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q-Dirichlet type L-functions with weight α

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Abstract

The aim of this paper is to construct q-Dirichlet type L-functions with weight α. We give the values of these functions at negative integers. These values are related to the generalized q-Bernoulli numbers with weight α.

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1 Introduction

Recently Kim, Simsek, Yang and also many mathematicians have studied a two-variable Dirichlet L-function.

In this paper, we need the following standard notions: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$, $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, $\mathbb{Z}^- = \{-1, -2, \ldots\}$. Also, as usual $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. We assume that $\ln(z)$ denotes the principal branch of the multi-valued function $\ln(z)$ with the imaginary part $\Im(\ln(z))$ constrained by $-\pi < \Im(\ln(z)) \leq \pi$.

In this paper, we study the two-variable Dirichlet L-function with weight α. We give some properties of this function. We also give explicit values of this function at negative integers which are related to the generalized Bernoulli polynomials and numbers with weight α.

Throughout this presentation, we use the following standard notions: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$, $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, $\mathbb{Z}^- = \{-1, -2, \ldots\}$. Also, as usual $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

Let χ be a primitive Dirichlet character with conductor $f \in \mathbb{N}$. The Dirichlet L-function is defined as follows:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \tag{1}$$

where $s \in \mathbb{C}$ ($\Re(s) > 1$) (see [1–22] and the references cited in each of earlier works). The function $L(s, \chi)$ is analytically continued to the complex s-plane, one has

$$L(1 - n, \chi) = -\frac{B_{n,\chi}}{n}, \tag{2}$$

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where $n \in \mathbb{Z}^+$ and $B_{n,\chi}$ denotes the usual generalized Bernoulli numbers, which are defined by means of the following generating function (see [1–22]):

$$\sum_{a=0}^{f-1} \frac{\chi(a)e^{at}t}{e^{ft}-1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}.$$

2 Two-variable q-Dirichlet L-function with weight α

The following generating functions are given by Kim *et al.* [3] and are related to the generalized Bernoulli polynomials with weight α as follows:

$$F_q^{(\alpha)}(x,t,\chi) = \frac{\alpha t}{[\alpha]_q} \sum_{m=0}^{\infty} q^{\alpha(x+m)} \chi(m) e^{t[x+m]_{q^\alpha}} = \sum_{n=0}^{\infty} \tilde{B}_{n,\chi,q}^{(\alpha)}(x) \frac{t^n}{n!}, \tag{3}$$

where

$$q \in \mathbb{C} \quad \left(\left|q^\alpha\right| < 1\right).$$

Remark 2.1 By substituting $\chi \equiv 1$ into (3), we have

$$F_q^{(\alpha)}(x,t) = \frac{\alpha t}{[\alpha]_q} \sum_{m=0}^{\infty} q^{\alpha(x+m)} e^{t[x+m]_{q^\alpha}} = \sum_{n=0}^{\infty} \tilde{B}_{n,\chi,q}^{(\alpha)}(x) \frac{t^n}{n!},$$

which is defined by Kim [12].

Remark 2.2 By substituting $\alpha = 1$ into (3), we have

$$\lim_{q\to 1} \tilde{B}_{n,\chi,q}^{(\alpha)}(x) = B_{n,\chi}(x),$$

where $B_{n,\chi}(x)$ denotes generalized Bernoulli polynomials attached to Drichlet character χ with conductor $f \in \mathbb{N}$ (see [1–22]).

By applying the derivative operator

$$\left.\frac{\partial^k}{\partial t^k} F_q^{(\alpha)}(x,t)\right|_{t=0}$$

to (3), we obtain

$$\frac{k\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{\alpha(x+m)} \chi(m) [m+x]_{q^\alpha}^{k-1} = \tilde{B}_{k,\chi,q}^{(\alpha)}(x), \tag{4}$$

where

$$\left|q^\alpha\right| < 1.$$

Observe that when $\chi \equiv 1$ in (4), one can obtain recurrence relation for the polynomial $\tilde{B}_{k,q}^{(\alpha)}(x)$.

By using (4), we define a *two-variable* q-Dirichlet L-function with weight α as follows.

Definition 2.3 Let $s, q \in \mathbb{C}$ ($|q^{\alpha}| < 1$). The two-variable q-Dirichlet L-functions with weight α are defined by

$$\tilde{L}_q^{(\alpha)}(s, \chi|x) = \frac{-\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} \frac{q^{\alpha(x+m)}\chi(m)}{[m+x]_{q^{\alpha}}^{s}}. \tag{5}$$

Remark 2.4 Substituting $x = 1$ into (5), then the q-Dirichlet L-functions with weight α are defined by

$$\tilde{L}_q^{(\alpha)}(s, \chi|1) = \frac{-\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} \frac{q^{\alpha(m+1)}\chi(m)}{(1+q^{\alpha}[m])^{s}}.$$

Remark 2.5 By applying the Mellin transformation to (3), Kim *et al.* [12] defined two-variable q-Dirichlet L-functions with weight α as follows: Let $|q| < 1$ and $\Re(s) > 0$, then

$$\tilde{L}_q^{(\alpha)}(s, \chi|x) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} F_q^{(\alpha)}(x, -t)\, dt \quad \left(\min\left\{\Re(s), \Re(x)\right\} > 0\right).$$

For $x = 1$, by using (5), we obtain the following corollary.

Corollary 2.6 *Let $q, s \in \mathbb{C}$. We assume that $\Re(q) < \frac{1}{2}$ and $|q^{\alpha}| < 1$. Then we have*

$$\tilde{L}_q^{(\alpha)}(s, \chi|1) = \frac{-\alpha(1-q^{\alpha})^{s}}{[\alpha]_q} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \binom{n+s-1}{n} \chi(m) q^{\alpha n(m+1)}.$$

Remark 2.7 Substituting $\alpha = 1$ into (5) and then $q \to 1$, we have

$$\begin{aligned} \tilde{L}(s, \chi|x) &= -\sum_{m=0}^{\infty} \frac{\chi(m)}{(m+x)^{s}} \\ &= -L(s, \chi|x), \end{aligned}$$

which gives us a two-variable Dirichlet L-function (see [6, 11, 16, 18–20, 22]). Substituting $x = 1$ into the above equation, one has (2).

Theorem 2.8 *Let $k \in \mathbb{Z}^{+}$. Then we have*

$$\tilde{L}_q^{(\alpha)}(1-k, \chi|x) = -\frac{\tilde{B}_{k,\chi,q}^{(\alpha)}(x)}{k}. \tag{6}$$

Proof By substituting $s = 1-k$ with $k \in \mathbb{Z}^{+}$ into (5), we have

$$\tilde{L}_q^{(\alpha)}(1-k, \chi|x) = \frac{-\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{\alpha(x+m)}\chi(m)[m+x]_{q^{\alpha}}^{k-1}.$$

Combining (4) with the above equation, we arrive at the desired result. □

Remark 2.9 If $q \to 1$, then (6) reduces to (1).

Remark 2.10 Substituting $\chi = 1$ into (5), we modify Kim's *et al.* zeta function as follows (see [12]):

$$-\tilde{\zeta}_q^{(\alpha)}(s,x) = \tilde{L}_q^{(\alpha)}(s,1|x) = \frac{-\alpha}{[\alpha]_q}\sum_{m=1}^{\infty}\frac{q^{\alpha[m+x]}}{[m+x]_{q^\alpha}^s} \quad \bigl(\Re(s) > 1\bigr). \tag{7}$$

This function gives us Hurwitz-type zeta functions with weight α. It is well known that this function interpolates the q-Bernoulli polynomials with weight α at negative integers, which is given by the following lemma.

Lemma 2.11 *Let $n \in \mathbb{Z}^+$. Then we have*

$$\tilde{\zeta}_q^{(\alpha)}(1-n,x) = -\frac{\tilde{B}_{n,q}^{(\alpha)}(x)}{n}. \tag{8}$$

Now we are ready to give relationship between (7) and (5). Substituting $m = a + kn$, where $a = 0,1,\ldots,k$; $n = 0,1,2,\ldots$ into (5), we obtain

$$\begin{aligned}
\tilde{L}_q^{(\alpha)}(s,\chi|x) &= \frac{-\alpha}{[\alpha]_q}\sum_{a=0}^{k}\sum_{n=0}^{\infty}\frac{q^{\alpha(x+a+kn)}\chi(a+kn)}{[a+kn+x]_{q^\alpha}^s} \\
&= \frac{-\alpha}{[\alpha]_q}\sum_{a=0}^{k}q^{\alpha(x+a)}\chi(a)\sum_{n=0}^{\infty}\frac{q^{kn\alpha}}{[k]_{q^\alpha}^s[\frac{a+x}{k}+n]_{q^{\alpha k}}^s} \\
&= \frac{-\alpha}{[\alpha]_q[k]_{q^\alpha}^s}\frac{[\alpha]_{q^{\alpha k}}}{\alpha^{k\alpha}}\sum_{a=0}^{k}q^{\alpha(x+a)}\chi(a)\tilde{\zeta}_{q^{k\alpha}}^{(k\alpha)}\left(s,\frac{a+x}{k}\right).
\end{aligned}$$

Therefore, we have the following theorem.

Theorem 2.12 *The following relation holds true*:

$$\tilde{L}_q^{(\alpha)}(s,\chi|x) = \frac{-\alpha^{1-k\alpha}[\alpha]_{q^{\alpha k}}}{[\alpha]_q[k]_{q^\alpha}^s}\sum_{a=0}^{k}q^{\alpha(x+a)}\chi(a)\tilde{\zeta}_{q^{k\alpha}}^{(k\alpha)}\left(s,\frac{a+x}{k}\right). \tag{9}$$

By substituting $s = 1-n$ with $n \in \mathbb{Z}^+$ into (9) and combining with (8) and (6), we give explicitly a formula of the generalized Bernoulli polynomials with weight α by the following theorem.

Theorem 2.13 *The following formula holds true*:

$$\tilde{B}_{n,\chi,q}^{(\alpha)}(x) = \frac{\alpha^{1-k\alpha}[\alpha]_{q^{\alpha k}}}{[\alpha]_q[k]_{q^\alpha}^{1-n}}\sum_{a=0}^{k}q^{\alpha(x+a)}\chi(a)\tilde{B}_{n,q}^{(\alpha)}\left(\frac{a+x}{k}\right). \tag{10}$$

By using (10), we obtain the following corollary.

Corollary 2.14 *The following formula holds true*:

$$\tilde{B}_{n,\chi,q}^{(\alpha)}(x) = \frac{\alpha^{1-k\alpha}[\alpha]_{q^{\alpha k}}[k]_{q^\alpha}^{n-1}}{[\alpha]_q}\sum_{a=0}^{k}\sum_{j=0}^{n}\binom{n}{j}q^{\alpha(x+a)}\chi(a)\left(\frac{a+x}{k}\right)^{n-j}\tilde{B}_{j,q}^{(\alpha)}.$$

Competing interests
The author declares that she has no competing interests.

Authors' contributions
The author completed the paper herself. The author read and approved the final manuscript.

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