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A note on the (h,q)-zeta-type function with weight α

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Abstract
The objective of this paper is to derive the symmetric property of an (h,q)-zeta function with weight α. By using this property, we give some interesting identities for (h,q)-Genocchi polynomials with weight α. As a result, our applications possess a number of interesting properties which we state in this paper.

MSC: 11S80; 11B68

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1 Introduction

Recently, Kim has developed a new method by using the q-Volkenborn integral (or p-adic q-integral on $\mathbb{Z}_p$) and has added weight to q-Bernoulli numbers and polynomials and investigated their interesting properties (see [1]). He also showed that these polynomials are closely related to weighted q-Bernstein polynomials and derived novel properties of q-Bernoulli numbers with weight α by using the symmetric property of weighted q-Bernstein polynomials with the help of the q-Volkenborn integral (for more details, see [2]). Afterward, Araci *et al.* have introduced weighted (h,q)-Genocchi polynomials and defined (h,q)-zeta-type function with weight α by applying the Mellin transformation to the generating function of the (h,q)-Genocchi polynomials with weight α which interpolates for (h,q)-Genocchi polynomials with weight α at negative integers (for details, see [3]). In this paper, we also consider a (h,q)-zeta-type function with weight α and derive some interesting properties.

We firstly list some notations as follows.

Imagine that p is a fixed odd prime. Throughout this work, $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote by the ring of integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Also, we denote $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ and $\exp(x) = e^x$. Let $v_p : \mathbb{C}_p \to \mathbb{Q} \cup \{\infty\}$ ($\mathbb{Q}$ is the field of rational numbers) denote the p-adic valuation of $\mathbb{C}_p$ normalized so that $v_p(p) = 1$. The absolute value on $\mathbb{C}_p$ will be denoted as $|\cdot|$, and $|x|_p = p^{-v_p(x)}$ for $x \in \mathbb{C}_p$. When one speaks of q-extensions, q is considered in many ways, *e.g.*, as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume $|1-q|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. We use the following notation:

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.1}$$

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We want to note that $\lim_{q\to 1}[x]_q = x$; *cf.* [1–23].

For a fixed positive integer d, set

$$X = X_d = \lim_{\overleftarrow{n}} \mathbb{Z}/dp^n\mathbb{Z},$$

$$X^* = \bigcup_{\substack{0<a<dp\\(a,p)=1}} a + dp\mathbb{Z}_p$$

and

$$a + dp^n\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^n}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^n$ (see [1–23]).

The following p-adic q-Haar distribution was defined by Kim:

$$\mu_q(x + p^n\mathbb{Z}_p) = \frac{q^x}{[p^n]_q}$$

for any positive n (see [12, 13]).

Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable functions on $\mathbb{Z}_p$. We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$ if the difference quotient

$$F_f(x,y) = \frac{f(x) - f(y)}{x - y}$$

has a limit $f'(a)$ as $(x,y) \to (a,a)$ and denote this by $f \in UD(\mathbb{Z}_p)$. In [12] and [13], the p-adic q-integral of the function $f \in UD(\mathbb{Z}_p)$ is defined by Kim as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(\xi)\, d\mu_q(\xi) = \lim_{n\to\infty} \sum_{\xi=0}^{p^n-1} f(\xi)\mu_q(\xi + p^n\mathbb{Z}_p). \tag{1.2}$$

The bosonic integral is considered as the bosonic limit $q \to 1$, $I_1(f) = \lim_{q\to 1} I_q(f)$. Similarly, the p-adic fermionic integration on $\mathbb{Z}_p$ is defined by Kim [8] as follows:

$$I_{-q}(f) = \lim_{q\to -q} I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x). \tag{1.3}$$

By using a fermionic p-adic q-integral on $\mathbb{Z}_p$, (h,q)-Genocchi polynomials are defined by [3]

$$\begin{aligned}\frac{\widetilde{G}^{(\alpha,h)}_{n+1,q}(x)}{n+1} &= \int_{\mathbb{Z}_p} q^{(h-1)\xi}\,[x+\xi]^n_{q^\alpha}\, d\mu_{-q}(\xi)\\ &= \lim_{n\to\infty} \frac{1}{[p^n]_{-q}} \sum_{\xi=0}^{p^n-1} (-1)^\xi\, [x+\xi]^n_{q^\alpha}\, q^{h\xi}.\end{aligned} \tag{1.4}$$

For $x = 0$ in (1.4), we have $\widetilde{G}_{n,q}^{(\alpha,h)}(0) := \widetilde{G}_{n,q}^{(\alpha,h)}$ are called (h,q)-Genocchi numbers with weight α which is defined by

$$\widetilde{G}_{0,q}^{(\alpha,h)} = 0 \quad \text{and} \quad q^h \frac{\widetilde{G}_{m+1}^{(\alpha,h)}(1)}{m+1} + \frac{\widetilde{G}_{m+1}^{(\alpha,h)}}{m+1} = \begin{cases} [2]_q & \text{if } m = 0, \\ 0 & \text{if } m \neq 0. \end{cases}$$

By (1.4), we have a distribution formula for (h,q)-Genocchi polynomials, which is shown by [3]

$$\widetilde{G}_{n+1,q}^{(\alpha,h)}(x) = \frac{[2]_q}{[2]_{q^a}} [a]_{q^\alpha}^n \sum_{j=0}^{a-1} (-1)^j q^{jh} \widetilde{G}_{n+1,q^a}^{(\alpha,h)}\left(\frac{x+j}{a}\right).$$

By applying some elementary methods, we will give symmetric properties of weighted (h,q)-Genocchi polynomials and a weighted (h,q)-zeta-type function. Consequently, our applications seem to be interesting and worthwhile for further works of many mathematicians in analytic numbers theory.

2 On the (h,q)-zeta-type function

In this part, we firstly recall the (h,q)-zeta-type function with weight α which is derived in [3] as follows:

$$\widetilde{\zeta}_q^{(\alpha,h)}(s,x) = [2]_q \sum_{m=0}^{\infty} \frac{(-1)^m q^{mh}}{[m+x]_{q^\alpha}^s}, \tag{2.1}$$

where $q \in \mathbb{C}$, $h \in \mathbb{N}$ and $\Re(s) > 1$. It is clear that the special case $h = 0$ and $q \to 1$ in (2.1) reduces to the ordinary Hurwitz-Euler zeta function. Now, we consider (2.1) in the following form:

$$\widetilde{\zeta}_{q^a}^{(\alpha,h)}\left(s, bx + \frac{bj}{a}\right) = [2]_{q^a} \sum_{m=0}^{\infty} \frac{(-1)^m q^{mah}}{[m + bx + \frac{bj}{a}]_{q^{a\alpha}}^s}.$$

By applying some basic operations to the above identity, that is, for any positive integers m and b, there exist unique non-negative integers k and i such that $m = bk + i$ with $0 \leq i \leq b-1$. For $a \equiv 1 (\operatorname{mod} 2)$ and $b \equiv 1 (\operatorname{mod} 2)$. Thus, we can compute as follows:

$$\begin{aligned} &\widetilde{\zeta}_{q^a}^{(\alpha,h)}\left(s, bx + \frac{bj}{a}\right) \\ &= [a]_{q^\alpha}^s [2]_{q^a} \sum_{m=0}^{\infty} \frac{(-1)^m q^{mah}}{[ma + abx + bj]_{q^{a\alpha}}^s} \\ &= [a]_{q^\alpha}^s [2]_{q^a} \sum_{m=0}^{\infty} \sum_{i=0}^{b-1} \frac{(-1)^{i+mb} q^{(i+mb)ah}}{[(i+mb)a + abx + bj]_{q^{a\alpha}}^s} \\ &= [a]_{q^\alpha}^s [2]_{q^a} \sum_{i=0}^{b-1} (-1)^i q^{iah} \sum_{m=0}^{\infty} \frac{(-1)^m q^{mbah}}{[ab(m+x) + ai + bj]_{q^\alpha}^s}. \end{aligned} \tag{2.2}$$

From this, we can easily discover the following:

$$\begin{aligned}&\sum_{j=0}^{a-1}(-1)^j q^{jbh}\widetilde{\zeta}_{q^a}^{(\alpha,h)}\left(s, bx+\frac{bj}{a}\right)\\&\quad=[a]_{q^\alpha}^s[2]_{q^a}\sum_{j=0}^{a-1}(-1)^j q^{jbh}\sum_{i=0}^{b-1}(-1)^i q^{iah}\sum_{m=0}^{\infty}\frac{(-1)^m q^{mbah}}{[ab(m+x)+ai+bj]_{q^\alpha}^s}.\end{aligned}\tag{2.3}$$

Replacing a by b and j by i in (2.2), we have the following:

$$\widetilde{\zeta}_{q^b}^{(\alpha,h)}\left(s, ax+\frac{ai}{b}\right)=[b]_{q^\alpha}^s[2]_{q^b}\sum_{j=0}^{a-1}(-1)^j q^{jbh}\sum_{m=0}^{\infty}\frac{(-1)^m q^{mbah}}{[ab(m+x)+ai+bj]_{q^\alpha}^s}.$$

By considering the above identity in (2.3), we can easily state the following theorem.

Theorem 1 *The following identity is true*:

$$\frac{[2]_{q^b}}{[a]_{q^\alpha}^s}\sum_{i=0}^{a-1}(-1)^i q^{ibh}\widetilde{\zeta}_{q^a}^{(\alpha,h)}\left(s, bx+\frac{bi}{a}\right)=\frac{[2]_{q^a}}{[b]_{q^\alpha}^s}\sum_{i=0}^{b-1}(-1)^i q^{iah}\widetilde{\zeta}_{q^b}^{(\alpha,h)}\left(s, ax+\frac{ai}{b}\right).$$

Now, setting $b=1$ in Theorem 1, we have the following distribution formula:

$$\widetilde{\zeta}_{q}^{(\alpha,h)}(s, ax)=\frac{[2]_q}{[2]_{q^a}[a]_{q^\alpha}^s}\sum_{i=0}^{a-1}(-1)^i q^{ih}\widetilde{\zeta}_{q^a}^{(\alpha,h)}\left(s, x+\frac{i}{a}\right).\tag{2.4}$$

Putting $a=2$ in (2.4) leads to the following corollary.

Corollary 1 *The following identity holds true*:

$$\widetilde{\zeta}_{q}^{(\alpha,h)}(s, 2x)=\frac{[2]_q}{[2]_{q^2}[2]_{q^\alpha}^s}\left(\widetilde{\zeta}_{q^2}^{(\alpha,h)}(s,x)-q^h\widetilde{\zeta}_{q^2}^{(\alpha,h)}\left(s, x+\frac{1}{2}\right)\right).$$

Taking $s=-m$ into Theorem 1, we have the symmetric property of (h,q)-Genocchi polynomials by the following theorem.

Theorem 2 *The following identity is true*:

$$[2]_{q^b}[a]_{q^\alpha}^{m-1}\sum_{j=0}^{a-1}(-1)^i q^{ibh}\widetilde{G}_{m,q^a}^{(\alpha,h)}\left(bx+\frac{bi}{a}\right)=[2]_{q^a}[b]_{q^\alpha}^{m-1}\sum_{i=0}^{b-1}(-1)^i q^{iah}\widetilde{G}_{m,q^b}^{(\alpha,h)}\left(ax+\frac{ai}{b}\right).$$

Now also, setting $b=1$ and replacing x by $\frac{x}{a}$ in the above theorem, we can rewrite the following (h,q)-Genocchi polynomials with weight α:

$$\widetilde{G}_{n,q}^{(\alpha,h)}(x)=\frac{[2]_q}{[2]_{q^a}}[a]_{q^\alpha}^{n-1}\sum_{i=0}^{a-1}(-1)^i q^{ih}\widetilde{G}_{n,q^a}^{(\alpha,h)}\left(\frac{x+i}{a}\right)\quad(2\nmid a).$$

Due to Araci *et al.* [3], we develop as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{G}_{n,q}^{(\alpha,h)}(x+y)\frac{t^n}{n!} &= [2]_q t\sum_{m=0}^{\infty}(-1)^m q^{mh}e^{t[x+y+m]_{q^\alpha}} \\ &= [2]_q t\sum_{m=0}^{\infty}(-1)^m q^{mh}e^{t[y]_{q^\alpha}}e^{(q^{\alpha y}t)[x+m]_{q^\alpha}} \\ &= \left(\sum_{n=0}^{\infty}[y]_{q^\alpha}^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}q^{\alpha(n-1)y}\widetilde{G}_{n,q}^{(\alpha,h)}(x)\frac{t^n}{n!}\right).\end{aligned}$$

By using the Cauchy product, we see that

$$\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}q^{\alpha(j-1)y}\widetilde{G}_{j,q}^{(\alpha,h)}(x)[y]_{q^\alpha}^{n-j}\right)\frac{t^n}{n!}.$$

Thus, by comparing the coefficients of $\frac{t^n}{n!}$, we state the following corollary.

Corollary 2 *The following equality holds true*:

$$\widetilde{G}_{n,q}^{(\alpha,h)}(x+y)=\sum_{j=0}^{n}\binom{n}{j}q^{\alpha(j-1)y}\widetilde{G}_{j,q}^{(\alpha,h)}(x)[y]_{q^\alpha}^{n-j}. \tag{2.5}$$

By using Theorem 2 and (2.5), we readily derive the following symmetric relation after some applications.

Theorem 3 *The following equality holds true*:

$$\begin{aligned}&[2]_{q^b}\sum_{i=0}^{m}\binom{m}{i}[a]_{q^\alpha}^{i-1}[b]_{q^\alpha}^{m-i}\widetilde{G}_{i,q^a}^{(\alpha,h)}(bx)\widetilde{S}_{m-i:q^b,h+i-1}^{(\alpha)}(a) \\ &\quad= [2]_{q^a}\sum_{i=0}^{m}\binom{m}{i}[b]_{q^\alpha}^{i-1}[a]_{q^\alpha}^{m-i}\widetilde{G}_{i,q^b}^{(\alpha,h)}(ax)\widetilde{S}_{m-i:q^a,h+i-1}^{(\alpha)}(b),\end{aligned}$$

where $\widetilde{S}_{m:q,i}^{(\alpha)}(a)=\sum_{j=0}^{a-1}(-1)^j q^{ji}[j]_{q^\alpha}^m$.

When $q\to 1$ into Theorem 3, it leads to the following corollary.

Corollary 3 *The following identity holds true*:

$$\begin{aligned}&\sum_{i=0}^{m}\binom{m}{i}a^{i-1}b^{m-i}G_i(bx)S_{m-i}(a) \\ &\quad= \sum_{i=0}^{m}\binom{m}{i}b^{i-1}a^{m-i}G_i(ax)S_{m-i}(b),\end{aligned}$$

where $S_m(a) = \sum_{j=0}^{a-1}(-1)^j j^m$ *and* $G_n(x)$ *are called the ordinary Genocchi polynomials which are defined via the following generating function*:

$$\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = \frac{2t}{e^t+1}e^{xt}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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