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Some fixed-point results on (generalized) Bruck-Reilly ∗-extensions of monoids

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Abstract

In this paper, we determine necessary and sufficient conditions for Bruck-Reilly and generalized Bruck-Reilly ∗-extensions of arbitrary monoids to be *regular, coregular* and *strongly π-inverse*. These *semigroup classes* have applications in various field of mathematics, such as matrix theory, discrete mathematics and *p*-adic analysis (especially in operator theory). In addition, while regularity and coregularity have so many applications in the meaning of boundaries (again in operator theory), inverse monoids and Bruck-Reilly extensions contain a mixture fixed-point results of algebra, topology and geometry within the purposes of this journal.
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1 Introduction and preliminaries

In combinatorial group and semigroup theory, for a finitely generated semigroup (monoid), a fundamental question is to find its presentation with respect to some (irreducible) system of generators and relators, and then classify it with respect to semigroup classes. In this sense, in [1], the authors obtained a presentation for the *Bruck-Reilly extension*, which was studied previously by Bruck [2], Munn [3] and Reilly [4]. In different manners, this extension has been considered as a fundamental construction in the theory of semigroups. In detail, many classes of regular semigroups are characterized by Bruck-Reilly extensions; for instance, any bisimple regular *w*-semigroup is isomorphic to a Reilly extension of a group [4] and any simple regular *w*-semigroup is isomorphic to a Bruck-Reilly extension of a finite chain of groups [5, 6]. After that, in another important paper [7], the author obtained a new monoid, namely the *generalized Bruck-Reilly ∗-extension*, and presented the structure of the ∗-bisimple type *A w*-semigroup. Later on, in [8], the authors studied the structure theorem of the ∗-bisimple type *A w^2*-semigroups as the generalized Bruck-Reilly ∗-extension. Moreover, in a joint work [9], it has been recently defined a presentation for the generalized Bruck-Reilly ∗-extension and then obtained a Gröbner-Shirshov basis of this new construction. As we depicted in the abstract of this paper, Bruck-Reilly, its general version generalized Bruck-Reilly ∗-extension of monoids and semigroup classes are not only important in combinatorial algebra but also in linear algebra, discrete mathematics and topology. So these semigroup classes, regular, coregular, inverse and strongly π-inverse, are the most studied classes in algebra.

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In this paper, as a next step of these above results, we investigate *regularity, coregularity* and *strongly π-inverse* properties over Bruck-Reilly and generalized Bruck-Reilly ∗-extensions of monoids. We recall that regularity and strongly π-inverse properties have been already studied for some other special extensions (semidirect and wreath products) of monoids [10, 11]. We further recall that these two important properties have been also investigated for the semidirect product version of Schützenberger products of any two monoids [12, 13]. However, there are not yet such investigations concerning coregularity. As we depicted in the abstract, *semigroup classes* have important applications in various fields of mathematics, such as matrix theory, discrete mathematics and p-adic analysis (especially in operator theory). In addition, while regularity and coregularity have so many applications in the meaning of boundaries (again in operator theory), inverse monoids and Bruck-Reilly extensions contain a mixture of algebra, topology and geometry within the purposes of this journal.

Now let us present the following fundamental material that will be needed in this paper. We refer the reader to [14–16] for more detailed knowledge.

An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $axa = a$. The semigroup S is called *regular* if all its elements are regular. Groups are of course regular semigroups, but the class of regular semigroups is vastly more extensive than the class of groups (see [16]). Further, to have an inverse element can also be important in a semigroup. Therefore, we call S is an *inverse semigroup* if every element has exactly one inverse. The well-known examples of inverse semigroups are groups and semilattices. An element $a \in S$ is called *coregular* and b its *coinverse* if $a = aba = bab$. A semigroup S is said to be *coregular* if each element of S is coregular [17]. In addition, let $E(S)$ and Reg S be the set of *idempotent* and *regular* elements, respectively. We then say that S is called π-*regular* if, for every $s \in S$, there is an $m \in \mathbb{N}$ such that $s^m \in$ Reg S. Moreover, if S is π-regular and the set $E(S)$ is a commutative subsemigroup of S, then S is called *strongly π-inverse semigroup* [16]. We recall that Reg S is an inverse subsemigroup of a strongly π-inverse semigroup S.

2 Bruck-Reilly extensions of monoids

Let us suppose that A is a monoid with an endomorphism θ defined on it such that $A\theta$ is in the $\mathcal{H}$-class [16] of the identity 1_A of A. Also, let $\mathbb{N}^0$ denotes the set of nonnegative integers. Hence, the set $\mathbb{N}^0 \times A \times \mathbb{N}^0$ with the multiplication

$$(m, a, n)(m', a', n') = (m - n + t, (a\theta^{t-n})(a'\theta^{t-m'}), n' - m' + t),$$

where $t = \max(n, m')$ and θ^0 is the identity map on A, forms a monoid with identity $(0, 1_A, 0)$. Then this monoid is called the *Bruck-Reilly extension* of A determined by θ [2–4] and denoted by $BR(A, \theta)$.

In the above references, the authors used $BR(A, \theta)$ to prove that every semigroup embeds in a simple monoid, and to characterize special classes of inverse semigroups. In [3, Theorem 3.1], Munn showed that $BR(A, \theta)$ is an inverse semigroup if and only if A is inverse. So, the following result is a direct consequence of this theorem.

Corollary 1 *Let A be an arbitrary monoid. Then $BR(A, \theta)$ is regular if and only if A is regular.*

For $m, m' \in \mathbb{N}^0$ and $a, a' \in A$, since

$$(m, a, m)(m', a', m') = (m - m + t, (a\theta^{t-m})(a'\theta^{t-m'}), m' - m' + t) = (t, b, t)$$

with $t = \max(m, m')$ and $b = (a\theta^{t-m})(a'\theta^{t-m'}) \in A$, the set $\{(m, a, m) | a \in A, m \in \mathbb{N}^0\}$ becomes a subsemigroup of $BR(A, \theta)$. Thus, we further have the following lemma.

Lemma 1 *Let $(m, a, n) \in BR(A, \theta)$. If (m, a, n) is coregular then $m = n$.*

Proof Let $(m, a, n) \in BR(A, \theta)$. Then there exists $(m', a', n') \in BR(A, \theta)$ such that

$$(m, a, n)(m', a', n')(m, a, n) = (m, a, n) \quad \text{and} \quad (m', a', n')(m, a, n)(m', a', n') = (m, a, n).$$

We have

$$(m, a, n)(m', a', n')(m, a, n) = (m - n - n' + m' + s', b, n - m + s')$$

for some $b \in A$, where $s = \max(n, m')$ and $s' = \max(n' - m' + s, m)$. This implies $m = m - n - n' + m' + s'$ and $n = n - m + s'$, in other words $m + m' = n + n'$. Further, for some $c \in A$, we have

$$(m', a', n')(m, a, n)(m', a', n') = (m' - n' - n + m + S', c, n' - m' + S'),$$

where $S = \max(n', m)$ and $S' = \max(n - m + S, m')$. This gives $m = m' - n' - n + m + S'$, $n = n' - m' + S'$, and consequently, $n' = m'$. Together with $m + m' = n + n'$, we obtain $m = n$ as required. □

Lemma 1 shows that a coregular element in $BR(A, \theta)$ and its coinverse belongs to

$$\{(m, a, m) | a \in A, m \in \mathbb{N}^0\}.$$

Now we can present the following result.

Theorem 1 *Let A be a monoid. Then $A' = \{(m, a, m) | a \in A, m \in \mathbb{N}^0\} \leq BR(A, \theta)$ is coregular if and only if A is coregular.*

Proof Assume that $A' \leq BR(A, \theta)$ is a coregular monoid. For $(0, a, 0) \in BR(A, \theta)$, there exists $(m', a', m') \in BR(A, \theta)$ such that

$$(0, a, 0)(m', a', m')(0, a, 0) = (m', (a\theta^{m'})a'(a\theta^{m'}), m') = (0, a, 0) \tag{1}$$

and

$$(m', a', m')(0, a, 0)(m', a', m') = (m', a'(a\theta^{m'})a', m') = (0, a, 0). \tag{2}$$

By (1) and (2), we clearly have $m' = 0$, and hence, $aa'a = a$ and $a'aa' = a$. So A is coregular.

Conversely, let $(m, a, m) \in BR(A, \theta)$. Then there is an $a' \in A$ with $aa'a = a$ and $a'aa' = a$. Thus, for $(m, a', m) \in BR(A, \theta)$, we get

$$(m, a, m)(m, a', m)(m, a, m) = (m, aa'a, m) = (m, a, m)$$

and

$$(m, a', m)(m, a, m)(m, a', m) = (m, a'aa', m) = (m, a, m).$$

Therefore, $A' = \{(m, a, m) | a \in A, m \in \mathbb{N}^0\} \leq BR(A, \theta)$ is coregular. □

In [3, Theorem 3.1], it is proved that:

- (m, a, n) is an idempotent element in $BR(A, \theta)$ if and only if $m = n$ and a is an idempotent element in A.

This result will be used in the proof of the following theorem.

Theorem 2 *$BR(A, \theta)$ is strongly π-inverse if and only if A is regular and the idempotents in A commute.*

Proof Let $BR(A, \theta)$ be strongly π-inverse, and let $a \in A$. Also let us consider the element $(0, a, 1)$ in $BR(A, \theta)$. Then there exists an element $r \in \mathbb{N}$ with $(0, a, 1)^r \in \operatorname{Reg} BR(A, \theta)$. It is actually a routine matter to show that $(0, a, 1)^r = (0, a(a\theta)^{r-1}, r)$. Moreover, there exists an element $a' \in A$ such that $(r, a', 0)$ is an inverse of $(0, a(a\theta)^{r-1}, r)$ (see [3]). Therefore,

$$(0, a(a\theta)^{r-1}, r) = (0, a(a\theta)^{r-1}, r)(r, a', 0)(0, a(a\theta)^{r-1}, r)$$
$$= (0, a(a\theta)^{r-1}a', r)(0, a(a\theta)^{r-1}, r) = (0, a(a\theta)^{r-1}a'a(a\theta)^{r-1}, r).$$

This shows that

$$a(a\theta)^{r-1} = a(a\theta)^{r-1}a'a(a\theta)^{r-1}. \tag{3}$$

By the assumption given in the beginning of this section, since $a\theta$ is in the $\mathcal{H}$-class of the element 1_A, we obtain $a\theta$ is a group element, and so there is an inverse element $((a\theta)^{r-1})^{-1}$. Thus, by (3), we get $a = a(a\theta)^{r-1}a'a$; in other words, $a \in \operatorname{Reg} A$. Consequently, A is regular. Now, let us also show that the elements in $E(A)$ are commutative. But this is quite clear by the fact that the idempotents in $BR(A, \theta)$ commute if and only if the idempotents in A commute (see [3, Theorem 3.1(5)]).

Conversely, let us suppose that A is regular and the idempotents in A commute. Then $BR(A, \theta)$ is regular, where π-regular by Corollary 1. Moreover, again by [3, Theorem 3.1(5)], $E(BR(A, \theta))$ is a commutative subsemigroup, which is required to $BR(A, \theta)$ satisfy strongly π-inverse property, hence the result. □

3 The generalized Bruck-Reilly ∗-extension of monoids

Suppose that A is an arbitrary monoid having H_1^* and H_1 as the $\mathcal{H}^*$- and $\mathcal{H}$- classes containing the identity element 1_A of A. Moreover, let us assume that β and γ are morphisms

from A into H_1^* and, for an element u in H_1, let λ_u be the inner automorphism of H_1^* defined by $x \mapsto uxu^{-1}$ such that $\gamma \lambda_u = \beta \gamma$.

Now one can consider the set $S = \mathbb{N}^0 \times \mathbb{N}^0 \times A \times \mathbb{N}^0 \times \mathbb{N}^0$ into a semigroup with a multiplication

$$(m,n,v,p,q)\big(m',n',v',p',q'\big)$$
$$= \begin{cases} (m,n-p+d,(v\beta^{d-p})(v'\beta^{d-n'}),p'-n'+d,q'), & \text{if } q = m', \\ (m,n,v(((u^{-n'}(v'\gamma)u^{p'})\gamma^{q-m'-1})\beta^p),p,q'-m'+q), & \text{if } q > m', \\ (m-q+m',n',(((u^{-n}(v\gamma)u^p)\gamma^{m'-q-1})\beta^{n'})v',p',q'), & \text{if } q < m', \end{cases} \tag{4}$$

where $d = \max(p,n')$ and β^0, γ^0 are interpreted as the identity map of A, and also u^0 is interpreted as the identity 1_A of A. In [8], Yu Shung and Li-Min Wang showed that S is a monoid with the identity $(0,0,1_A,0,0)$. In fact, this new monoid $S = \mathbb{N}^0 \times \mathbb{N}^0 \times A \times \mathbb{N}^0 \times \mathbb{N}^0$ is denoted by $GBR^*(A;\beta,\gamma;u)$ and called *generalized Bruck-Reilly $*$-extension* of A determined by the morphisms β, γ and the element u.

The following lemmas were established in [8].

Lemma 2 *If $(m,n,v,p,q) \in GBR^*(A;\beta,\gamma;u)$, then (m,n,v,p,q) is an idempotent if and only if $m = q$, $n = p$ and v is idempotent.*

Lemma 3 *If $(m,n,v,p,q) \in GBR^*(A;\beta,\gamma;u)$, then (m,n,v,p,q) has an inverse*

$$\big(m',n',v',p',q'\big) \in S$$

if and only if v' is an inverse of v in A while $m' = q$, $n' = p$, $p' = n$ and $q' = m$.

Then we have an immediate consequence as in the following.

Corollary 2 *Let A be a monoid. Then $GBR^*(A;\beta,\gamma;u)$ is regular if and only if A is regular.*

In this section, we mainly characterize the properties coregularity and strongly π-inverse over the generalized Bruck-Reilly $*$-extensions of monoids. More specifically, for a given monoid A, we determine the maximal submonoid of $GBR^*(A;\beta,\gamma;u)$, which can be held coregularity if A satisfies particular properties.

Our first observation is the following.

Lemma 4 *The set $\mathcal{L} := \{(m,n,v,n,m)|v \in A, m,n \in \mathbb{N}^0\}$ is a submonoid of $GBR^*(A;\beta,\gamma;u)$.*

Proof By considering the multiplication in (4), the proof can be seen easily. □

It turns out that all coregular elements in $GBR^*(A;\beta,\gamma;u)$ belong to the submonoid $\mathcal{L}$.

Lemma 5 *Let $(m,n,v,p,q) \in GBR^*(A;\beta,\gamma;u)$. If (m,n,v,p,q) is coregular then $m = q$ and $n = p$.*

Proof Let $(m, n, v, p, q) \in GBR^*(A; \beta, \gamma; u)$ be a coregular element. Then there exists an element $(m', n', v', p', q') \in GBR^*(A; \beta, \gamma; u)$ such that

$$(m, n, v, p, q)(m', n', v', p', q')(m, n, v, p, q) = (m, n, v, p, q) \quad \text{and}$$

$$(m', n', v', p', q')(m, n, v, p, q)(m', n', v', p', q') = (m, n, v, p, q).$$

Let $q = m'$ and $q' = m$. Then we have

$$(m, n, v, p, q)(m', n', v', p', q')(m, n, v, p, q)$$
$$= (m, n + n' - p - p' + z', w, p - n + z', q)$$

for some $w \in A$, where $z = \max(p, n')$ and $z' = \max(p' - n' + z, n)$. This implies that

$$n = n + n' - p - p' + z' \tag{5}$$

and

$$p = p - n + z'. \tag{6}$$

By (6), we have $n = z'$. Applying this in (5), we get $n + n' - p - p' + n = n$ and thus $n + n' = p + p'$. Further, we have

$$(m', n', v', p', q')(m, n, v, p, q)(m', n', v', p', q')$$
$$= (m', n' + n - p' - p + Z', w', p' - n' + Z', q')$$

for some $w' \in A$, where $Z = \max(p', n)$ and $Z' = \max(p - n + Z, n')$. This implies that $m = m'$, $q = q'$,

$$n = n' + n - p' - p + Z' \tag{7}$$

and

$$p = p' - n' + Z'. \tag{8}$$

By writing the equality (8) in (7), we get $n' = p'$. Together with $n + n' = p + p'$, we obtain $n = p$. By assuming $q = m'$, we also get $m = q$.

Now let $(x, y, t, z, w)(x', y', t', z', w') = (x'', y'', t'', z'', w'')$. Then it is easy to verify that $x'' \geq x', x$ and $y'' \geq y', y$. If $w \neq x'$, we can easily see that $x'' > x', x$ or $y'' > y', y$. This shows that $(m, n, a, p, q)(m', n', a', p', q')(m, n, a, p, q) \neq (m, n, a, p, q)$ if $q \neq m'$ or $q' \neq m$. Hence, $q \neq m'$ or $q' \neq m$ is not possible. □

Then we have the following result.

Theorem 3 *Let A be a monoid. Then the submonoid $\mathcal{L}$ of $GBR^*(A; \beta, \gamma; u)$ is coregular if and only if A is coregular.*

Proof Suppose that $\mathcal{L} = \{(m, n, v, n, m) | v \in A, m, n \in \mathbb{N}^0\} \leq GBR^*(A; \beta, \gamma; u)$ is coregular. For each $(m', 0, v, 0, m')$ in $\mathcal{L}$, there exists an element $(m', n', v', n', m') \in \mathcal{L}$ such that

$$(m', 0, v, 0, m')(m', n', v', n', m')(m', 0, v, 0, m') = (m', n', (v\beta^{n'})v'(v\beta^{n'}), n', m')$$
$$= (m', 0, v, 0, m') \tag{9}$$

and

$$(m', n', v', n', m')(m', 0, v, 0, m')(m', n', v', n', m') = (m', n', v'(v\beta^{n'})v', n', m')$$
$$= (m', 0, v, 0, m'). \tag{10}$$

By (9) and (10), we obtain $n' = 0$, and hence $vv'v = v$ and $v'vv' = v$. So, A is coregular.

Conversely, let A be a coregular monoid and $(m, n, v, n, m) \in \mathcal{L}$. Then there exists an element $v' \in A$ with $vv'v = v$ and $v'vv' = v$. Therefore, for $(m, n, v', n, m) \in \mathcal{L}$, we get

$$(m, n, v, n, m)(m, n, v', n, m)(m, n, v, n, m) = (m, n, vv'v, n, m) = (m, n, v, n, m),$$
$$(m, n, v', n, m)(m, n, v, n, m)(m, n, v', n, m) = (m, n, v'vv', n, m) = (m, n, v, n, m).$$

Hence, $\mathcal{L} \leq GBR^*(A; \beta, \gamma; u)$ is a coregular monoid, as desired. $\qquad\square$

In the final theorem, we consider strongly π-inverse property.

Theorem 4 *$GBR^*(A; \beta, \gamma; u)$ is strongly π-inverse if and only if A is regular and the idempotents in A commute.*

Proof We will follow the same format as in the proof of Theorem 2. So, let us suppose that $GBR^*(A; \beta, \gamma; u)$ is a strongly π-inverse monoid, and let $a \in A$. Then, for $(0, 0, a, 1, 0) \in GBR^*(A; \beta, \gamma; u)$, there is an element $r \in \mathbb{N}$ with $(0, 0, a, 1, 0)^r \in \text{Reg } GBR^*(A; \beta, \gamma; u)$. It is easily seen that $(0, 0, a, 1, 0)^r = (0, 0, a(a\beta)^{r-1}, r, 0)$. Moreover, there is an element $a' \in A$ such that $(0, r, a', 0, 0)$ is an inverse of $(0, 0, a(a\beta)^{r-1}, r, 0)$ by Lemma 3. From here, we have

$$(0, 0, a(a\beta)^{r-1}, r, 0) = (0, 0, a(a\beta)^{r-1}, r, 0)(0, r, a', 0, 0)(0, 0, a(a\beta)^{r-1}, r, 0)$$
$$= (0, 0, a(a\beta)^{r-1}a'a(a\beta)^{r-1}, r, 0).$$

This actually shows that

$$a(a\beta)^{r-1} = a(a\beta)^{r-1}a'a(a\beta)^{r-1}. \tag{11}$$

At the same time, since $a\beta$ is in the $\mathcal{H}^*$-class of the 1_A, there exists an inverse element $((a\beta)^{r-1})^{-1}$. Thus, by (11), we get $a = a(a\beta)^{r-1}a'a$, in other words, $a \in \text{Reg } A$. Hence, A is regular. Now, let us show that the elements in $E(A)$ are commutative to conclude the necessity part of the proof. To do that, consider any two elements v_1 and v_2 in $E(A)$. Thus, $(0, 0, v_1, 0, 0), (0, 0, v_2, 0, 0) \in E(GBR^*(A; \beta, \gamma; u))$ (by Lemma 2) and we have

$$(0, 0, v_1v_2, 0, 0) = (0, 0, v_1, 0, 0)(0, 0, v_2, 0, 0)$$
$$= (0, 0, v_2, 0, 0)(0, 0, v_1, 0, 0) = (0, 0, v_2v_1, 0, 0).$$

So, $v_1v_2 = v_2v_1$, as required.

Conversely, let us suppose that A is regular. Then $GBR^*(A; \beta, \gamma; u)$ is regular, where π-regular by Corollary 2. Now we need to show that the elements in $E(GBR^*(A; \beta, \gamma; u))$ commute. To do that, let us take $(m, n, e, n, m), (m', n', e', n', m') \in E(GBR^*(A; \beta, \gamma; u))$, and thus $ee' = e'e$ by Lemma 2. Now, by considering the multiplication $(m, n, e, n, m)(m', n', e', n', m')$ as defined in (4), we have the following cases.

Case (i): If $m = m'$, then we get

$$(m, n, e, n, m)(m', n', e', n', m') = (m, d', (e\beta^{d'-n})(e'\beta^{d'-n'}), d', m')$$

and

$$(m', n', e', n', m')(m, n, e, n, m) = (m', d', (e'\beta^{d'-n'})(e\beta^{d'-n}), d', m'),$$

respectively, where $d' = \max(n, n')$. Since $e, e' \in E(A)$, we deduce that both $e'\beta^{d'-n'}$ and $e\beta^{d'-n}$ are the elements of $E(A)$, in other words,

$$(e'\beta^{d'-n'})(e\beta^{d'-n}) = (e\beta^{d'-n})(e'\beta^{d'-n'}).$$

Thus, $(m, n, e, n, m)(m', n', e', n', m') = (m', n', e', n', m')(m, n, e, n, m)$.

Case (ii): If $m < m'$ or $m > m'$, then we get

$$(m, n, e, n, m)(m', n', e', n', m') = (m', n', (((u^{-n}(e\gamma)u^n)\gamma^{m'-m-1})\beta^{n'})e', n', m'),$$

$$(m', n', e', n', m')(m, n, e, n, m) = (m', n', e'(((u^{-n}(e\gamma)u^n)\gamma^{m'-m-1})\beta^{n'}), n', m')$$

or

$$(m, n, e, n, m)(m', n', e', n', m') = (m, n, (((u^{-n'}(e'\gamma)u^{n'})\gamma^{m-m'-1})\beta^n)e', n, m),$$

$$(m', n', e', n', m')(m, n, e, n, m) = (m, n, e'(((u^{-n'}(e'\gamma)u^{n'})\gamma^{m-m'-1})\beta^n), n, m),$$

respectively. Since $((u^{-n}(e\gamma)u^n)\gamma^{m'-m-1})\beta^{n'}, ((u^{-n'}(e'\gamma)u^{n'})\gamma^{m-m'-1})\beta^n \in E(A)$, we clearly obtain $(m, n, e, n, m)(m', n', e', n', m') = (m', n', e', n', m')(m, n, e, n, m)$.

Hence, the result.   □

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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