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GENERALIZED INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. In this paper, we prove some general inequalities for convex functions and give Ostrowski, Hadamard and Simpson type results for a special case of these inequalities.

1. Introduction

The function $f:[a,b]\to\mathbb{R}$, is said to be convex, if we have

$$f(tx+(1-t)y)\leqslant tf(x)+(1-t)f(y)$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. For more information see the papers [3], [1], [2].

Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function and let $a,b\in I$, with $a<b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\,dx\leqslant\frac{f(a)+f(b)}{2}$$

is known in the literature as Hadamard's inequality.

In 1928 Ostrowski proved the following famous inequality:

THEOREM 1. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) and its derivative $f':(a,b)\to\mathbb{R}$ be bounded on (a,b), that is, $\|f'\|_\infty:=\sup_{t\in(a,b)}|f'(x)|<\infty$. Then for any $x\in[a,b]$, the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leqslant\left|\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right|(b-a)\|f'\|_\infty.$$

The inequality is sharp in the sense that the constant $1/4$ cannot be replaced by a smaller one. In the rest of this section we list known results which we will generalize in the following section.

Sarikaya et al. obtained following Simpson type inequalities in [5].

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THEOREM 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differential mapping on I^o such that $f' \in L_1[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{5(b-a)}{72}\left[|f'(a)|+|f'(b)|\right]. \tag{1}$$

THEOREM 3. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differential mapping on I^o such that $f' \in L_1[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$, $q \geqslant 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right| \qquad (2)\\ &\leqslant \frac{(b-a)}{72}(5)^{1-\frac{1}{q}}\left\{\left[\frac{61\,|f'(b)|^q+29\,|f'(a)|^q}{18}\right]^{\frac{1}{q}}\right.\\ &\quad\left.+\left[\frac{61\,|f'(a)|^q+29\,|f'(b)|^q}{18}\right]^{\frac{1}{q}}\right\}. \end{aligned}$$

Estimation for the difference between the middle and the leftmost term in the Hadamard inequality was proved by Kirmaci in [8].

THEOREM 4. *Let $f : I^o \to \mathbb{R}$ be a differential mapping on I^o, $a,b \in I^o$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leqslant \frac{b-a}{4}\left(\frac{|f'(a)|+|f'(b)|}{2}\right). \tag{3}$$

Similarly, bound for the difference between the middle and the right most term in the Hadamard inequality was considered by Dragomir and Agarwal in [7] and has the following forms.

THEOREM 5. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differential mapping on I^o, $a,b \in I^o$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (4)\\ &\leqslant \frac{(b-a)}{8}\left[|f'(a)|+|f'(b)|\right]. \end{aligned}$$

THEOREM 6. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differential mapping on I^o, $a,b \in I^o$ with $a < b$ and let $p > 1$. If the new mapping $|f'|^{p/(p-1)}$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \tag{5}$$
$$\leqslant \frac{(b-a)}{2(p+1)^{1/p}}\left[\frac{|f'(a)|^{p/p-1}+|f'(b)|^{p/p-1}}{2}\right]^{(p-1)/p}.$$

In this paper we give inequalities involving m harmonic polynomials and a function f such that $\left|f^{(n)}\right|^q$ is convex for some $q \geqslant 1$ and $n \in \mathbb{N}$. After each general inequality we obtain a result related to particular case $n = 1$, $m = 2$ and point out that for some especial values of our variables h and c, these results become the known results given in the above text.

2. Main results

In the further text m and n are fixed integers, $\sigma := \{a = x_0 < x_1 < x_2 < \ldots < x_m = b\}$ is a division of an interval $[a,b]$ with $m+1$ nodes, $\sigma' := \{0 = s_0 < s_1 < \ldots < s_m = 1\}$ is a corresponding division of the interval $[0,1]$ connected with σ by relation $s_j = \frac{x_j - a}{b-a}$. Let $\{P_{jk}\}_{k\in\mathbb{N}}$, $j = 1,\ldots,m$ be harmonic sequences of polynomials, i.e. $P'_{jk} = P_{j,k-1}$, $k \in \mathbb{N}$, $P_{j0} = 1$, $j = 1,\ldots,m$. Let us define a kernel S_n as

$$S_n(t,\sigma) = \begin{cases} P_{1n}(t), & t \in [a,x_1] \\ P_{2n}(t), & t \in (x_1,x_2] \\ \vdots & \\ P_{mn}(t), & t \in (x_{m-1},b]. \end{cases}$$

In paper [4] Pečarić and Varošanec gave the following identity for n-times differentiable function f on $[a,b]$:

$$(-1)^n \int_a^b S_n(x,\sigma) f^{(n)}(x)\,dx = \int_a^b f(t)\,dt + \sum_{k=1}^n (-1)^k \Big[P_{mk}(b) f^{(k-1)}(b) \tag{6}$$
$$+ \sum_{j=1}^{m-1} \left[P_{jk}(x_j) - P_{j+1,k}(x_j)\right] f^{(k-1)}(x_j) - P_{1k}(a) f^{(k-1)}(a)\Big].$$

For further applications of this identity see [4], [6].

Let us denote the right-hand side of the above identity by I_m. Using substitution

$x = tb + (1-t)a$ on the left handside of (6) we get

$$(-1)^n \int_a^b S_n(x,\sigma) f^{(n)}(x)\,dx$$
$$= (-1)^n (b-a) \int_0^1 S_n(tb+(1-t)a,\sigma) f^{(n)}(tb+(1-t)a)\,dt.$$

Using notation $C_n(t,\sigma') = (b-a)S_n(tb+(1-t)a)$ identity (6) becomes

$$(-1)^n \int_0^1 C_n\left(t,\sigma'\right) f^{(n)}(tb+(1-t)a)\,dt = I_m. \tag{7}$$

Also, in the further text we use abbreviations $K_{jn}(t) = (b-a)P_{jn}(tb+(1-t)a)$, $j = 1,\ldots,m$, *i.e.*

$$C_n\left(t,\sigma'\right) = \begin{cases} K_{1n}(t), & t \in [0,s_1] \\ K_{2n}(t), & t \in (s_1,s_2] \\ \quad\vdots & \\ K_{mn}(t), & t \in (s_{m-1},1]. \end{cases}$$

THEOREM 7. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be an n-times differentiable function on I°, $a,b \in I^\circ$, $a<b$ and $\left|f^{(n)}\right|^q$ be convex on $[a,b]$ for some $q \geqslant 1$ such that $S_n(t,\sigma) f^{(n)}(t)$ is integrable on $[a,b]$. Then*

$$|I_m| \leqslant \sum_{j=1}^m \left(\int_{s_{j-1}}^{s_j} \left|K_{jn}(t)\right| dt\right)^{1-\frac{1}{q}} \left\{\left|f^{(n)}(b)\right|^q \left(\int_{s_{j-1}}^{s_j} t\left|K_{jn}(t)\right| dt\right)\right. \tag{8}$$
$$\left. + \left|f^{(n)}(a)\right|^q \left(\int_{s_{j-1}}^{s_j} (1-t)\left|K_{jn}(t)\right| dt\right)\right\}^{\frac{1}{q}}.$$

Proof. Using (7) and the power-mean inequality for $q \geqslant 1$ we have

$$|I_m| \leqslant \int_0^1 \left|C_n\left(t,\sigma'\right)\right| \left|f^{(n)}(tb+(1-t)a)\right| dt$$
$$= \sum_{j=1}^m \int_{s_{j-1}}^{s_j} \left|K_{jn}(t)\right| \left|f^{(n)}(tb+(1-t)a)\right| dt$$
$$\leqslant \sum_{j=1}^m \left(\int_{s_{j-1}}^{s_j} \left|K_{jn}(t)\right| dt\right)^{1-\frac{1}{q}} \left(\int_{s_{j-1}}^{s_j} \left|K_{jn}(t)\right| \left|f^{(n)}(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|f^{(n)}\right|^q$ is convex we have

$$
\begin{aligned}
|I_m| &\leqslant \sum_{j=1}^{m}\left(\int_{s_{j-1}}^{s_j}\left|K_{jn}(t)\right|dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_{s_{j-1}}^{s_j}\left|K_{jn}(t)\right|\left(t\left|f^{(n)}(b)\right|^q+(1-t)\left|f^{(n)}(a)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&=\sum_{j=1}^{m}\left(\int_{s_{j-1}}^{s_j}\left|K_{jn}(t)\right|dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\left|f^{(n)}(b)\right|^q\int_{s_{j-1}}^{s_j}t\left|K_{jn}(t)\right|dt+\left|f^{(n)}(a)\right|^q\int_{s_{j-1}}^{s_j}(1-t)\left|K_{jn}(t)\right|dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

So, we deduce the desired result. □

The case when $q=1$ is significant, so we write that result as the following corollary.

COROLLARY 1. *If f satisfies assumptions of Theorem 7 with $q=1$ which also means $\left|f^{(n)}\right|$ is convex on $[a,b]$, then*

$$
|I_m| \leqslant \left|f^{(n)}(b)\right|\sum_{j=1}^{m}\int_{s_{j-1}}^{s_j}t\left|K_{jn}(t)\right|dt+\left|f^{(n)}(a)\right|\sum_{j=1}^{m}\int_{s_{j-1}}^{s_j}(1-t)\left|K_{jn}(t)\right|dt. \tag{9}
$$

COROLLARY 2. *If f satisfies assumptions of Theorem 7 with $n=1,\ m=2$, then*

$$
\begin{aligned}
&\left|h[f(a)+f(b)]+(1-2h)f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \qquad (10)\\
&\leqslant (b-a)\left\{\left(\frac{c^2}{2}-hc+h^2\right)^{1-\frac{1}{q}}\left[\lambda_1\left|f'(b)\right|^q+\lambda_2\left|f'(a)\right|^q\right]^{\frac{1}{q}}\right. \\
&\quad+\left(\frac{c^2+1}{2}-(c+h)(1-h)\right)^{1-\frac{1}{q}}\left[\lambda_3\left|f'(b)\right|^q+\lambda_4\left|f'(a)\right|^q\right]^{\frac{1}{q}},
\end{aligned}
$$

where $x \in [a,b]$, $h \in \left[0, \frac{1}{2}\right]$, $c = \frac{x-a}{b-a}$ *such that* $h \leqslant c \leqslant 1-h$ *and*

$$\lambda_1 = \frac{1}{6}\left(2c^3 - 3c^2h + 2h^3\right)$$
$$\lambda_2 = \frac{1}{6}\left(-6ch + 6h^2 - 2h^3 - 2c^3 + 3c^2 + 3c^2h\right)$$
$$\lambda_3 = \frac{1}{6}\left(2c^3 + 3c^2h - 3c^2 - 2h^3 + 6h^2 - 3h + 1\right)$$
$$\lambda_4 = \frac{1}{6}\left(2 - 2c^3 + 6c^2 - 3c^2h + 6ch - 6c + 2h^3 - 3h\right).$$

Proof. Let us consider a division $\sigma = \{a \leqslant x \leqslant b\}$ of an interval $[a,b]$. Let us define the kernel $S_1(t,\sigma)$ as

$$S_1(t,\sigma) = \begin{cases} P_{11}(t) = t - (1-h)a - hb, t \in [a,x] \\ P_{21}(t) = t - ha - (1-h)b, t \in (x,b] \end{cases}.$$

Polynomials P_{11} and P_{21} are obviously harmonic. Denote by $c := \frac{x-a}{b-a}$. Then we consider a corresponding division $\sigma' = \{0 \leqslant c \leqslant 1\}$ of the interval $[0,1]$ and the kernel $C_1(t,\sigma')$ is equal to

$$C_1(t,\sigma') = \begin{cases} K_{11}(t) = (b-a)^2(t-h), \quad t \in [0,c] \\ K_{21}(t) = (b-a)^2(t-1+h), t \in (c,1] \end{cases}.$$

Putting in Theorem 7 the kernel $C_1(t,\sigma')$ after simple calculation with taking into account the condition $h \leqslant c \leqslant 1-h$ that is,

$$\int_0^c |t-h|\,dt = \int_0^h (h-t)\,dt + \int_h^c (t-h)\,dt = \frac{c^2}{2} - ch + h^2$$

and

$$\int_c^1 |t-1+h|\,dt = \int_c^{1-h} (1-h-t)\,dt + \int_{1-h}^1 (t-1+h)\,dt = \frac{c^2+1}{2} - (c+h)(1-h),$$

we get the desired inequality. □

REMARK 1. If we set $h = \frac{1}{6}$ and $c = \frac{1}{2}$ in (10) we obtain inequality (2).

REMARK 2. For different selections of parameters h and c in (10) we obtain the following Ostrowski, Simpson and Hadamard type inequalities.

(i) For the case $q = 1$, $h = 0$ we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right|$$
$$\leqslant \frac{b-a}{6}\left[\left(1 - 3c^2 + 4c^3\right)\left|f'(b)\right| + \left(2 - 6c + 9c^2 - 4c^3\right)\left|f'(a)\right|\right]$$

which is an Ostowski type inequality. Furthermore, if $c=\frac{1}{2}$ we get inequality (3).

(ii) For the case $q=1,\ h=\frac{1}{6}$ we obtain the following Simpson type inequality

$$\left|\frac{1}{6}\left[f(a)+4f(x)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leqslant \frac{(b-a)}{6}\left[\left(\frac{2}{3}-3c^2+4c^3\right)\left|f'(b)\right|+\left(\frac{5}{3}-6c+9c^2-4c^3\right)\left|f'(a)\right|\right]$$

which for $c=\frac{1}{2}$ collapses to inequality (1).

(iii) If $q=1,\ h=c=\frac{1}{2}$ we obtain the following Hadamard type inequality (4).

THEOREM 8. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be an n-times differentiable mapping on I°, $a,b\in I^\circ,\ a<b$, and $\left|f^{(n)}\right|^q$ is convex on $[a,b]$ for some $q>1,\ S_n(t,\sigma)f^{(n)}(t)$ is integrable on $[a,b]$. Then the following inequality holds:*

$$|I_m|\leqslant\left(\sum_{j=1}^{m}\int_{s_{j-1}}^{s_j}\left|K_{jn}(t)\right|^p dt\right)^{\frac{1}{p}}\left(\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{2}\right)^{\frac{1}{q}},\tag{11}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From (7) and property of modulus we have

$$|I_m|=\left|\int_0^1 C_n\left(t,\sigma'\right)f^{(n)}(tb+(1-t)a)\,dt\right|$$
$$\leqslant\int_0^1\left|C_n\left(t,\sigma'\right)f^{(n)}(tb+(1-t)a)\right|dt.$$

By using Hölder inequality we have

$$\left|\int_0^1 C_n\left(t,\sigma'\right)f^{(n)}(tb+(1-t)a)\,dt\right|$$
$$\leqslant\left(\int_0^1\left|C_n\left(t,\sigma'\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|f^{(n)}\right|^q$ is convex, by Hadamard inequality we get

$$\int_0^1 \left|f^{(n)}(tb+(1-t)a)\right|^q dt \leqslant \frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{2}.$$

Also we have

$$\int_0^1 \left|C_n\left(t,\sigma'\right)\right|^p dt = \sum_{j=1}^{m} \int_{s_{j-1}}^{s_j} \left|K_{jn}(t)\right|^p dt.$$

By combining these we deduce the desired result. □

COROLLARY 3. *If f satisfies assumptions of Theorem 8 with $n=1,\ m=2$, then*

$$\left|h[f(a)+f(b)]+(1-2h)f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{12}$$

$$\leqslant (b-a)\left(\frac{2h^{1+p}+(c-h)^{1+p}+(1-c-h)^{1+p}}{1+p}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}},$$

where $x\in[a,b],\ h\in\left[0,\frac{1}{2}\right],\ c=\frac{x-a}{b-a}$ such that $h\leqslant c\leqslant 1-h$.

Proof. Putting in Theorem 8 the kernel $C_1\left(t,\sigma'\right)$ defined as in the proof of Corollary 2, after simple calculation we get desired inequality. □

REMARK 3. For different selections of the parameters h and c in (12) we obtain the following Ostrowski, Hadamard and Simpson type inequalities

(i) For the case $h=0$ we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \leqslant (b-a)\left(\frac{c^{1+p}+(1-c)^{1+p}}{1+p}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

(ii) For the case $h=\frac{1}{2}$ and $c=\frac{1}{2}$ we obtain (5).

(iii) For $h=\frac{1}{6}$ and $c=\frac{1}{2}$ we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\leqslant \frac{(b-a)}{6}\left(\frac{2^{1+p}+1}{3(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

THEOREM 9. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be an* n*-times differentiable mapping on* I°, $a, b \in I^{\circ}$, $a < b$, *and* $\left|f^{(n)}\right|^q$ *is convex on* $[a,b]$ *for some* $q > 1$, $S_n(t,\sigma) f^{(n)}(t)$ *is integrable on* $[a,b]$. *Then the following inequality holds:*

$$|I_m| \leqslant \sum_{j=1}^{m} \left(\int_{s_{j-1}}^{s_j} \left|K_{jn}(t)\right|^p dt \right)^{\frac{1}{p}} \left(\frac{\left|f^{(n)}(x_j)\right|^q + \left|f^{(n)}(x_{j-1})\right|^q}{2} \right)^{\frac{1}{q}}, \tag{13}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using (7), property of modulus and Hölder Inequality we have

$$\begin{aligned} |I_m| &\leqslant \int_0^1 \left|C_n(t,\sigma') f^{(n)}(tb + (1-t)a)\right| dt. \\ &\leqslant \sum_{j=1}^{m} \int_{s_{j-1}}^{s_j} \left|K_{jn}(t) f^{(n)}(tb + (1-t)a)\right| dt \\ &\leqslant \sum_{j=1}^{m} \left(\int_{s_{j-1}}^{s_j} \left|K_{jn}(t)\right|^p dt \right)^{\frac{1}{p}} \left(\int_{s_{j-1}}^{s_j} \left|f^{(n)}(tb + (1-t)a)\right|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $\left|f^{(n)}\right|^q$ is convex, by using Hadamard inequality we get

$$\int_{s_{j-1}}^{s_j} \left|f^{(n)}(tb + (1-t)a)\right|^q dt \leqslant \frac{\left|f^{(n)}(x_j)\right|^q + \left|f^{(n)}(x_{j-1})\right|^q}{2}.$$

So, this implies (13). □

COROLLARY 4. *If* f *satisfies assumptions of Theorem 8 with* $n = 1$, $m = 2$, *then*

$$\begin{aligned} &\left| h[f(a) + f(b)] + (1-2h) f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right| \qquad (14) \\ &\leqslant (b-a) \left\{ \left(\frac{h^{1+p} + (c-h)^{1+p}}{1+p} \right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(x)|^q}{2} \right)^{\frac{1}{q}} \right. \\ &\qquad \left. + \left(\frac{h^{1+p} + (1-c-h)^{1+p}}{1+p} \right)^{\frac{1}{p}} \left(\frac{|f'(b)|^q + |f'(x)|^q}{2} \right)^{\frac{1}{q}} \right\}, \end{aligned}$$

where $x \in [a,b]$, $h \in \left[0, \frac{1}{2}\right]$, $c = \frac{x-a}{b-a}$ *such that* $h \leqslant c \leqslant 1-h$.

Proof. Putting in Theorem 9 the kernel $C_n(t,\sigma')$ defined as in the proof of Corollary 2, after simple calculation we get desired inequality. □

REMARK 4. For different selections of the parameters h and c in (14) we obtain the following Ostrowski, Hadamard and Simpson type inequalities

(i) For the case $h=0$ we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leqslant (b-a)\left\{\left(\frac{c^{1+p}}{1+p}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(1-c)^{1+p}}{1+p}\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

(ii) For the case $h=\frac{1}{2}$ and $c=\frac{1}{2}$ we get the following inequality

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leqslant \frac{(b-a)}{2}\left(\frac{1}{2(1+p)}\right)^{\frac{1}{p}}\left\{\left(\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

(iii) For $h=\frac{1}{6}$ and $c=\frac{1}{2}$ we have the following Simpson type inequality

$$\left|\frac{1}{6}\left[f(a)+4f(\frac{a+b}{2})+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leqslant \frac{(b-a)}{6}\left(\frac{2^{1+p}+1}{6(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

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