$$D_\lambda^0 F(z) = F(z), \tag{9}$$

$$D_\lambda^1 F(z) = (1-\lambda) D_\lambda^0 F(z) + \lambda \left[z\left(D_\lambda^0 F(z)\right)_z + \overline{z}\left(D_\lambda^0 F(z)\right)_{\overline{z}} \right], \quad \lambda \geq 0,$$
$$D_\lambda^n F(z) = D_\lambda^1 \left(D_\lambda^{n-1} F(z)\right), \quad (n \in \mathbb{N}). \tag{10}$$

If F is given by (3), then from (10) we see that

$$D_\lambda^n F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} \left[1 + (j-1)\lambda + 2(k-1)\lambda\right]^n \times a_{j,p-k+1} z^j + \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} \left[1 + (j-1)\lambda + 2(k-1)\lambda\right]^n \times \overline{b}_{j,p-k+1} \overline{z}^j, \quad \left(a_{1,p} = 1, \left|b_{1,p}\right| < 1\right). \tag{11}$$

When $p = 1$, we get the generalized Salagean operator for harmonic univalent functions defined by Li and Liu [14].

Denote by $SH_p(n, \lambda, \alpha)$ the class of functions F of the form (3) which satisfy the condition

$$\operatorname{Re}\left\{ \frac{D_\lambda^{n+1} F(z)}{D_\lambda^n F(z)} \right\} \geq \alpha, \quad 0 \leq \alpha < 1, \tag{12}$$

where $D_\lambda^n F(z)$ is defined by (11).

We let the subclass $\overline{SH}_p$ of SH_p consist of functions F of the form (3) which include $f_{p-k+1} = h_{p-k+1} + \overline{g}_{p-k+1}$, where

$$h_p(z) = z - \sum_{j=2}^{\infty} \left|a_{j,p}\right| z^j,$$
$$h_{p-k+1}(z) = -\sum_{j=1}^{\infty} \left|a_{j,p-k+1}\right| z^j, \quad (k \geq 2), \tag{13}$$

$$g_{p-k+1}(z) = -\sum_{j=1}^{\infty} \left|b_{j,p-k+1}\right| z^j, \quad (k \geq 1). \tag{14}$$

Define $\overline{SH}_p(n, \lambda, \alpha) := SH_p(n, \lambda, \alpha) \cap \overline{SH}_p$.

The main object of the paper is to introduce a class of p-harmonic functions by using the generalized Salagean operator which was defined by Li and Liu [14]. We investigate necessary and sufficient coefficient conditions, extreme points, distortion bounds, and convex combination of the class.

2. Main Results

Theorem 1. *Let F be a p-harmonic function given by* (3). *Furthermore, let*

$$\sum_{k=1}^{p} \sum_{j=1}^{\infty} \left[1 + (j-1)\lambda + 2(k-1)\lambda - \alpha\right] \times \left[1 + (j-1)\lambda + 2(k-1)\lambda\right]^n \left[\left|a_{j,p-k+1}\right| + \left|b_{j,p-k+1}\right|\right] \leq 2(1-\alpha), \tag{15}$$

where $\lambda \geq 1$, $n \in \mathbb{N}$, $0 \leq \alpha < 1$, and $a_{1,p} = 1$. Then F is sense preserving, p-harmonic, and univalent in U and $F \in SH_p(n, \lambda, \alpha)$.

Proof. Suppose $z_1, z_2 \in U$ and $z_1 \neq z_2$, so that $|z_1| \leq |z_2| < 1$:

$$\left|F(z_1) - F(z_2)\right| \geq \left|f_p(z_1) - f_p(z_2)\right| - \left|\sum_{k=2}^{p} \left[\left|z_1\right|^{2(k-1)} f_{p-k+1}(z_1) - \left|z_2\right|^{2(k-1)} f_{p-k+1}(z_2)\right]\right|$$
$$\geq \left|z_1 - z_2\right| \left[1 - \sum_{j=2}^{\infty} \frac{\left|z_1^j - z_2^j\right|}{\left|z_1 - z_2\right|} \left|a_{j,p}\right| - \sum_{j=1}^{\infty} \frac{\left|z_1^j - z_2^j\right|}{\left|z_1 - z_2\right|} \left|b_{j,p}\right| \right.$$

$$
\begin{aligned}
&\quad - \sum_{k=2}^{p}\sum_{j=1}^{\infty} |z_2|^{2(k-1)} \frac{\left|z_1^j - z_2^j\right|}{|z_1 - z_2|} \\
&\quad \times \left[\left|a_{j,p-k+1}\right| + \left|b_{j,p-k+1}\right|\right] \Bigg] \\
&> |z_1 - z_2| \left[1 - \sum_{j=2}^{\infty} j\left|a_{j,p}\right| - \sum_{j=1}^{\infty} j\left|b_{j,p}\right| \right. \\
&\quad \left. - \sum_{k=2}^{p}\sum_{j=1}^{\infty} j\left[\left|a_{j,p-k+1}\right| + \left|b_{j,p-k+1}\right|\right] \right] \\
&\geq |z_1 - z_2| \left[2 - \sum_{k=1}^{p}\sum_{j=1}^{\infty} \Big(\left[1 + (j-1)\lambda + 2(k-1)\lambda - \alpha\right] \right. \\
&\quad \times \left[1 + (j-1)\lambda + 2(k-1)\lambda\right]^n \Big) \\
&\quad \left. \times (1-\alpha)^{-1} \left[\left|a_{j,p-k+1}\right| + \left|b_{j,p-k+1}\right|\right] \right] \\
&\geq 0,
\end{aligned} \tag{16}
$$

which proves univalence.

In order to prove that F is sense preserving, we need to show that $|F_z(z)| - |F_{\overline{z}}(z)| > 0$:

$$
\begin{aligned}
&\left|F_z(z)\right| - \left|F_{\overline{z}}(z)\right| \\
&= \left| 1 + \sum_{j=2}^{\infty} j a_{j,p} z^{j-1} + \sum_{k=2}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} (j+k-1) a_{j,p-k+1} z^{j-1} \right. \\
&\quad \left. + \sum_{k=1}^{p} \frac{|z|^{2(k-1)}}{z} \sum_{j=1}^{\infty} (k-1) \overline{b}_{j,p-k+1} \overline{z}^j \right| \\
&\quad - \left| \sum_{k=1}^{p} \frac{|z|^{2(k-1)}}{\overline{z}} \sum_{j=1}^{\infty} (k-1) a_{j,p-k+1} z^j \right. \\
&\quad \left. + \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} (j+k-1) \overline{b}_{j,p-k+1} \overline{z}^{j-1} \right| \\
&> 2 - \sum_{k=1}^{p}\sum_{j=1}^{\infty} \left[j + 2(k-1)\right] \left[\left|a_{j,p-k+1}\right| + \left|b_{j,p-k+1}\right|\right] \\
&\geq 2 - \sum_{k=1}^{p}\sum_{j=1}^{\infty} \Big(\left[1 + (j-1)\lambda + 2(k-1)\lambda - \alpha\right] \\
&\quad \times \left[1 + (j-1)\lambda + 2(k-1)\lambda\right]^n \Big) \\
&\quad \times (1-\alpha)^{-1} \left[\left|a_{j,p-k+1}\right| + \left|b_{j,p-k+1}\right|\right] \\
&\geq 0,
\end{aligned} \tag{17}
$$

for all $z \in U$.

Using the fact that $\operatorname{Re} w \geq \alpha$ if and only if $|1 - \alpha + w| \geq |1 + \alpha - w|$, it suffices to show that

$$
\begin{aligned}
&\left|(1-\alpha) D_\lambda^n F(z) + D_\lambda^{n+1} F(z)\right| \\
&\quad - \left|(1+\alpha) D_\lambda^n F(z) - D_\lambda^{n+1} F(z)\right| \geq 0.
\end{aligned} \tag{18}
$$

Substituting for $D_\lambda^n F$ in (18), we obtain

$$
\begin{aligned}
&\left|(1-\alpha) D_\lambda^n F(z) + D_\lambda^{n+1} F(z)\right| \\
&\quad - \left|(1+\alpha) D_\lambda^n F(z) - D_\lambda^{n+1} F(z)\right| \\
&\geq 2(1-\alpha)|z| \\
&\quad - 2\sum_{j=2}^{\infty} \left[1 + (j-1)\lambda - \alpha\right] \left[1 + (j-1)\lambda\right]^n \left|a_{j,p}\right| |z|^j \\
&\quad - 2\sum_{j=1}^{\infty} \left[1 + (j-1)\lambda - \alpha\right] \left[1 + (j-1)\lambda\right]^n \left|b_{j,p}\right| |z|^j \\
&\quad - 2\sum_{k=2}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} \left[1 + (j-1)\lambda + 2(k-1)\lambda - \alpha\right] \\
&\quad \times \left[1 + (j-1)\lambda + 2(k-1)\lambda\right]^n \\
&\quad \times \left[\left|a_{j,p-k+1}\right| + \left|b_{j,p-k+1}\right|\right] |z|^j \\
&> 2(1-\alpha)|z| \left[2 - \sum_{k=1}^{p}\sum_{j=1}^{\infty} \Big(\left[1 + (j-1)\lambda + 2(k-1)\lambda - \alpha\right] \right. \\
&\quad \times \left[1 + (j-1)\lambda + 2(k-1)\lambda\right]^n \Big) \\
&\quad \left. \times (1-\alpha)^{-1} \left[\left|a_{j,p-k+1}\right| + \left|b_{j,p-k+1}\right|\right] \right].
\end{aligned} \tag{19}
$$

This last expression is nonnegative by (15), and so the proof is complete. □

Theorem 2. *Let F be given by* (13) *and* (14). *Then* $F \in \overline{SH}_p(n, \lambda, \alpha)$ *if and only if*

$$
\begin{aligned}
&\sum_{k=1}^{p}\sum_{j=1}^{\infty} \left[1 + (j-1)\lambda + 2(k-1)\lambda - \alpha\right] \\
&\quad \times \left[1 + (j-1)\lambda + 2(k-1)\lambda\right]^n \\
&\quad \times \left[\left|a_{j,p-k+1}\right| + \left|b_{j,p-k+1}\right|\right] \\
&\leq 2(1-\alpha),
\end{aligned} \tag{20}
$$

where $\lambda \geq 1$, $n \in \mathbb{N}$, $0 \leq \alpha < 1$, *and* $a_{1,p} = 1$.

Proof. The "if" part follows from Theorem 1 upon noting that $\overline{SH}_p(n, \lambda, \alpha) \subset SH_p(n, \lambda, \alpha)$. For the "only if" part, we show that $F \notin \overline{SH}_p(n, \lambda, \alpha)$ if the condition (20) does not hold.

Note that a necessary and sufficient condition for F given by (13) and (14) to be in $\overline{SH}_p(n,\lambda,\alpha)$ is that the condition (12) should be satisfied.

This is equivalent to $\operatorname{Re}\{A(z)/B(z)\} \geq 0$, where

$$\begin{aligned}
A(z) &= (1-\alpha) z \\
&\quad - \sum_{j=2}^{\infty} (1+(j-1)\lambda-\alpha)[1+(j-1)\lambda]^n |a_{j,p}| z^j \\
&\quad - \sum_{j=1}^{\infty} (1+(j-1)\lambda-\alpha)[1+(j-1)\lambda]^n |b_{j,p}| \overline{z}^j \\
&\quad - \sum_{k=2}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} [1+(j-1)\lambda+2(k-1)\lambda-\alpha] \\
&\quad\quad \times [1+(j-1)\lambda+2(k-1)\lambda]^n \\
&\quad\quad \times [|a_{j,p-k+1}| z^j + |b_{j,p-k+1}| \overline{z}^j], \\
B(z) &= z - \sum_{j=2}^{\infty} [1+(j-1)\lambda]^n |a_{j,p}| z^j \\
&\quad - \sum_{j=1}^{\infty} [1+(j-1)\lambda]^n |b_{j,p}| \overline{z}^j \\
&\quad - \sum_{k=2}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} [1+(j-1)\lambda+2(k-1)\lambda]^n \\
&\quad\quad \times [|a_{j,p-k+1}| z^j + |b_{j,p-k+1}| \overline{z}^j].
\end{aligned} \tag{21}$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis, where $0 \leq z = r < 1$ we must have

$$\begin{aligned}
&\Bigg((1-\alpha) - \sum_{j=2}^{\infty}(1+(j-1)\lambda-\alpha)[1+(j-1)\lambda]^n |a_{j,p}| r^{j-1} \\
&\quad - \sum_{j=1}^{\infty} (1+(j-1)\lambda-\alpha)[1+(j-1)\lambda]^n |b_{j,p}| r^{j-1}\Bigg) \\
&\times \Bigg(1 - \sum_{j=2}^{\infty}[1+(j-1)\lambda]^n |a_{j,p}| - \sum_{j=1}^{\infty}[1+(j-1)\lambda]^n |b_{j,p}| \\
&\quad - \sum_{k=2}^{p}\sum_{j=1}^{\infty} [1+(j-1)\lambda+2(k-1)\lambda]^n \\
&\quad \times [|a_{j,p-k+1}| + |b_{j,p-k+1}|]\Bigg)^{-1} \\
&- \Bigg(\sum_{k=2}^{p}\sum_{j=1}^{\infty} [1+(j-1)\lambda+2(k-1)\lambda-\alpha] \\
&\quad \times [1+(j-1)\lambda+2(k-1)\lambda]^n \\
&\quad \times [|a_{j,p-k+1}| + |b_{j,p-k+1}|] r^{j+2k-3}\Bigg) \\
&\times \Bigg(1 - \sum_{j=2}^{\infty} [1+(j-1)\lambda]^n |a_{j,p}| \\
&\quad - \sum_{j=1}^{\infty} [1+(j-1)\lambda]^n |b_{j,p}| \\
&\quad - \sum_{k=2}^{p}\sum_{j=1}^{\infty} [1+(j-1)\lambda+2(k-1)\lambda]^n \\
&\quad \times [|a_{j,p-k+1}| + |b_{j,p-k+1}|]\Bigg)^{-1} \\
&\geq 0.
\end{aligned} \tag{22}$$

If the condition (20) does not hold, then the numerator in (22) is negative for r is sufficiently close to 1. Hence there exist $z_0 = r_0$ in (0, 1) for which the quotient in (22) is negative. This contradicts the required condition for $F \in \overline{SH}_p(n,\lambda,\alpha)$ and so the proof is complete. □

Theorem 3. *Let F be given by* (13) *and* (14). *Then $F \in \overline{SH}_p(n,\lambda,\alpha)$ if and only if*

$$F(z) = \sum_{k=1}^{p}\sum_{j=1}^{\infty} \left(X_{j,p-k+1} h_{j,p-k+1}(z) + Y_{j,p-k+1} g_{j,p-k+1}(z)\right), \tag{23}$$

where

$$\begin{aligned}
h_{1,p}(z) &= z, \\
h_{j,p}(z) &= z - \frac{2(1-\alpha)}{(1+(j-1)\lambda-\alpha)[1+(j-1)\lambda]^n} z^j, \quad (j \geq 2), \\
g_{j,p}(z) &= z - \frac{2(1-\alpha)}{(1+(j-1)\lambda-\alpha)[1+(j-1)\lambda]^n} \overline{z}^j, \quad (j \geq 1), \\
h_{j,p-k+1}(z) &= z - |z|^{2(k-1)} (2(1-\alpha)) \\
&\quad \times \big([1+(j-1)\lambda+2(k-1)\lambda-\alpha]
\end{aligned}$$

$$\times\left[1+(j-1)\lambda+2(k-1)\lambda\right]^{n}\Big)^{-1}z^{j},\quad (j\geq 1,\ 2\leq k\leq p),$$

$$g_{j,p-k+1}(z) = z-|z|^{2(k-1)}(2(1-\alpha))\times\Big(\left[1+(j-1)\lambda+2(k-1)\lambda-\alpha\right]\times\left[1+(j-1)\lambda+2(k-1)\lambda\right]^{n}\Big)^{-1}\overline{z}^{j}\quad (j\geq 1,\ 2\leq k\leq p), \tag{24}$$

and $\sum_{k=1}^{p}\sum_{j=1}^{\infty}(X_{j,p-k+1}+Y_{j,p-k+1})=1$, $X_{j,p-k+1}\geq 0$, $Y_{j,p-k+1}\geq 0$.

In particular, the extreme points of $\overline{SH}_p(n,\lambda,\alpha)$ *are* $\{h_{j,p-k+1}(z)\}$ *and* $\{g_{j,p-k+1}(z)\}$, *where* $j\geq 1$ *and* $1\leq k\leq p$.

Proof. For functions F of the form (13) and (14) we have

$$\begin{aligned}F(z) &= \sum_{k=1}^{p}\sum_{j=1}^{\infty}\left(X_{j,p-k+1}h_{j,p-k+1}(z)+Y_{j,p-k+1}g_{j,p-k+1}(z)\right)\\ &= z-\sum_{j=2}^{\infty}(2(1-\alpha))\left((1+(j-1)\lambda-\alpha)\left[1+(j-1)\lambda\right]^{n}\right)^{-1}\times X_{j,p}z^{j}\\ &\quad-\sum_{j=1}^{\infty}(2(1-\alpha))\left((1+(j-1)\lambda-\alpha)\left[1+(j-1)\lambda\right]^{n}\right)^{-1}\times Y_{j,p}\overline{z}^{j}\\ &\quad-\sum_{k=2}^{p}|z|^{2(k-1)}\times\sum_{j=1}^{\infty}(2(1-\alpha))\Big(\left[1+(j-1)\lambda+2(k-1)\lambda-\alpha\right]\times\left[1+(j-1)\lambda+2(k-1)\lambda\right]^{n}\Big)^{-1}\\ &\quad\times\left[X_{j,p-k+1}z^{j}+Y_{j,p-k+1}\overline{z}^{j}\right].\end{aligned} \tag{25}$$

Then

$$\begin{aligned}&\sum_{j=2}^{\infty}\frac{(1+(j-1)\lambda-\alpha)\left[1+(j-1)\lambda\right]^{n}}{2(1-\alpha)}\times\left(\frac{2(1-\alpha)}{(1+(j-1)\lambda-\alpha)\left[1+(j-1)\lambda\right]^{n}}X_{j,p}\right)\\ &+\sum_{j=1}^{\infty}\frac{(1+(j-1)\lambda-\alpha)\left[1+(j-1)\lambda\right]^{n}}{2(1-\alpha)}\times\left(\frac{2(1-\alpha)}{(1+(j-1)\lambda-\alpha)\left[1+(j-1)\lambda\right]^{n}}Y_{j,p}\right)\\ &+\sum_{k=2}^{p}\sum_{j=1}^{\infty}\Big(\left[1+(j-1)\lambda+2(k-1)\lambda-\alpha\right]\times\left[1+(j-1)\lambda+2(k-1)\lambda\right]^{n}\Big)(2(1-\alpha))^{-1}\\ &\quad\times\Big((2(1-\alpha))\Big(\left[1+(j-1)\lambda+2(k-1)\lambda-\alpha\right]\times\left[1+(j-1)\lambda+2(k-1)\lambda\right]^{n}\Big)^{-1}\times\left[X_{j,p-k+1}+Y_{j,p-k+1}\right]\Big)\\ &=\sum_{j=2}^{\infty}X_{j,p}+\sum_{j=1}^{\infty}Y_{j,p}+\sum_{k=2}^{p}\sum_{j=1}^{\infty}\left[X_{j,p-k+1}+Y_{j,p-k+1}\right]=1-X_{1,p}\leq 1,\end{aligned} \tag{26}$$

and so $F\in\overline{SH}_p(n,\lambda,\alpha)$. Conversely, if $F\in\overline{SH}_p(n,\lambda,\alpha)$, then

$$\begin{aligned}\left|a_{j,p}\right| &\leq\frac{2(1-\alpha)}{(1+(j-1)\lambda-\alpha)\left[1+(j-1)\lambda\right]^{n}},\quad (j\geq 2),\\ \left|a_{j,p-k+1}\right| &\leq(2(1-\alpha))\Big(\left[1+(j-1)\lambda+2(k-1)\lambda-\alpha\right]\times\left[1+(j-1)\lambda+2(k-1)\lambda\right]^{n}\Big)^{-1},\quad (j\geq 1,\ 2\leq k\leq p),\\ \left|b_{j,p}\right| &\leq\frac{2(1-\alpha)}{(1+(j-1)\lambda-\alpha)\left[1+(j-1)\lambda\right]^{n}},\quad (j\geq 2),\\ \left|b_{j,p-k+1}\right| &\leq(2(1-\alpha))\Big(\left[1+(j-1)\lambda+2(k-1)\lambda-\alpha\right]\times\left[1+(j-1)\lambda+2(k-1)\lambda\right]^{n}\Big)^{-1},\quad (j\geq 1,\ 2\leq k\leq p).\end{aligned} \tag{27}$$

Set

$$X_{j,p} = \left((1+(j-1)\lambda-\alpha) \times [1+(j-1)\lambda]^n\right)(2(1-\alpha))^{-1}|a_{j,p}|, \quad (j \geq 2),$$

$$Y_{j,p} = \left((1+(j-1)\lambda-\alpha)[1+(j-1)\lambda]^n\right) \times (2(1-\alpha))^{-1}|b_{j,p}|, \quad (j \geq 1),$$

$$X_{j,p-k+1} = \left([1+(j-1)\lambda+2(k-1)\lambda-\alpha] \times [1+(j-1)\lambda+2(k-1)\lambda]^n\right) \times (2(1-\alpha))^{-1}|a_{j,p-k+1}|, \quad (j \geq 1,\ 2 \leq k \leq p),$$

$$Y_{j,p-k+1} = \left([1+(j-1)\lambda+2(k-1)\lambda-\alpha] \times [1+(j-1)\lambda+2(k-1)\lambda]^n\right) \times (2(1-\alpha))^{-1}|b_{j,p-k+1}|, \quad (j \geq 1,\ 2 \leq k \leq p),$$

$$X_{1,p} = 1-\sum_{j=2}^{\infty} X_{j,p}-\sum_{j=1}^{\infty} Y_{j,p} - \sum_{k=2}^{p}\sum_{j=1}^{\infty}\left[X_{j,p-k+1}+Y_{j,p-k+1}\right], \tag{28}$$

where $X_{1,p} \geq 0$. Then, as required, we obtain

$$F(z) = \sum_{k=1}^{p}\sum_{j=1}^{\infty}\left(X_{j,p-k+1}h_{j,p-k+1}(z)+Y_{j,p-k+1}g_{j,p-k+1}(z)\right). \tag{29}$$

□

Theorem 4. *Let $F \in \overline{SH}_p(n,\lambda,\alpha)$. Then for $|z| = r < 1$ we have*

$$|F(z)| \leq \left(1+|b_{1,p}|+\sum_{k=2}^{p}\left(|a_{1,p-k+1}|+|b_{1,p-k+1}|\right)\right)r + \left(\frac{2(1-\alpha)}{(1+\lambda-\alpha)[1+\lambda]^n} - \sum_{k=1}^{p}\frac{[1+2(k-1)\lambda-\alpha][1+2(k-1)\lambda]^n}{(1+\lambda-\alpha)[1+\lambda]^n} \times \left[|a_{1,p-k+1}|+|b_{1,p-k+1}|\right]\right)r^2,$$

$$|F(z)| \geq \left(1-|b_{1,p}|-\sum_{k=2}^{p}\left(|a_{1,p-k+1}|+|b_{1,p-k+1}|\right)\right)r - \left(\frac{2(1-\alpha)}{(1+\lambda-\alpha)[1+\lambda]^n} - \sum_{k=1}^{p}\frac{[1+2(k-1)\lambda-\alpha][1+2(k-1)\lambda]^n}{(1+\lambda-\alpha)[1+\lambda]^n} \times \left[|a_{1,p-k+1}|+|b_{1,p-k+1}|\right]\right)r^2. \tag{30}$$

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Let $F \in \overline{SH}_p(n,\lambda,\alpha)$. Taking the absolute value of F we have

$$\begin{aligned}
|F(z)| &\leq \left(\sum_{k=1}^{p}\left(|a_{1,p-k+1}|+|b_{1,p-k+1}|\right)\right)r + \left(\sum_{k=1}^{p}\sum_{j=2}^{\infty}\left(|a_{j,p-k+1}|+|b_{j,p-k+1}|\right)\right)r^2 \\
&\leq \left(\sum_{k=1}^{p}\left(|a_{1,p-k+1}|+|b_{1,p-k+1}|\right)\right)r + \frac{2(1-\alpha)r^2}{(1+\lambda-\alpha)[1+\lambda]^n} \\
&\quad \times \left(\sum_{k=1}^{p}\sum_{j=2}^{\infty}\left([1+(j-1)\lambda+2(k-1)\lambda-\alpha] \times [1+(j-1)\lambda+2(k-1)\lambda]^n\right) \times (2(1-\alpha))^{-1}\left[|a_{j,p-k+1}|+|b_{j,p-k+1}|\right]\right) \\
&\leq \left(1+|b_{1,p}|+\sum_{k=2}^{p}\left(|a_{1,p-k+1}|+|b_{1,p-k+1}|\right)\right)r + \frac{2(1-\alpha)r^2}{(1+\lambda-\alpha)[1+\lambda]^n} \\
&\quad \times \left(1-\sum_{k=1}^{p}\frac{[1+2(k-1)\lambda-\alpha][1+2(k-1)\lambda]^n}{2(1-\alpha)} \times \left[|a_{1,p-k+1}|+|b_{1,p-k+1}|\right]\right).
\end{aligned} \tag{31}$$

□

The following covering result follows from the left-hand inequality in Theorem 4.

Corollary 5. *Let F of the form* (13) *and* (14) *be so that* $F \in \overline{SH}_p(n, \lambda, \alpha)$. *Then*

$$\begin{aligned}\Big\{ w : |w| < {} & \frac{(1+\lambda-\alpha)[1+\lambda]^n-(1-\alpha)}{(1+\lambda-\alpha)[1+\lambda]^n} \\ & + \frac{(1-\alpha)-(1+\lambda-\alpha)[1+\lambda]^n}{(1+\lambda-\alpha)[1+\lambda]^n}\left|b_{1,p}\right| \\ & + \sum_{k=2}^{p}\Big([1+2(k-1)\lambda-\alpha][1+2(k-1)\lambda]^n \\ & \quad -(1+\lambda-\alpha)[1+\lambda]^n\Big)\left((1+\lambda-\alpha)[1+\lambda]^n\right)^{-1} \\ & \quad \times\left(\left|a_{1,p-k+1}\right|+\left|b_{1,p-k+1}\right|\right)\Big\} \subset F(U).\end{aligned} \tag{32}$$

Theorem 6. *The class $\overline{SH}_p(n, \lambda, \alpha)$ is closed under convex combinations.*

Proof. Let $F_i \in \overline{SH}_p(n, \lambda, \alpha)$ for $i = 1, 2, \ldots$, where F_i is given by

$$\begin{aligned}F_i(z) = {} & z - \sum_{j=2}^{\infty}\left|a_{ij,p}\right| z^j - \sum_{j=1}^{\infty}\left|b_{ij,p}\right| \overline{z}^j \\ & - \sum_{k=2}^{p}|z|^{2(k-1)}\sum_{j=1}^{\infty}\left[\left|a_{ij,p-k+1}\right| z^j + \left|b_{ij,p-k+1}\right| \overline{z}^j\right].\end{aligned} \tag{33}$$

Then by (20),

$$\begin{aligned}\sum_{k=1}^{p}\sum_{j=1}^{\infty} & \Big([1+(j-1)\lambda+2(k-1)\lambda-\alpha] \\ & \times[1+(j-1)\lambda+2(k-1)\lambda]^n\Big) \\ & \times(2(1-\alpha))^{-1}\left[\left|a_{ij,p-k+1}\right|+\left|b_{ij,p-k+1}\right|\right] \le 1.\end{aligned} \tag{34}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \le t_i \le 1$, the convex combination of F_i may be written as

$$\begin{aligned}\sum_{i=1}^{\infty} t_i F_i(z) & = z - \sum_{j=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i\left[\left|a_{ij,p}\right| z^j + \left|b_{ij,p}\right| \overline{z}^j\right]\right) \\ & - \sum_{k=2}^{p}|z|^{2(k-1)}\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i\left[\left|a_{ij,p-k+1}\right| z^j + \left|b_{ij,p-k+1}\right| \overline{z}^j\right]\right).\end{aligned} \tag{35}$$

Then by (34),

$$\begin{aligned}\sum_{k=1}^{p}\sum_{j=1}^{\infty} & \Big([1+(j-1)\lambda+2(k-1)\lambda-\alpha] \\ & \times[1+(j-1)\lambda+2(k-1)\lambda]^n\Big)(2(1-\alpha))^{-1} \\ & \times\left(\sum_{i=1}^{\infty} t_i\left[\left|a_{ij,p-k+1}\right|+\left|b_{ij,p-k+1}\right|\right]\right) \\ & = \sum_{i=1}^{\infty} t_i\Bigg[\sum_{k=1}^{p}\sum_{j=1}^{\infty}\Big([1+(j-1)\lambda+2(k-1)\lambda-\alpha] \\ & \qquad \times[1+(j-1)\lambda+2(k-1)\lambda]^n\Big) \\ & \qquad \times(2(1-\alpha))^{-1}\left[\left|a_{ij,p-k+1}\right|+\left|b_{ij,p-k+1}\right|\right]\Bigg] \\ & \le \sum_{i=1}^{\infty} t_i = 1.\end{aligned} \tag{36}$$

This is the condition required by (20) and so $\sum_{i=1}^{\infty} t_i F_i(z) \in \overline{SH}_p(n, \lambda, \alpha)$. □

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