

SIMPSON TYPE INEQUALITIES VIA φ -CONVEXITY

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ABSTRACT. In this paper, we obtain some Simpson type inequalities for functions whose derivatives in absolute value are φ -convex.

1. INTRODUCTION AND PRELIMINARIES

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is a four times continuously differentiable mapping on (a, b) and $\|f^{(4)}\|_{\infty} = \sup |f^{(4)}(x)| < \infty$. The following inequality

$$\begin{aligned} & \left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b-a)^4 \end{aligned}$$

is well known in the literature as Simpson's inequality.

For some results about Simpson inequality see [3]-[7].

In [3], Alomari et al. proved some inequalities of Simpson type for s -convex functions by using the following Lemma.

Lemma 1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous mapping on I° where $a, b \in I$ with $a < b$. Then the following equality holds:*

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & = (b-a) \int_0^1 p(t) f'(tb + (1-t)a) dt, \end{aligned}$$

where

$$p(t) = \begin{cases} t - \frac{1}{6}, & t \in [0, \frac{1}{2}) \\ t - \frac{5}{6}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Let $f, \varphi : K \rightarrow \mathbb{R}$, where K is a nonempty closed set in \mathbb{R}^n , be continuous functions. We recall the following results, which are due to Noor and Noor [1], Noor [2] as follows:

Definition 1. *Let $u \in K$. Then the set K is said to be φ -convex at u with respect to φ , if*

$$u + te^{i\varphi}(v-u) \in K, \quad \forall u, v \in K, \quad t \in [0, 1].$$

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Remark 1. We would like to mention that the Definition 1 of a φ -convex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point u which is contained in K . We don't require that the point v should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that v should be an end point of the path for every pair of points, $u, v \in K$, then $e^{i\varphi}(v - u) = v - u$ if and only if, $\varphi = 0$, and consequently φ -convexity reduces to convexity. Thus, it is true that every convex set is also an φ -convex set, but the converse is not necessarily true.

Definition 2. The function f on the φ -convex set K is said to be φ -convex with respect to φ , if

$$f(u + te^{i\varphi}(v - u)) \leq (1 - t)f(u) + tf(v), \quad \forall u, v \in K, \quad t \in [0, 1].$$

The function f is said to be φ -concave if and only if $-f$ is φ -convex. Note that every convex function is a φ -convex function, but the converse is not true.

The following inequality is known as the Hölder inequality[8]:

Theorem 1. Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on $[a, b]$ and if $|f|^p$ and $|g|^q$ are integrable functions on $[a, b]$ then

$$\int_a^b |f(x)g(x)| dx \leq \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx \right)^{\frac{1}{q}},$$

with equality holding if and only if $A|f(x)|^p = B|g(x)|^q$ almost everywhere, where A and B are constants.

2. SIMPSON TYPE INEQUALITIES FOR φ -CONVEX FUNCTIONS

Throughout this section, let $K = [a, a + e^{i\varphi}(b - a)]$ and $0 \leq \varphi \leq \frac{\pi}{2}$

We used the following Lemma to obtain our main results.

Lemma 2. Let $K \subset \mathbb{R}$ be a φ -convex subset and $f : K \rightarrow (0, \infty)$ be a differentiable function on K° (the interior of K), $a, b \in K$ with $a < a + e^{i\varphi}(b - a)$. If f' is integrable on $[a, a + e^{i\varphi}(b - a)]$, following equality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + e^{i\varphi}(b - a)}{2}\right) + f(a + e^{i\varphi}(b - a)) \right] - \frac{1}{e^{i\varphi}(b - a)} \int_a^{a + e^{i\varphi}(b - a)} f(x) dx \right| \\ &= e^{i\varphi}(b - a) \int_0^1 p(t) f'(a + te^{i\varphi}(b - a)) dt, \end{aligned}$$

where

$$p(t) = \begin{cases} t - \frac{1}{6}, & t \in [0, \frac{1}{2}) \\ t - \frac{5}{6}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Proof. Since K is a φ -convex set, for $a, b \in K$ and $t \in [0, 1]$ we have $a + e^{i\varphi}(b-a) \in K$. Integrating by parts implies that

$$\begin{aligned}
& \int_0^{\frac{1}{2}} \left(t - \frac{1}{6}\right) f'(a + te^{i\varphi}(b-a)) dt + \int_{\frac{1}{2}}^1 \left(t - \frac{5}{6}\right) f'(a + te^{i\varphi}(b-a)) dt \\
&= \left(t - \frac{1}{6}\right) \frac{f(a + te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)} \Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{f(a + te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)} dt \\
&\quad + \left(t - \frac{5}{6}\right) \frac{f(a + te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)} \Big|_{\frac{1}{2}}^1 - \int_{\frac{1}{2}}^1 \frac{f(a + te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)} dt \\
&= \frac{1}{6e^{i\varphi}(b-a)} \left[f(a) + 4f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) + f(a + e^{i\varphi}(b-a)) \right] \\
&\quad - \frac{1}{e^{i\varphi}(b-a)} \left[\int_0^{\frac{1}{2}} f(a + te^{i\varphi}(b-a)) dt + \int_{\frac{1}{2}}^1 f(a + te^{i\varphi}(b-a)) dt \right].
\end{aligned}$$

If we change the variable $x = a + te^{i\varphi}(b-a)$ and multiply the resulting equality with $e^{i\varphi}(b-a)$ we get the desired result. \square

Theorem 2. Let $f : K \rightarrow (0, \infty)$ be a differentiable function on K° . If $|f'|$ is φ -convex function on K° and $a, b \in K$ with $a < a + e^{i\varphi}(b-a)$. Then, the following inequality holds:

$$\begin{aligned}
& \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) + f(a + e^{i\varphi}(b-a)) \right] - \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x) dx \right| \\
&\leq \frac{5}{72} e^{i\varphi}(b-a) [|f'(a)| + |f'(b)|].
\end{aligned}$$

Proof. From Lemma 2 and using the φ -convexity of $|f'|$ we have

$$\begin{aligned}
& \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) + f(a + e^{i\varphi}(b-a)) \right] - \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x) dx \right| \\
&\leq e^{i\varphi}(b-a) \left\{ \int_0^{\frac{1}{2}} \left|t - \frac{1}{6}\right| |f'(a + te^{i\varphi}(b-a))| dt + \int_{\frac{1}{2}}^1 \left|t - \frac{5}{6}\right| |f'(a + te^{i\varphi}(b-a))| dt \right\} \\
&\leq e^{i\varphi}(b-a) \left\{ \int_0^{\frac{1}{6}} \left(\frac{1}{6} - t\right) [(1-t)|f'(a)| + t|f'(b)|] dt \right. \\
&\quad + \int_{\frac{1}{6}}^{\frac{1}{2}} \left(t - \frac{1}{6}\right) [(1-t)|f'(a)| + t|f'(b)|] dt \\
&\quad + \int_{\frac{1}{2}}^{\frac{5}{6}} \left(\frac{5}{6} - t\right) [(1-t)|f'(a)| + t|f'(b)|] dt \\
&\quad \left. + \int_{\frac{5}{6}}^1 \left(t - \frac{5}{6}\right) [(1-t)|f'(a)| + t|f'(b)|] dt \right\} \\
&= \frac{5}{72} e^{i\varphi}(b-a) [|f'(a)| + |f'(b)|]
\end{aligned}$$

which completes the proof. \square

Theorem 3. Let $f : K \rightarrow (0, \infty)$ be a differentiable function on K° , $a, b \in K$ with $a < a + e^{i\varphi}(b - a)$. If $|f'|^q$ is φ -convex function on K° for some fixed $q > 1$ then the following inequality holds

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) + f(a + e^{i\varphi}(b-a)) \right] - \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x) dx \right| \\ & \leq e^{i\varphi}(b-a) \left(\frac{1 + 2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \\ & \quad \times \left\{ \left(\frac{3}{8} |f'(a)|^q + \frac{1}{8} |f'(b)|^q \right)^{\frac{1}{q}} + \left(\frac{1}{8} |f'(a)|^q + \frac{3}{8} |f'(b)|^q \right)^{\frac{1}{q}} \right\} \end{aligned}$$

where $p = \frac{q}{q-1}$.

Proof. From Lemma 2 and using the Hölder inequality, we have

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) + f(a + e^{i\varphi}(b-a)) \right] - \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x) dx \right| \\ & \leq e^{i\varphi}(b-a) \left\{ \left(\int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} |f'(a + te^{i\varphi}(b-a))|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right|^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 |f'(a + te^{i\varphi}(b-a))|^q dt \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Since $|f'|^q$ is φ -convex, we obtain

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) + f(a + e^{i\varphi}(b-a)) \right] - \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x) dx \right| \\ & \leq e^{i\varphi}(b-a) \left\{ \left(\int_0^{\frac{1}{6}} \left(\frac{1}{6} - t \right)^p dt + \int_{\frac{1}{6}}^{\frac{1}{2}} \left(t - \frac{1}{6} \right)^p dt \right)^{\frac{1}{p}} \right. \\ & \quad \times \left(\int_0^{\frac{1}{2}} [(1-t)|f'(a)|^q + t|f'(b)|^q] dt \right)^{\frac{1}{q}} \\ & \quad + \left(\int_{\frac{1}{2}}^{\frac{5}{6}} \left(\frac{5}{6} - t \right)^p dt + \int_{\frac{5}{6}}^1 \left(t - \frac{5}{6} \right)^p dt \right)^{\frac{1}{p}} \\ & \quad \left. \times \left(\int_{\frac{1}{2}}^1 [(1-t)|f'(a)|^q + t|f'(b)|^q] dt \right)^{\frac{1}{q}} \right\} \\ & = e^{i\varphi}(b-a) \left(\frac{1 + 2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \\ & \quad \times \left\{ \left(\frac{3}{8} |f'(a)|^q + \frac{1}{8} |f'(b)|^q \right)^{\frac{1}{q}} + \left(\frac{1}{8} |f'(a)|^q + \frac{3}{8} |f'(b)|^q \right)^{\frac{1}{q}} \right\} \end{aligned}$$

which is the desired. \square

Theorem 4. *Under the assumptions of Theorem 3, we have the following inequality*

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f \left(\frac{2a + e^{i\varphi}(b-a)}{2} \right) + f(a + e^{i\varphi}(b-a)) \right] - \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x) dx \right| \\ & \leq e^{i\varphi}(b-a) \left(\frac{2(1+2^{p+1})}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}. \end{aligned}$$

Proof. From Lemma 2, φ -convexity of $|f'|^q$ and using the Hölder inequality, we have

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f \left(\frac{2a + e^{i\varphi}(b-a)}{2} \right) + f(a + e^{i\varphi}(b-a)) \right] - \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x) dx \right| \\ & \leq e^{i\varphi}(b-a) \left[\int_0^1 |p(t)| |f'(a + te^{i\varphi}(b-a))| dt \right] \\ & \leq e^{i\varphi}(b-a) \left(\int_0^1 |p(t)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(a + te^{i\varphi}(b-a))|^q dt \right)^{\frac{1}{q}} \\ & \leq e^{i\varphi}(b-a) \left(\int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right|^p dt + \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 [(1-t)|f'(a)|^q + t|f'(b)|^q] dt \right)^{\frac{1}{q}} \\ & = e^{i\varphi}(b-a) \left(\frac{2(1+2^{p+1})}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}} \end{aligned}$$

where we used the fact that

$$\int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right|^p dt = \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right|^p dt = \frac{(1+2^{p+1})}{6^{p+1}(p+1)}.$$

The proof is completed. \square

Theorem 5. *Let $f : K \rightarrow (0, \infty)$ be a differentiable function on K° , $a, b \in K$ with $a < a + e^{i\varphi}(b-a)$. If $|f'|^q$ is φ -convex function on K° for some fixed $q \geq 1$ then the following inequality holds*

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f \left(\frac{2a + e^{i\varphi}(b-a)}{2} \right) + f(a + e^{i\varphi}(b-a)) \right] - \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x) dx \right| \\ & \leq e^{i\varphi}(b-a) \left(\frac{5}{72} \right)^{1-\frac{1}{q}} \\ & \quad \times \left\{ \left(\frac{61|f'(a)|^q + 29|f'(b)|^q}{1296} \right)^{\frac{1}{q}} + \left(\frac{29|f'(a)|^q + 61|f'(b)|^q}{1296} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Proof. From Lemma 2 and using the power-mean inequality, we have

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{2a + e^{i\varphi}(b-a)}{2}\right) + f(a + e^{i\varphi}(b-a)) \right] - \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x) dx \right| \\ & \leq e^{i\varphi}(b-a) \\ & \quad \times \left\{ \left(\int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right|^{1-\frac{1}{q}} dt \right)^{\frac{1}{q}} \left(\int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right| |f'(a + te^{i\varphi}(b-a))|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right|^{1-\frac{1}{q}} dt \right)^{\frac{1}{q}} \left(\int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right| |f'(a + te^{i\varphi}(b-a))|^q dt \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Since $|f'|^q$ is φ -convex function we have

$$\begin{aligned} & \int_0^{\frac{1}{2}} \left| t - \frac{1}{6} \right| |f'(a + te^{i\varphi}(b-a))|^q dt \\ & \leq \int_0^{\frac{1}{6}} \left(\frac{1}{6} - t \right) [(1-t)|f'(a)|^q + t|f'(b)|^q] dt \\ & \quad + \int_{\frac{1}{6}}^{\frac{1}{2}} \left(t - \frac{1}{6} \right) [(1-t)|f'(a)|^q + t|f'(b)|^q] dt \\ & = \frac{61|f'(a)|^q + 29|f'(b)|^q}{1296} \end{aligned}$$

and

$$\begin{aligned} & \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right| |f'(a + te^{i\varphi}(b-a))|^q dt \\ & \leq \int_{\frac{1}{2}}^{\frac{5}{6}} \left(\frac{5}{6} - t \right) [(1-t)|f'(a)|^q + t|f'(b)|^q] dt \\ & \quad + \int_{\frac{5}{6}}^1 \left(t - \frac{5}{6} \right) [(1-t)|f'(a)|^q + t|f'(b)|^q] dt \\ & = \frac{29|f'(a)|^q + 61|f'(b)|^q}{1296}. \end{aligned}$$

Combining all the above inequalities gives us the desired result. \square

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