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On the Diophantine equation $(x+1)^k + (x+2)^k + ... + (2x)^k = y^n$

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ABSTRACT

In this work, we give upper bounds for n on the title equation. Our results depend on assertions describing the precise exponents of 2 and 3 appearing in the prime factorization of $T_k(x) = (x+1)^k + (x+2)^k + ... + (2x)^k$. Further, on combining Baker's method with the explicit solution of polynomial exponential congruences (see e.g. [6]), we show that for $2 \leq x \leq 13, k \geq 1, y \geq 2$ and $n \geq 3$ the title equation has no solutions.

1. Introduction

Let x and k be positive integers. Write

$$S_k(x) = 1^k + 2^k + ... + x^k$$

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for the sum of the k-th powers of the first x positive integers. The Diophantine equation

$$S_k(x) = y^n \ , \tag{1.1}$$

in unknown positive integers k, n, x, y with $n \geq 2$ has a rich history. In 1875, the classical question of Lucas [12] was whether equation (1.1) has only the solutions $x = y = 1$ and $x = 24$, $y = 70$ for $(k, n) = (2, 2)$. In 1918, Watson [21] solved equation (1.1) with $(k, n) = (2, 2)$. In 1956, Schäffer [17] considered equation (1.1). He showed, for fixed $k \geq 1$ and $n \geq 2$, that (1.1) possesses at most finitely many solutions in positive integers x and y, unless

$$(k, n) \in \{(1, 2), (3, 2), (3, 4), (5, 2)\} \tag{1.2}$$

where, in each case, there are infinitely many such solutions. There are several effective and ineffective results concerning equation (1.2), see the survey paper [8]. Schäffer's conjectured that (1.2) has the unique non-trivial (i.e. $(x, y) \neq (1, 1)$) solution, namely $(k, n, x, y) = (2, 2, 24, 70)$. In 2004, Jacobson, Pintér, Walsh [10] and Bennett, Győry, Pintér [3], proved that the Schäffer's conjecture is true if $2 \leq k \leq 58$, k is even $n = 2$ and $2 \leq k \leq 11$, n is arbitrary, respectively. In 2007, Pintér [15], proved that the equation

$$S_k(x) = y^{2n}, \text{ in positive integers } \ x, y, n \text{ with } n > 2 \tag{1.3}$$

has only the trivial solution $(x, y) = (1, 1)$ for odd values of k, with $1 \leq k < 170$.

In 2015, Hajdu [9], proved that Schäffer's conjecture holds under certain assumptions on x, letting all the other parameters free. He also proved that the conjecture is true if $x \equiv 0, 3 \pmod 4$ and $x < 25$. The main tools in the proof of this result were the 2-adic valuation of $S_k(x)$ and local methods for polynomial-exponential congruences. Recently Bérczes, Hajdu, Miyazaki and Pink [6], provided all solutions of equation (1.1) with $1 \leq x < 25$ and $n \geq 3$.

Now we consider the Diophantine equation

$$(x + 1)^k + (x + 2)^k + ... + (x + d)^k = y^n \tag{1.4}$$

for fixed positive integers k and d.

In 2013, Zhang and Bai [2], considered the Diophantine equation (1.4) with $k = 2$. They first proved that all integer solutions of equation (1.4) such that $n > 1$ and $d = x$ are $(x, y) = (0, 0)$, $(x, y, n) = (1, \pm 2, 2), (2, \pm 5, 2), (24, \pm 182, 2)$ or $(x, y) = (-1, -1)$ with $2 \nmid n$. Secondly, they showed that if $p \equiv \pm 5 \pmod{12}$ is prime, $p \mid d$ and $v_p(d) \not\equiv 0 \pmod n$, then equation (1.4) has no integer solution (x, y) with $k = 2$. In 2014, the equation

$$(x - 1)^k + x^k + (x + 1)^k = y^n \quad x, y, n \in \mathbb{Z}, \quad n \geq 2, \tag{1.5}$$

was solved completely by Zhang [22], for $k = 2, 3, 4$ and the next year, Bennett, Patel and Siksek [4], extend Zhang's result, completely solving equation (1.5) in the cases $k = 5$ and $k = 6$. In 2016, Bennett, Patel and Siksek [5], considered the equation (1.4). They gave the integral solutions to the equation (1.4) using linear forms in logarithms, sieving and Frey curves where $k = 3$, $2 \leq d \leq 50$, $x \geq 1$ and n is prime.

Let $k \geq 2$ be even, and let r be a non-zero integer. Recently, Patel and Siksek [14], showed that for almost all $d \geq 2$ (in the sense of natural density), the equation

$$x^k + (x+r)^k + ... + (x+(d-1)r)^k = y^n, \quad x, y, n \in \mathbb{Z}, \quad n \geq 2$$

has no solutions. Let $k, l \geq 2$ be fixed integers. More recently, Soydan [20], considered the equation

$$(x+1)^k + (x+2)^k + ... + (lx)^k = y^n, \quad x, y \geq 1, n \in \mathbb{Z}, \quad n \geq 2 \tag{1.6}$$

in integers. He proved that the equation (1.6) has only finitely many solutions in positive integers, x, y, k, n, where l is even, $n \geq 2$ and $k \neq 1, 3$. He also showed that the equation (1.6) has infinitely many solutions where $n \geq 2$, l is even and $k = 1, 3$.

In this paper, we are interested in the integer solutions of the equation

$$T_k(x) = y^n \tag{1.7}$$

where

$$T_k(x) = (x+1)^k + (x+2)^k + ... + (2x)^k \tag{1.8}$$

for positive integer k. We provide upper bounds for n and give some results about equation (1.7).

2. The main results

Our main results provide upper bounds for the exponent n in equation (1.7) in terms of 2 and 3-valuations v_2 and v_3 of some functions of x and x, k. Further, on combining Theorem 2.1 with Baker's method and with a version of the local method (see e.g. [6]), we show that for $2 \leq x \leq 13, k \geq 1, y \geq 2$ and $n \geq 3$ equation (1.7) has no solutions.

For a prime p and an integer m, let $v_p(m)$ denote the highest exponent v such that $p^v | m$.

Theorem 2.1. (*i*) *Assume first that* $x \equiv 0 \pmod 4$. *Then for any solution* (k, n, x, y) *of equation* (1.7), *we get*

$$n \leq \begin{cases} v_2(x) - 1, & \text{if } k = 1 \text{ or } k \text{ is even,} \\ 2v_2(x) - 2, & \text{if } k \geq 3 \text{ is odd.} \end{cases}$$

(ii) Assume that $x \equiv 1 \pmod 4$ *and* $k = 1$*, then for any solution* (k, n, x, y) *of equation* (1.7)*, we get* $n \leq v_2(3x+1) - 1$.

Suppose next that $x \equiv 1, 5 \pmod 8$ *and* $x \not\equiv 1 \pmod{32}$ *with* $k \neq 1$*. Then for any solution* (k, n, x, y) *of equation* (1.7)*, we get*

$$n \leq \begin{cases} v_2(7x+1) - 1, & \text{if } x \equiv 1 \pmod 8 \text{ and } k{=}2, \\ v_2((5x+3)(3x+1)) - 2, & \text{if } x \equiv 1 \pmod 8 \text{ and } k = 3, \\ v_2(3x+1), & \text{if } x \equiv 5 \pmod 8 \text{ and } k \geq 3 \text{ is odd}, \\ 1, & \text{if } x \equiv 5 \pmod 8 \text{ and } k \geq 2 \text{ is even}, \\ 2, & \text{if } x \equiv 9 \pmod{16} \text{ and } k \geq 4 \text{ is even}, \\ 3, & \text{if } x \equiv 9 \pmod{16} \text{ and } k \geq 5 \text{ is odd} \\ & \text{or} \\ & \text{if } x \equiv 17 \pmod{32} \text{ and } k \geq 4 \text{ is even}, \\ 4, & \text{if } x \equiv 17 \pmod{32} \text{ and } k \geq 5 \text{ is odd}. \end{cases}$$

(iii) Suppose now that $x \equiv 0 \pmod 3$ *and* k *is odd or* $x \equiv 0, 4 \pmod 9$ *and* $k \geq 2$ *is even. Then for any solution* (k, n, x, y) *of equation* (1.7)*,*

$$n \leq \begin{cases} v_3(x), & \text{if } x \equiv 0 \pmod 3 \text{ and } k = 1, \\ v_3(x) - 1, & \text{if } x \equiv 0 \pmod 9 \text{ and } k \geq 2 \text{ is even}, \\ v_3(kx^2), & \text{if } x \equiv 0 \pmod 3 \text{ and } k > 3 \text{ is odd}, \\ v_3(x^2(5x+3)), & \text{if } x \equiv 0 \pmod 3 \text{ and } k = 3, \\ v_3(2x+1) - 1, & \text{if } x \equiv 4 \pmod 9 \text{ and } k \geq 2 \text{ is even}. \end{cases}$$

Theorem 2.2. *Assume that* $x \equiv 1, 4 \pmod 8$ *or* $x \equiv 4, 5 \pmod 8$*. Then Eq.* (1.7) *has no solution with* $k = 1$ *or* $k \geq 2$ *is even, respectively.*

Theorem 2.3. *Consider equation* (1.7) *in positive integer unknowns* (x, k, y, n) *with* $2 \leq x \leq 13$*,* $k \geq 1$*,* $y \geq 2$ *and* $n \geq 3$*. Then equation* (1.7) *has no solutions.*

3. Auxiliary results

3.1. Bernoulli polynomials

The Bernoulli polynomials $B_q(x)$ are defined by

$$\frac{ze^{zx}}{e^z - 1} = \sum_{q=0}^{\infty} \frac{B_q(x) z^q}{q!}, \quad |z| < 2\pi.$$

Their expansion around the origin is given by

$$B_q(x) = \sum_{i=0}^{q} \binom{q}{i} B_i x^{q-i}, \tag{3.1}$$

where $B_n = B_n(0)$ for $(n = 0, 1, 2, ...)$ are the Bernoulli numbers. For the following properties of Bernoulli Polynomials, we refer to Haynsworth and Goldberg, [1], pp. 804–805 (see also Rademacher, [16]):

$$B_k = B_k(0), \quad k = 0, 1, 2, ... \tag{3.2}$$

$$B_{2k+1}(0) = B_{2k+1}(1) = B_{2k+1} = 0, \quad k = 1, 2, ... \tag{3.3}$$

$$B_k(1-x) = (-1)^k B_k(x) \tag{3.4}$$

$$B_k(x) + B_k(x + \frac{1}{2}) = 2^{1-k} B_k(2x) \tag{3.5}$$

$$(-1)^{k+1} B_{2k+1}(x) > 0, \quad k = 1, 2, ... \quad 0 < x < \frac{1}{2} \tag{3.6}$$

The polynomials $S_k(x)$ are strongly connected to the Bernoulli polynomials since $S_k(x)$ may be expressed as

$$S_k(x) = \frac{1}{k+1}(B_{k+1}(x+1) - B_{k+1}(0)). \tag{3.7}$$

3.2. Decomposition of the polynomials $S_k(x)$ and $T_k(x)$

We start by stating some well-known properties of the polynomial $S_k(x)$ which we will need later; see e.g. [16] for details.

If $k = 1$, then $S_1(x) = \frac{x(x+1)}{2}$ while, if $k > 1$, we can write

$$S_k(x) = \begin{cases} \frac{1}{C_k} x^2 (x+1)^2 R_k(x), & \text{if } k > 1 \text{ is odd}, \\ \frac{1}{C_k} x(x+1)(2x+1) R_k(x), & \text{if } k > 1 \text{ is even}, \end{cases}$$

where C_k is a positive integer and $R_k(x)$ is a polynomial with integer coefficients.

By $T_k(x) = S_k(2x) - S_k(x)$ the polynomial $T_k(x)$ is also in a strong connection with the Bernoulli polynomials. This connection is shown in the below Lemma.

Lemma 3.1.

$$T_k(x) = \frac{B_{k+1}(2x+1) - B_{k+1}(x+1)}{k+1} \tag{3.8}$$

where $B_q(x)$ is the q-th Bernoulli polynomial defined by (3.1).

Proof. It is an application of the equality

$$\sum_{n=M}^{N-1} n^k = \frac{1}{k+1}\{B_{k+1}(N) - B_{k+1}(M)\}$$

which is given by Rademacher in [16], pp. 3–4. □

Secondly, applying Lemma 3.1 to equation (1.7), we have the following:

Lemma 3.2. *If $k = 1$, then $T_1(x) = \frac{x(3x+1)}{2}$, while for $k > 1$ we can write*

(*i*) $T_k(x) = \frac{1}{D_k} x(2x+1)M_k$, *if $k \geq 2$ is even,*
(*ii*) $T_k(x) = \frac{1}{D_k} x^2(3x+1)M_k$, *if $k > 1$ is odd*

where D_k is a positive integer and $M_k(x)$ is a polynomial with integer coefficients.

Proof. (*i*) Firstly we prove that $x = 0$ and $x = -\frac{1}{2}$ are roots of the polynomial $T_k(x)$ where $k \geq 2$ is even. By (3.8), we have

$$T_k(x) = \frac{B_{k+1}(2x+1) - B_{k+1}(x+1)}{k+1}. \tag{3.9}$$

It is clear that $x = 0$ and $x = -\frac{1}{2}$ satisfy (3.9) by using (3.3) and (3.4).

Secondly we show that $x = 0$ and $x = -\frac{1}{2}$ are simple roots of $T_k(x)$ for $k \geq 2$ even. Since for the Bernoulli polynomials $B_n(x)$ we have

$$\frac{dB_n(x)}{dx} = nB_{n-1}(x)$$

We may write

$$T_k'(x) = (k+1)(2B_k(2x+1) - B_k(x+1)) \tag{3.10}$$

and

$$T_k''(x) = k(k+1)(4B_{k-1}(2x+1) - B_{k-1}(x+1)). \tag{3.11}$$

If $k \geq 2$ is even, then $T_k'(0) = (k+1)(2B_k(1) - B_k(1)) = (k+1)B_k(1) \neq 0$. So $x = 0$ is a simple root of $T_k(x)$. Similarly, since $T_k'(-\frac{1}{2}) = (k+1)(2B_k(0) - B_k(\frac{1}{2})) \neq 0$ it follows that $x = -\frac{1}{2}$ is the simple root of $T_k(x)$ where k is even.

(*ii*) Now by (3.4) and (3.9) we see that $x = -\frac{1}{3}$ is a root of $T_k(x)$ whenever $k > 1$ is odd. Using (3.4), (3.6), (3.8) and (3.9), we have $T_k(-\frac{1}{3}) = 0 \neq T_k'(-\frac{1}{3})$. So $x = -\frac{1}{3}$ is a simple root of $T_k(x)$ where k is odd. Similarly we can show that $x = 0$ is a double root of $T_k(x)$ if k is odd. So, the proof is completed. □

3.3. Congruence properties of $S_k(x)$

In this subsection we give some useful lemmas which will be used to prove some of our main results.

Lemma 3.3. *([19], Lemma 1) If p is a prime, $d,\ q \in \mathbb{N}$, $k \in \mathbb{Z}^+$, $m_1 \in p^d\mathbb{N} \cup \{0\}$ and $m_2 \in p^d\mathbb{N} \cup \{0\}$, then*

$$S_k(qm_1 + m_2) \equiv qS_k(m_1) + S_k(m_2) \pmod{p^d}. \tag{3.12}$$

Proof. The proof is similar to Lemma 1 in [19]. □

Lemma 3.4. *([9], Lemma 3.2) Let x be a positive integer. Then we have*

$$v_3(S_k(x)) = \begin{cases} v_3(x(x+1)), & \text{if } k = 1, \\ v_3(x(x+1)(2x+1)) - 1, & \text{if } k \text{ is even}, \\ 0, & \text{if } x \equiv 1 \pmod 3 \text{ and } k \geq 3 \text{ is odd}, \\ v_3(kx^2(x+1)^2) - 1, & \text{if } x \equiv 0, 2 \pmod 3 \text{ and } k \geq 3 \text{ is odd}. \end{cases}$$

Lemma 3.5. *([19], Theorem 3) Let p be an odd prime and let m and k be positive integers.*

(i) *For some integer $d \geq 1$, we can write*

$$m = qp^d + r\frac{p^d - 1}{p - 1} = qp^d + rp^{d-1} + rp^{d-2} + \cdots + rp^0,$$

where $r \in \{0, 1, ..., p-1\}$ and $0 \leq q \not\equiv r \equiv m \pmod p$.

(ii) *In the case of $m \equiv 0 \pmod p$, we have*

$$S_k(m) \equiv \begin{cases} -p^{d-1} \pmod{p^d}, & \text{if } p - 1 \mid k, \\ 0 \pmod{p^d}, & \text{if } p - 1 \nmid k. \end{cases}$$

(iii) *In the case of $m \equiv -1 \pmod p$, we have*

$$S_k(m) \equiv \begin{cases} -p^{d-1}(q+1) \pmod{p^d}, & \text{if } p - 1 \mid k, \\ 0 \pmod{p^d}, & \text{if } p - 1 \nmid k. \end{cases}$$

(iv) *In the case of $m \equiv \frac{p-1}{2} \pmod p$, we have*

$$S_k(m) \equiv \begin{cases} -p^{d-1}(q+\frac{1}{2}) \pmod{p^d}, & \text{if } p - 1 \mid k, \\ 0 \pmod{p^d}, & \text{if } p - 1 \nmid k. \end{cases}$$

3.4. Linear forms in logarithms

For an algebraic number α of degree d over $\mathbb{Q}$, we define the *absolute logarithmic height* of α by the following formula:

$$\mathrm{h}(\alpha) = \frac{1}{d}\left(\log|a_0| + \sum_{i=1}^{d}\log\max\{1, |\alpha^{(i)}|\}\right),$$

where a_0 is the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$, and $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(d)}$ are the conjugates of α in the field of complex numbers.

Let α_1 and α_2 be multiplicatively independent algebraic numbers with $|\alpha_1| \geq 1$ and $|\alpha_2| \geq 1$. Consider the linear form in two logarithms:

$$\Lambda = b_2 \log\alpha_2 - b_1 \log\alpha_1,$$

where $\log\alpha_1, \log\alpha_2$ are any determinations of the logarithms of α_1, α_2 respectively, and b_1, b_2 are positive integers.

We shall use the following result due to Laurent [11].

Lemma 3.6 *([11], Theorem 2). Let ρ and μ be real numbers with $\rho > 1$ and $1/3 \leq \mu \leq 1$. Set*

$$\sigma = \frac{1+2\mu-\mu^2}{2}, \quad \lambda = \sigma\log\rho.$$

Let a_1, a_2 be real numbers such that

$$a_i \geq \max\{1,\ \rho|\log\alpha_i| - \log|\alpha_i| + 2D\mathrm{h}(\alpha_i)\} \qquad (i = 1, 2),$$

where

$$D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}] \,/\, [\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}].$$

Let h be a real number such that

$$h \geq \max\left\{D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log\lambda + 1.75\right) + 0.06,\ \lambda,\ \frac{D\log 2}{2}\right\}.$$

We assume that

$$a_1 a_2 \geq \lambda^2.$$

Put

$$H = \frac{h}{\lambda} + \frac{1}{\sigma}, \quad \omega = 2 + 2\sqrt{1+\frac{1}{4H^2}}, \quad \theta = \sqrt{1+\frac{1}{4H^2}} + \frac{1}{2H}.$$

Table 1
Bounding n and k under the indicated conditions.

x	n_0 $(y > 4x^2)$	n_1 $(y > 10^6)$	k_1 $(y \leq 4x^2)$
2	7,500	3,200	45,000
3	21,000	10,000	120,000
6	94,000	53,000	540,000
7	128,000	74,200	740,000
10	253,000	157,000	1,450,000
11	301,000	190,000	1,750,000

Then we have

$$\log|\Lambda| \geq -Ch'^2 a_1 a_2 - \sqrt{\omega\theta}h' - \log\left(C'h'^2 a_1 a_2\right)$$

with

$$h' = h + \frac{\lambda}{\sigma}, \quad C = C_0 \frac{\mu}{\lambda^3 \sigma}, \quad C' = \sqrt{\frac{C\sigma\omega\theta}{\lambda^3\mu}},$$

where

$$C_0 = \left(\frac{\omega}{6} + \frac{1}{2}\sqrt{\frac{\omega^2}{9} + \frac{8\lambda\omega^{5/4}\theta^{1/4}}{3\sqrt{a_1 a_2}H^{1/2}} + \frac{4}{3}\left(\frac{1}{a_1} + \frac{1}{a_2}\right)\frac{\lambda\omega}{H}}\right)^2.$$

3.5. A Baker type estimate

Let $A = \{2, 3, 6, 7, 10, 11\}$ and consider equation (1.7) with $x \in A$. The following lemma provides sharp upper bounds for the solutions n, k of the equation (1.7) and will be used in the proof of Theorem 2.3.

Lemma 3.7. *Let $A = \{2, 3, 6, 7, 10, 11\}$ and consider equation (1.7) with $x \in A$ in integer unknowns (k, y, n) with $k \geq 83, y \geq 2$ and $n \geq 3$ a prime. Then for $y > 4x^2$ we have $n \leq n_0$, for $y > 10^6$ even $n \leq n_1$ holds, and for $y \leq 4x^2$ we have $k \leq k_1$, where $n_0 = n_0(x), n_1 = n_1(x)$ and $k_1 = k_1(x)$ are given in Table 1.*

Proof. In the course of the proof we will always assume that $x \in A$ and we distinguish three cases according to $y > 4x^2$, $y > 10^6$ or $y \leq 4x^2$.

Case I. $y > 4x^2$

We may suppose, without loss of generality, that n is large enough, that is

$$n > n_0. \tag{3.13}$$

Further, by $k \geq 83$ we easily deduce that for every $x \in A$ we have

$$(x+1)^k + (x+2)^k + \cdots + (2x)^k < 2 \cdot (2x)^k, \tag{3.14}$$

and

$$(x+1)^k + (x+2)^k + \cdots + (2x-1)^k < 2 \cdot (2x-1)^k. \tag{3.15}$$

Since $y > 4x^2$ by (1.7), (3.14) and $x \geq 2$ we get that

$$k \geq 2n. \tag{3.16}$$

Using (3.16) and the fact that n is odd we may write k in the form

$$k = Bn + r \text{ with } B \geq 1,\ 0 \leq |r| \leq \frac{n-1}{2}. \tag{3.17}$$

We show that in (3.17) we have $r \neq 0$. On the contrary, suppose $r = 0$. Then, using (1.7) and (3.15) we infer by (3.17) that

$$\begin{aligned} 2(2x-1)^k &> (x+1)^k + (x+2)^k + \cdots + (2x-1)^k = y^n - (2x)^k = y^n - (2x)^{Bn} \\ &= (y-(2x)^B)(y^{n-1} + \cdots + (2x)^{B(n-1)}) \geq (2x)^{B(n-1)}. \end{aligned}$$

Hence

$$n < \frac{\log(2x)}{\log\left(\frac{2x}{2x-1}\right)} + \frac{\log 2}{B \log\left(\frac{2x}{2x-1}\right)}.$$

This together with $x \leq 11$ and $B \geq 1$ implies $n < 82$, which contradicts (3.13). Thus, $r \neq 0$.

On dividing equation (1.7), by y^n we obviously get

$$1 - \frac{(2x)^k}{y^n} = \frac{s}{y^n}, \tag{3.18}$$

where $s = (x+1)^k + \ldots + (2x-1)^k$. Using (3.17) and (3.18) we infer that

$$\left|(2x)^r \cdot \left(\frac{(2x)^B}{y}\right)^n - 1\right| = \frac{s}{y^n}. \tag{3.19}$$

Put

$$\Lambda_r = \begin{cases} r\log(2x) - n\log\frac{y}{(2x)^B} & \text{if } r > 0, \\ |r|\log(2x) - n\log\frac{(2x)^B}{y} & \text{if } r < 0. \end{cases} \tag{3.20}$$

In what follows we find upper and lower bounds for $\log|\Lambda_r|$. We distinguish two subcases according to

$$1-\frac{(2x)^k}{y^n}\geq 0.795 \text{ or } 1-\frac{(2x)^k}{y^n}<0.795,$$

respectively. If $1-\frac{(2x)^k}{y^n}\geq 0.795$ then by (1.7) and (3.14) we immediately obtain a contradiction, so we may assume that the latter case holds.

It is well known (see Lemma B.2 of [18]) that for every $z\in\mathbb{R}$ with $|z-1|<0.795$ one has

$$|\log z|<2|z-1|. \tag{3.21}$$

On applying inequality (3.21) with $z=(2x)^k/y^n$ we get by (3.18), (3.19), (3.20) and $(2x)^k\neq y^n$ that

$$|\Lambda_r|<\frac{2s}{y^n}. \tag{3.22}$$

Observe that (1.7) implies

$$k<\frac{n\log y}{\log 2x}. \tag{3.23}$$

Thus by (3.22), (3.15) and (3.23) we infer that

$$\log|\Lambda_r|<-\frac{\log\left(\frac{2x}{2x-1}\right)}{\log(2x)}(\log y)\,n+\log 4. \tag{3.24}$$

Next, for a lower bound for $\log|\Lambda_r|$, we shall use Lemma 3.6 with

$$(\alpha_1,\alpha_2,b_1,b_2)=\begin{cases}\left(\frac{y}{(2x)^B},2x,n,r\right) & \text{if } r>0,\\ \left(\frac{(2x)^B}{y},2x,n,|r|\right) & \text{if } r<0.\end{cases}$$

Using (1.7) and (3.14) one can easily check that $\alpha_1>1$ and $\alpha_2>1$. We show that α_1,α_2 are multiplicatively independent. Assume the contrary. Then the set of prime factors of y coincides with that of $2x$. This implies that y must be even. But for $x\in A$ we easily see that y is odd, which is a contradiction, proving that α_1 and α_2 are multiplicatively independent.

Now, we apply Lemma 3.6 for every $x\in A$ with

$$(\rho,\mu)=(7.7,0.57). \tag{3.25}$$

In what follows we shall derive upper bounds for the quantities

$$\rho|\log\alpha_i|-\log|\alpha_i|+2D\mathrm{h}(\alpha_i),\quad (i=1,2)$$

occurring in Lemma 3.6. Since $D=1$ and $\alpha_2>1$, for $i=2$ we get

$$\rho|\log\alpha_2| - \log|\alpha_2| + 2D\mathrm{h}(\alpha_2) = (\rho+1)\log 2x. \tag{3.26}$$

For $i = 1$ we obtain

$$\rho|\log\alpha_1| - \log|\alpha_1| + 2D\mathrm{h}(\alpha_1) < \frac{\rho+1}{2}\log 2x + 2\log y. \tag{3.27}$$

To verify that (3.27) is valid we shall estimate $\log\alpha_1$ and $\mathrm{h}(\alpha_1)$ from above, by using equation (1.7), i.e. $s + (2x)^{Bn+r} = y^n$. Observe

$$\mathrm{h}(\alpha_1) = \mathrm{h}\left(\frac{(2x)^B}{y}\right) \le \log\max\{(2x)^B, y\} = \begin{cases}\log y & \text{if } r > 0,\\ \log(2x)^B & \text{if } r < 0.\end{cases}$$

If $r > 0$, then

$$\alpha_1^n = \left(\frac{y}{(2x)^B}\right)^n = (2x)^r + \frac{s}{(2x)^{Bn}} = (2x)^r\left(1 + \frac{s}{(2x)^k}\right) < 2(2x)^r \ (\text{as } s < (2x)^k),$$

so

$$\log\alpha_1 < \frac{\log 2}{n} + \frac{r}{n}\log(2x) \le \frac{\log 2}{n} + \frac{n-1}{2n}\log(2x),$$

whence

$$\rho|\log\alpha_1| - \log|\alpha_1| + 2D\mathrm{h}(\alpha_1) <$$
$$< \left(\frac{\log 2}{n\log(2x)} + \frac{n-1}{2n}\right)(\rho-1)\log(2x) + 2\log y$$

which by (3.25) and $x \ge 2$ clearly implies (3.27).

If $r < 0$, then

$$\alpha_1^n = \left(\frac{(2x)^B}{y}\right)^n = (2x)^{-r}\left(1 - \frac{s}{y^n}\right) < (2x)^{-r} = (2x)^{|r|},$$

so

$$\log\alpha_1 < \frac{|r|}{n}\log(2x) \le \frac{n-1}{2n}\log(2x),$$

and

$$\log(2x)^B = \log\alpha_1 + \log y < \frac{n-1}{2n}\log(2x) + \log y,$$

and we get

$$\rho|\log\alpha_1| - \log|\alpha_1| + 2D\mathrm{h}(\alpha_1) < \left(\frac{n-1}{2n}(\rho-1) + \frac{n-1}{n}\right)\log(2x) + 2\log y,$$

which by (3.13) again implies (3.27).

Table 2
Choosing the parameter $h = \log n + \varepsilon$ occurring in Lemma 3.6 if the case $y > 4x^2$.

x	2	3	6	7	10	11
ε	0.3560	0.0995	−0.2275	−0.2877	−0.4144	−0.4458

Table 3
Lower bounds for H and upper bounds for $\omega, \theta, C_0, C, C', h'$ occurring in Lemma 3.6 if $y > 4x^2$.

x	H	ω	θ	C_0	C	C'	h'
2	6.11	4.0067	1.0852	2.3688	0.2341	0.51	$\log n + 2.3974$
3	6.52	4.0059	1.0797	2.2241	0.2198	0.50	$\log n + 2.1409$
6	7.16	4.0049	1.0723	2.0867	0.2062	0.50	$\log n + 1.8139$
7	7.29	4.0047	1.0710	2.0662	0.2042	0.50	$\log n + 1.7537$
10	7.59	4.0044	1.0681	2.0271	0.2003	0.50	$\log n + 1.6270$
11	7.67	4.0043	1.0674	2.0182	0.1994	0.50	$\log n + 1.5956$

In view of (3.26) we can obviously take for every $x \in A$

$$a_2 = (\rho + 1)\log(2x), \tag{3.28}$$

while for the values a_1 we use the upper bound occurring in (3.27). Namely, we can take a_1 as

$$a_1 = \frac{\rho+1}{2}\log(2x) + 2\log y \quad \text{if} \quad x \in A \tag{3.29}$$

Since $\mu = 0.57$ we get

$$\sigma = 0.90755 \text{ and } \lambda = 0.90755 \log\rho, \tag{3.30}$$

whence by (3.25), (3.28), (3.29), (3.30) and $y > 4x^2$ we easily check that for every $x \in A$

$$a_1 a_2 > \lambda^2$$

holds. Now, we are going to derive an upper bound h for the quantity

$$\max\left\{D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log\lambda + 1.75\right) + 0.06,\ \lambda,\ \frac{D\log 2}{2}\right\}.$$

Using $D = 1$, (3.25), (3.28), (3.29), (3.30) and $y > 4x^2$, for the values of h occurring in Lemma 3.6 we obtain $h = \log n + \varepsilon$, with $\varepsilon = \varepsilon(x)$ given in Table 2.

Further, by (3.13) we easily check that for the above values of h assumptions of Lemma 3.6 concerning the parameter h are satisfied. Using (3.13) again we obtain a lower bound for H and hence upper bounds for ω and θ. Moreover, using these values of ω and θ by (3.25), (3.28), (3.29), (3.30) and $y > 4x^2$ for $x \in A$ we obtain Table 3.

By Lemma 3.6 we obtain

$$\log|\Lambda_r| > -C{h'}^2 a_1 a_2 - \sqrt{\omega\theta}h' - \log(C'{h'}^2 a_1 a_2), \tag{3.31}$$

Table 4
Choosing the parameter $h = \log n + \varepsilon$ occurring in Lemma 3.6 if $y > 10^6$.

x	2	3	6	7	10	11
ε	0.0324	−0.1870	−0.4620	−0.5122	−0.6079	−0.6339

Table 5
Lower bounds for H and upper bounds for $\omega, \theta, C_0, C, C', h'$ occurring in Lemma 3.6 if $y > 10^6$.

x	H	ω	θ	C_0	C	C'	h'
2	5.04	4.0099	1.1042	2.1846	0.1587	0.36	$\log n + 2.2943$
3	5.49	4.0083	1.0953	2.1067	0.1531	0.36	$\log n + 2.0749$
6	6.17	4.0066	1.0844	2.0259	0.1472	0.36	$\log n + 1.7999$
7	6.31	4.0063	1.0824	2.0130	0.1463	0.36	$\log n + 1.7497$
10	6.71	4.0056	1.0773	1.9871	0.1506	0.36	$\log n + 1.6222$
11	6.79	4.0055	1.0764	1.9813	0.1502	0.36	$\log n + 1.5962$

whence on comparing (3.24) with (3.31) we get

$$n < \left(\frac{Ch'^2 a_1 a_2}{\log y} + \frac{\sqrt{\omega\theta}}{\log y} h' + \frac{\log(C'h'^2 a_1 a_2)}{\log y} + \frac{\log 4}{\log y} \right) \frac{\log 2x}{\log \frac{2x}{2x-1}}. \tag{3.32}$$

Finally, using (3.28), (3.29) and $y > 4x^2$ for $x \in A$, by Table 3 we easily see that inequality (3.32) contradicts (3.13), proving the desired bounds for n in this case.

Case II. $y > 10^6$

We work as in the previous case. Namely, we apply Lemma 3.6 again, the only difference is that in this case for y we may write $y > 10^6$. We may suppose, without loss of generality, that n is large enough, that is

$$n > n_1. \tag{3.33}$$

Further, we choose $\mu = 0.57$ uniformly, and set

$$\rho = \begin{cases} 9.6 & \text{if} \quad x = 2, 3, 6, 7 \\ 9.3 & \text{if} \quad x = 10, 11. \end{cases} \tag{3.34}$$

As before, we may take a_1 and a_2 as in (3.29) and (3.28).

Thus by (3.34), (3.29), (3.28) and $y > 10^6$ for the values of h occurring in Lemma 3.6 we obtain $h = \log n + \varepsilon$, with $\varepsilon = \varepsilon(x)$ given in Table 4.

On combining (3.28), (3.29), (3.33), (3.34) with $y > 10^6$ and with Table 4 we obtain Table 5.

By Lemma 3.6 we obtain

$$\log |\Lambda_r| > -Ch'^2 a_1 a_2 - \sqrt{\omega\theta} h' - \log(C'h'^2 a_1 a_2), \tag{3.35}$$

whence on comparing (3.24) with (3.35) we obtain

$$n < \left(\frac{Ch'^2 a_1 a_2}{\log y} + \frac{\sqrt{\omega\theta}}{\log y}h' + \frac{\log(C'h'^2 a_1 a_2)}{\log y} + \frac{\log 4}{\log y}\right)\frac{\log 2x}{\log \frac{2x}{2x-1}}. \tag{3.36}$$

Finally, using (3.28), (3.29) and $y > 10^6$, by Table 5 we see that (3.36) contradicts (3.33), proving the validity of the desired bounds for n in this case.

Case III. $y \le 4x^2$

In order to obtain the desired upper bounds for k we may clearly assume that k is large, namely

$$k > k_1. \tag{3.37}$$

Since $y \le 4x^2$ we have by (1.7) that

$$n > \lfloor k/2 \rfloor. \tag{3.38}$$

Hence by (3.38), we can write

$$n = Bk + r \quad \text{with} \quad B \ge 1,\ 0 \le |r| \le \left\lfloor \frac{k}{2} \right\rfloor. \tag{3.39}$$

Further, using the same argument as in Case I, by $x \in A$ and $k \ge 83$ we may suppose that in (3.39) we have $r \ne 0$.

We divide our equation (1.7) by $(2x)^k$. Then, by (3.39) we infer

$$y^r \left(\frac{y^B}{2x}\right)^k - 1 = \frac{s}{x^k}, \tag{3.40}$$

where $s = (x+1)^k + 2^k + \cdots + (2x-1)^k$. Thus, $y^r \left(\frac{y^B}{2x}\right)^k > 1$. Put

$$\Lambda_r = b_2 \log \alpha_2 - b_1 \log \alpha_1, \tag{3.41}$$

where

$$(\alpha_1, \alpha_2, b_1, b_2) = \begin{cases} \left(\frac{2x}{y^B}, y, k, r\right) & \text{if } r > 0, \\ \left(\frac{y^B}{2x}, y, k, |r|\right) & \text{if } r < 0. \end{cases} \tag{3.42}$$

It is easy to see $\alpha_1 > 1$ and $\alpha_2 > 1$, moreover similarly to Case I we obtain that α_1 and α_2 are multiplicatively independent. We find upper and lower bounds for $\log|\Lambda_r|$. Since for every $z \in \mathbb{R}$ with $z > 1$ we have $|\log z| < |z - 1|$ it follows by (3.40), (3.41), (3.42) and (3.15) that

Table 6
Choosing the parameter $h = \log n + \varepsilon$ occurring in Lemma 3.6 if the case $y \leq 4x^2$.

x	2	3	6	7	10	11
ε	−0.1099	−0.3665	−0.6935	−0.7537	−0.8805	−0.9118

Table 7
Lower bounds for H and upper bounds for $\omega, \theta, C_0, C, C', h'$ occurring in Lemma 3.6 if $y \leq 4x^2$.

x	H	ω	θ	C_0	C	C'	h'
2	7.50	4.0045	1.0689	2.1294	0.2947	0.67	$\log k + 1.7145$
3	7.94	4.0040	1.0650	2.0435	0.2828	0.67	$\log k + 1.4580$
6	8.65	4.0034	1.0595	1.9620	0.2715	0.67	$\log k + 1.1309$
7	8.80	4.0033	1.0585	1.9498	0.2698	0.67	$\log k + 1.0708$
10	9.13	4.0030	1.0563	1.9265	0.2666	0.67	$\log k + 0.9440$
11	9.23	4.0030	1.0557	1.9212	0.2659	0.67	$\log k + 0.9127$

$$\log |\Lambda_r| < -k \log \left(\frac{2x}{2x-1} \right) + \log 2. \tag{3.43}$$

For a lower bound, we again use Lemma 3.6. We choose $\mu = 0.57$ uniformly, and we set for every $x \in A$

$$\rho = 6.2. \tag{3.44}$$

Moreover, using the same argument as in Case I by $y \leq 4x^2$ we may take

$$a_1 = 1.02 \cdot (\rho + 3) \log(2x), \tag{3.45}$$

and

$$a_2 = 2 \cdot (\rho + 1) \log 2x. \tag{3.46}$$

Since $\mu = 0.57$ we get

$$\sigma = 0.90755 \text{ and } \lambda = 0.90755 \log \rho, \tag{3.47}$$

whence by (3.44), (3.45), (3.46), (3.47) we easily check that for every $x \in A$

$$a_1 a_2 > \lambda^2$$

holds. Now, we are going to derive an upper bound h for the quantity

$$\max \left\{ D \left(\log \left(\frac{b_1}{a_2} + \frac{b_2}{a_1} \right) + \log \lambda + 1.75 \right) + 0.06, \ \lambda, \ \frac{D \log 2}{2} \right\}.$$

Using (3.44), (3.45), (3.46), (3.47) for h occurring in Lemma 3.6 we obtain $h = \log n + \varepsilon$, with $\varepsilon = \varepsilon(x)$ given in Table 6.

On combining (3.37), (3.44), (3.45), (3.46), (3.47) with Table 6 we obtain Table 7.

Further, on using Table 7 and Lemma 3.6 we obtain

$$\log|\Lambda_r| > -Ch'^2 a_1 a_2 - \sqrt{\omega\theta}h' - \log(C'h'^2 a_1 a_2), \tag{3.48}$$

whence, on comparing (3.43) with (3.48) we get

$$k < \frac{Ch'^2 a_1 a_2 + \sqrt{\omega\theta}h' + \log(2C'h'^2 a_1 a_2)}{\log\left(\frac{2x}{2x-1}\right)}.$$

Finally, using (3.44), (3.45), (3.46), by Table 7 we obtain the desired bounds for k in this case. Thus our lemma is proved. □

4. Formulas for $v_2(T_k(x)), v_3(T_k(x))$

For the proofs of our main results, we will need formulas for $v_2(T_k(x))$ and $v_3(T_k(x))$. The heart of the proof of Lemma 4.2 is the following lemma

Lemma 4.1. *For $q, k, t \geq 1$ and $q \equiv 1 \pmod 2$, we have*

$$v_2(T_k(2^t q)) = \begin{cases} t-1, & \text{if } k = 1 \text{ or } k \text{ is even,} \\ 2t-2, & \text{if } k \geq 3 \text{ is odd.} \end{cases}$$

Proof. We shall follow the proof of Lemma 1 of Macmillian–Sondow [13]. We induct on t. Now we introduce the following equality

$$T_k(x) = S_k(2x) - S_k(x), \tag{4.1}$$

which we will use frequently on this work. Since $S_k(2^2 q)$ is even and $S_k(2q)$ is odd, by using (4.1), we get $v_2(T_k(2q)) = 0$ and so Lemma 4.1 holds for $t = 1$. By Lemma 3.2 with $x = 2^t q$, it also holds for all $t \geq 1$ when $k = 1$. Now we assume inductively that (4.1) is true for fixed $t \geq 1$.

Let m be a positive integer, we can write the power sum $S_k(2m)$ as

$$\begin{aligned} S_k(2m) = m^k + \sum_{j=1}^{m}((m-j)^k + (m+j)^k) &= m^k + 2\sum_{j=1}^{m}\sum_{i=0}^{[\frac{k}{2}]}\binom{k}{2i}m^{k-2i}j^{2i} \\ &= m^k + 2\sum_{i=0}^{[\frac{k}{2}]}\binom{k}{2i}m^{k-2i}S_{2i}(m). \end{aligned} \tag{4.2}$$

By (4.1), putting $x = m$, we have

$$T_k(m) = S_k(2m) - S_k(m) \tag{4.3}$$

Now we consider (4.3) with $m = 2^t q$. If $k \geq 2$ is even, we extract the last terms of the summations of $S_k(2m)$ and $S_k(m)$, then we can write as

$$S_k(2^{t+1}q) = 2^{kt}q^k + 2^t \frac{S_k(2^t q)}{2^{t-1}} + 2^{2t+1} \sum_{i=0}^{\frac{k-2}{2}} \binom{k}{2i} 2^{t(k-2i-2)} q^{k-2i} S_{2i}(2^t q)$$

and

$$S_k(2^t q) = 2^{k(t-1)}q^k + 2^{t-1} \frac{S_k(2^{t-1}q)}{2^{t-2}} + 2^{2t-1} \sum_{i=0}^{\frac{k-2}{2}} \binom{k}{2i} 2^{(t-1)(k-2i-2)} q^{k-2i} S_{2i}(2^{t-1}q).$$

Hence we have

$$\begin{aligned} T_k(2^t q) = {} & 2^{k(t-1)}q^k(2^k - 1) + 2^{t-1}[2\frac{S_k(2^t q)}{2^{t-1}} - \frac{S_k(2^{t-1}q)}{2^{t-2}}] \\ & + 2^{2t+1} \sum_{i=0}^{\frac{k-2}{2}} \binom{k}{2i} 2^{t(k-2i-2)} q^{k-2i} S_{2i}(2^t q) \\ & - 2^{2t-1} \sum_{i=0}^{\frac{k-2}{2}} \binom{k}{2i} 2^{(t-1)(k-2i-2)} q^{k-2i} S_{2i}(2^{t-1}q). \end{aligned}$$

By the induction hypothesis, the fraction is actually an odd integer. Since $k(t-1) > t-1$, we get that $v_2(T_k(2^t q)) = t - 1$, as desired.

Now we consider the case $k \geq 3$ is odd. Similarly to the former case, we have

$$S_k(2^{t+1}q) = 2^{tk}q^k + 2^{2t}kq \frac{S_{k-1}(2^t q)}{2^{t-1}} + 2^{3t+1} \sum_{i=0}^{\frac{k-3}{2}} \binom{k}{2i} 2^{t(k-2i-3)} q^{k-2i} S_{2i}(2^t q)$$

and

$$S_k(2^t q) = 2^{k(t-1)}q^k + 2^{2t-2}kq \frac{S_{k-1}(2^{t-1}q)}{2^{t-2}} + 2^{3t-2} \sum_{i=0}^{\frac{k-3}{2}} \binom{k}{2i} 2^{(t-1)(k-2i-3)} q^{k-2i} S_{2i}(2^{t-1}q).$$

From here, we get

$$T_k(2^t q) = 2^{k(t-1)}q^k(2^k - 1) + 2^{2t-2}kq[\frac{2^2 S_{k-1}(2^t q)}{2^{t-1}} - \frac{S_{k-1}(2^{t-1}q)}{2^{t-2}}]$$

$$+ 2^{3t+1}\sum_{i=0}^{\frac{k-3}{2}}\binom{k}{2i}2^{t(k-2i-3)}q^{k-2i}S_{2i}(2^t q)$$

$$- 2^{3t-2}\sum_{i=0}^{\frac{k-3}{2}}\binom{k}{2i}2^{(t-1)(k-2i-3)}q^{k-2i}S_{2i}(2^{t-1} q).$$

Again by induction, the fraction is an odd integer.

Since $k(k-1) > 2(t-2)$ and k and q are odd, wee see that $v_2(T_k(2^t q)) = 2t-2$, as required. This completes the proof of Lemma. □

Lemma 4.2. (*i*) *Let x be a positive even integer. Then we have,*

$$v_2(T_k(x)) = \begin{cases} v_2(x)-1, & \text{if } k=1 \text{ or } k \text{ is even,} \\ 2v_2(x)-2, & \text{if } k \geq 3 \text{ is odd.} \end{cases}$$

(*ii*) *Let x be a positive odd integer. If x is odd and $k=1$, then for any solution (k,n,x,y) of (1.7) we get $v_2(T_k(x)) = v_2(3x+1)-1$.*

If $x \equiv 1, 5 \pmod 8$ and $x \not\equiv 1 \pmod{32}$ with $k \neq 1$, then we have

$$v_2(T_k(x)) = \begin{cases} v_2(7x+1)-1, & \text{if } x \equiv 1 \pmod 8 \text{ and } k=2, \\ v_2((5x+3)(3x+1))-2, & \text{if } x \equiv 1 \pmod 8 \text{ and } k=3, \\ v_2(3x+1), & \text{if } x \equiv 5 \pmod 8 \text{ and } k \geq 3 \text{ is odd,} \\ 1, & \text{if } x \equiv 5 \pmod 8 \text{ and } k \geq 2 \text{ is even,} \\ 2, & \text{if } x \equiv 9 \pmod{16} \text{ and } k \geq 4 \text{ is even,} \\ 3, & \text{if } x \equiv 9 \pmod{16} \text{ and } k \geq 5 \text{ is odd} \\ & \text{or} \\ & \text{if } x \equiv 17 \pmod{32} \text{ and } k \geq 4 \text{ is even,} \\ 4, & \text{if } x \equiv 17 \pmod{32} \text{ and } k \geq 5 \text{ is odd.} \end{cases}$$

If $x \equiv 3, 7 \pmod 8$, then for any solution (k,n,x,y) of (1.7), we obtain $v_2(T_k(x)) = 0$.

Proof. (*i*) Firstly, if $k \geq 2$ is even, since $2x+1$ is always odd, then we have

$$v_2(\frac{x(2x+1)}{2}) = v_2(x) - 1.$$

Putting $x = 2^t q$ where q is odd and $t \geq 1$, we get

$$v_2(\frac{2^t q(2^{t+1}q+1)}{2}) = v_2(2^{t-1}q) = t-1.$$

Secondly if we consider the case $k \geq 3$ is odd, then

$$v_2(\frac{x^2(3x+1)}{4}) = v_2(x^2) - 2.$$

Putting $x = 2^t q$, we have

$$v_2((2^t q)^2) - 2 = v_2(2^{2t}) - 2 = 2t - 2.$$

Finally, in the case of $k = 1$, we have

$$v_2(\frac{x(3x+1)}{2}) = v_2(x) - 1.$$

Set $x = 2^t q$, we have

$$v_2(2^t q) - 1 = t - 1.$$

So, the proof is completed.

(ii) Since $S_1(x) = \frac{x(x+1)}{2}$, $S_2(x) = \frac{x(x+1)(2x+1)}{6}$ and $S_3(x) = (\frac{x(x+1)}{2})^2$ for any positive integer x, by (4.1) if x is odd or $x \equiv 1 \pmod 8$, then the statement is automatic for $k = 1$ or $k = 2, 3$, respectively.

Next we consider the case $x \equiv 5 \pmod 8$ and $k \geq 3$ is odd. Since $3x+1 \equiv 0 \pmod 8$, we have $3x + 1 = 2^d r$ with $d \geq 3$, $2 \nmid r$. So we obtain

$$v_2(3x+1) = d \tag{4.4}$$

Since x is odd, $T_k(x)$ has exactly odd terms. Putting $x = \frac{2^d r-1}{3}$ in (1.8), we have

$$T_k(\frac{2^d r - 1}{3}) = (\frac{1}{3})^k[(2^d r+2)^k + (2^d r+5)^k + \cdots + (32^{d-1}r)^k + \cdots + (2^{d+1}r-2)^k] \tag{4.5}$$

which has $(32^{d-1}r)^k$ as the middle term of expansion. Considering (4.5) in modulo 2^d with $k(d-1) > d$, we obtain $T_k(\frac{2^d r-1}{3}) \equiv 0 \pmod{2^d}$. Then we have $v_2(T_k(\frac{2^d r-1}{3})) = v_2(2^d t) = d$ with $2 \nmid t$. By (4.4), the statement follows in this case, as well.

Now we consider the case $x \equiv 5 \pmod 8$ and $k \geq 2$ is even. We distinguish two cases. Assume first $k \geq 4$ is even. Using the polynomial

$$Q_k(x) = x^k + (x+1)^k + (x+2)^k + ... + 2^k(x-1)^k \tag{4.6}$$

and the equality

$$T_k(x) = Q_k(x) - x^k + (2x-1)^k + (2x)^k \tag{4.7}$$

we obtain

$$T_k(x) \equiv Q_k(x) \pmod 8.$$

Then we have

$$v_2(T_k(x)) = v_2(Q_k(x)) \tag{4.8}$$

Applying Lemma 4.2 (i) on the polynomial $Q_k(x)$ we obtain $v_2(Q_k(x)) = v_2(x-1) - 1$ and hence the statement follows also in this case. For the case $k = 2$, by (4.1) we get also $v_2(T_k(x)) = v_2(7x+1) - 1 = 1$.

Next we consider the case $x \equiv 9 \pmod{16}$ and $k \geq 5$ is odd. By (4.7) we have

$$T_k(x) \equiv Q_k(x) + 8 \pmod{16} \tag{4.9}$$

Using Lemma 4.2 (ii), we have $v_2(Q_k(x)) = 2v_2(x-1) - 2$. So, we get $v_2(Q_k(x)) = 4$ and

$$Q_k(x) = 2^4 t, \quad 2 \nmid t \tag{4.10}$$

By (4.9) and (4.10), the statement follows in this case.

Now we consider the case $x \equiv 9 \pmod{16}$ and $k \geq 4$ is even. By (4.7) we have

$$T_k(x) \equiv Q_k(x) \pmod{16} \tag{4.11}$$

Using Lemma 4.2 (i), we get $v_2(Q_k(x)) = v_2(x-1) - 1$. So we get $v_2(T_k(x)) = v_2(Q_k(x)) = 2$ with (4.11).

Next we consider the case $x \equiv 17 \pmod{32}$ and $k \geq 4$ is even. We distinguish two cases. If $k = 4$ then,

$$T_4(x) \equiv Q_4(x) + 16 \pmod{32} \tag{4.12}$$

Using Lemma 4.2 (i) we obtain $v_2(Q_4(x)) = 3$ and

$$Q_4(x) = 2^3 r, 2 \nmid r \tag{4.13}$$

By (4.12) and (4.13), we get $v_2(T_4(x)) = 3$. For the case $k \geq 6$ is even, by (4.7) we have

$$T_k(x) \equiv Q_k(x) \pmod{32}$$

Similar to the former cases, we obtain $v_2(T_k(x)) = 3$.

Now we consider $x \equiv 17 \pmod{32}$ and $k \geq 5$ is odd, by (4.7) we have

$$T_k(x) \equiv Q_k(x) + 16 \pmod{32} \tag{4.14}$$

By Lemma 4.2 (ii), we have $v_2(Q_k(x)) = 6$. With (4.14) similar to the former cases, we get $v_2(T_k(x)) = 4$.

Next we consider the case $x \equiv 3 \pmod 8$. By (4.7) we obtain

$$T_k(x) \equiv Q_k(x) + 2 \pmod 8$$

or

$$T_k(x) \equiv Q_k(x) \pmod 8$$

where k is odd or even, respectively. In both cases we obtain $v_2(Q_k(x)) = 0$ using Lemma 4.2. Then the statement follows in this case.

Now we consider the case $x \equiv 7 \pmod 8$. By (4.7) we get

$$T_k(x) \equiv Q_k(x) + 6 \pmod 8$$

or

$$T_k(x) \equiv Q_k(x) + 2 \pmod 8$$

where k is odd or even, respectively. In both cases, we get $v_2(Q_k(x)) = 0$ using Lemma 4.2. Then the statement follows in this case, as well. So, the proof of Lemma is completed. □

Lemma 4.3. *Assume that k is not even if $x \equiv 5 \pmod 9$. Then we have*

$$v_3(T_k(x)) = \begin{cases} v_3(x), & \text{if } k=1, \\ v_3(x) - 1, & \text{if } x \equiv 0 \pmod 3 \text{ and } k \geq 2 \text{ is even}, \\ v_3(kx^2), & \text{if } x \equiv 0 \pmod 3 \text{ and } k > 3 \text{ is odd}, \\ v_3(x^2(5x+3)), & \text{if } x \equiv 0 \pmod 3 \text{ and } k = 3, \\ 0, & \text{if } x \equiv \pm 1 \pmod 3 \text{ and } k \geq 3 \text{ is odd}, \\ 0, & \text{if } x \equiv 2, 8 \pmod 9 \text{ and } k \geq 2 \text{ is even}, \\ v_3(2x+1) - 1, & \text{if } x \equiv 1 \pmod 3 \text{ and } k \geq 2 \text{ is even}. \end{cases}$$

Proof. When $k = 1$, $T_1(x) = \frac{x(3x+1)}{2}$. Then statement is shown automatically.

When $x \equiv 0 \pmod 3$ and $k \geq 2$ is even, by (3.12) we have

$$S_k(2x) \equiv 2S_k(x) \pmod{3^d}, \text{ with p=3}. \tag{4.15}$$

Considering (4.1) in modulo 3^d, with (4.15) we have

$$T_k(x) \equiv S_k(x) \pmod{3^d}. \tag{4.16}$$

Using Lemma 3.5 (*ii*) and (4.16), we get $T_k(x) \equiv -3^{d-1}q \pmod{3^d}$. And hence $v_3(T_k(x)) = d - 1$. This is desired case.

When $x \equiv 0 \pmod 3$ and $k > 3$ is odd, writing $x = q3^d$ with $k = 3^\gamma k'$ and $q \nmid 3$, by Lemma 3.4 we have

$$v_3(S_k(2x)) = v_3(S_k(x)) = \gamma + 2d - 1 \tag{4.17}$$

Using (4.1) and (4.17), we get

$$T_k(x) \equiv 0 \pmod{3^{\gamma+2d}}.$$

And hence $v_3(T_k(x)) = v_3(kx^2) = \gamma + 2d$.

When $x \equiv 0 \pmod 3$ and $k = 3$, we have $T_3(x) = \frac{x^2(5x+3)(3x+1)}{4}$. Since $3x + 1 \equiv 1 \pmod 3$, the statement follows in this case.

When $x \equiv 1 \pmod 3$ and $k \geq 3$ is odd, using Lemma 3.4, $v_3(S_k(2x)) = v_3(kx^2(x+1)^2) - 1$ and $v_3(S_k(x)) = 0$. By (4.1) the statement follows in this case.

When $x \equiv 2 \pmod 3$ and $k \geq 3$ is odd, using Lemma 3.4, similar to the former case we obtain $v_3(T_k(x)) = 0$ with (4.2).

When $x \equiv 8 \pmod 9$ or $x \equiv 2 \pmod 9$ and $k \geq 2$ is even, by (4.1) and Lemma 3.4 we get $v_3(S_k(2x)) = v_3(2x(2x+1)(4x+1)) - 1$ or $v_3(S_k(2x)) = v_3(x(x+1)(2x+1)) - 1$, respectively. If $x \equiv 8 \pmod 9$, then $v_3(S_k(2x)) = 0$ and hence $v_3(T_k(x)) = v_3(S_k(2x))$. If $x \equiv 2 \pmod 9$, then $v_3(S_k(x)) = 0$ and hence $v_3(T_k(x)) = v_3(S_k(x))$.

Assume now that $x \equiv 1 \pmod 3$ and $k \geq 2$ is even. Applying Lemma 3.5 (iv), with (4.2) we obtain

$$T_k(x) \equiv 3^{d-1}(-\frac{1}{2}) \pmod{3^d}.$$

And hence $v_3(T_k(x)) = d - 1$. By Lemma 3.5, we write $x = q3^d + r\frac{3^d-1}{2}$ where $r \equiv 1 \pmod 3$, $0 \leq q \not\equiv r \equiv x \pmod 3$. So we get $2x+1 = 3^d(2q+1)$. Since $v_3(2x+1)-1 = d-1$, the statement follows in this case. So the proof is completed. □

5. Proofs of the main results

Now we are ready to prove our main results. We start with Theorem 2.1, since it will be used in the proofs of the other statements.

Proof of Theorem 2.1. (i) Since $x \equiv 0 \pmod 4$, by Lemma 4.2 we have $v_2(T_k(x)) > 0$, i.e. $T_k(x)$ is even. Thus if (1.7) satisfies, then $v_2(y) > 0$ and we have

$$nv_2(y) = v_2(y^n) = v_2(T_k(x)) = \begin{cases} v_2(x) - 1, & \text{if } k \text{ is even,} \\ 2v_2(x) - 2, & \text{if } k \text{ is odd,} \end{cases}$$

implying the statement in this case.

(ii) As now $x \equiv 1, 5 \pmod 8$ and $x \not\equiv 1 \pmod{32}$ with $k \neq 1$, Lemma 4.2 (ii) implies that $v_2(T_k(x)) > 0$. Hence (1.7) gives $v_2(y) > 0$ and we have

$$nv_2(y) = v_2(T_k(x))$$
$$= \begin{cases} v_2(7x+1)-1, & \text{if } x \equiv 1 \pmod 8 \text{ and k=2},\\ v_2((5x+3)(3x+1))-2, & \text{if } x \equiv 1 \pmod 8 \text{ and } k=3,\\ v_2(3x+1), & \text{if } x \equiv 5 \pmod 8 \text{ and } k \geq 3 \text{ is odd},\\ 1, & \text{if } x \equiv 5 \pmod 8 \text{ and } k \geq 2 \text{ is even},\\ 2, & \text{if } x \equiv 9 \pmod{16} \text{ and } k \geq 4 \text{ is even},\\ 3, & \text{if } x \equiv 9 \pmod{16} \text{ and } k \geq 5 \text{ is odd}\\ & \text{or}\\ & \text{if } x \equiv 17 \pmod{32} \text{ and } k \geq 4 \text{ is even},\\ 4, & \text{if } x \equiv 17 \pmod{32} \text{ and } k \geq 5 \text{ is odd}. \end{cases}$$

And if $x \equiv 1 \pmod 4$ and $k = 1$, then Lemma 4.2 (ii) also implies that $v_2(T_k(x)) > 0$. Hence (1.7) gives $v_2(y) > 0$ and we obtain $nv_2(y) = v_2(y^n) = v_2(T_k(x)) = v_2(3x+1)-1$. Implying the statement in this case, as well. So, the proof of the case (ii) is completed.

(iii) Suppose now that $x \equiv 0 \pmod 3$ and k is odd or $x \equiv 0, 1 \pmod 3$ and $k \geq 2$ is even, by Lemma 4.3 implies that $v_3(y) > 0$ and we have

$$nv_3(y) = v_3(T_k(x))$$
$$= \begin{cases} v_3(x), & \text{if } x \equiv 0 \pmod 3 \text{ and k=1},\\ v_3(x)-1, & \text{if } x \equiv 0 \pmod 3 \text{ and } k \geq 2 \text{ is even},\\ v_3(kx^2), & \text{if } x \equiv 0 \pmod 3 \text{ and } k > 3 \text{ is odd},\\ v_3(x^2(5x+3)), & \text{if } x \equiv 0 \pmod 3 \text{ and } k = 3,\\ v_3(2x+1)-1, & \text{if } x \equiv 1 \pmod 3 \text{ and } k \geq 2 \text{ is even}. \end{cases}$$

So, the proof of Theorem 2.1 is completed. □

Proof of Theorem 2.2. Observe that since $x \equiv 4 \pmod 8$, we have $v_2(T_k(x)) = v_2(x) - 1 = 1$. Hence if $k = 1$ or k is even then by part (i) of Theorem 2.1 we obtain $n \leq 1$, which is impossible. Since $x \equiv 5 \pmod 8$, we have $v_2(T_k(x)) = 1$. Hence if $k \geq 2$ is even then by part (ii) of Theorem 2.1 we obtain $n \leq 1$, which is impossible. Since $x \equiv 1 \pmod 8$, we have $v_2(T_k(x)) = v_2(3x+1) - 1 = 1$. Hence if $k = 1$ then by part (ii) of Theorem 2.1 we obtain $n \leq 1$, which is impossible. Thus, the proof is completed. □

Proof of Theorem 2.3. Let $2 \leq x \leq 13$ and consider equation (1.7) in unknown integers (k, y, n) with $k \geq 1, y \geq 2$ and $n \geq 3$. We distinguish two cases according to $x \in \{2, 3, 6, 7, 10, 11\}$ or $x \in \{4, 5, 8, 9, 12, 13\}$, respectively.

Assume first that $x \in \{2, 3, 6, 7, 10, 11\}$ is fixed. In this case for $k \le 83$ a direct computation shows that $T_k(x)$ is not a perfect nth power, so equation (1.7) has no solution. Now we assume that $k \ge 83$. Now we split the treatment into 3 subcases according to the size of y. If $y \le 4x^2$ then Lemma 3.7 shows that $k \le k_1$. Further, if $4x^2 < y \le 10^6$ then we get $n \le n_0$ by Lemma 3.7 and thus $T_k(x) \le 10^{6n_0}$, which in turn gives

$$k < \frac{6n_0 \log 10}{\log(2x)}.$$

So for each x under the assumption $y \le 10^6$ we get a bound for k and we check for each k below this bound and each $x \in \{2, 3, 6, 7, 10, 11\}$ if $T_k(x)$ has a prime factor p with $p \le y$. If not, then we are done, however, if such a p exists, then we also show, that for at least one such p we have $\nu_p(T_k(x)) \le 12$, which shows that $n \le 12$. For $y < 10^6$, $3 \le n \le 12$ we get again very good bound for k and a direct check will show that equation (1.7) has no solutions.

Now it is only left the case $y > 10^6$, in which case we get $n < n_1$ by Lemma 3.7, and for each fixed $3 \le n \le n_1$ we proceeded as follows. Recall that $x \in \{2, 3, 6, 7, 10, 11\}$ is fixed, and we also fixed $3 \le n \le n_1$. We took primes of the form $p := 2in + 1$ with $i \in \mathbb{Z}$ and we considered equation (1.7) locally modulo these primes. More precisely, we took the smallest such prime p_1 and put $o_1 := p_1 - 1$. Then for all values of $k = 1, \dots, o_1$ we checked whether $T_k(x) \pmod{p_1}$ is a perfect power or not, and we built the set $K(o_1)$ of all those values of $k \pmod{o_1}$ for which $T_k(x) \pmod{p}$ was a perfect power. In principle this provided a list of all possible values of $k \pmod{o_1}$ for which we might have a solution. Then we considered the next prime p_2 of the form $p_2 := 2in + 1$ with $i \in \mathbb{Z}$ and we defined $o_2 := \mathrm{LCM}(o_1, p_2 - 1)$. We expanded the set $K(o_1)$ to the set $K_0(o_2)$ of all those numbers $1, \dots, o_2$ which are congruent to elements of $K(o_1)$ modulo o_1. Then we considered equation (1.7) modulo p_2 and we excluded from the set $K_0(o_2)$ all those elements k for which $T_k(x) \pmod{p_2}$ is not a perfect power. This way we got the set $K(o_2)$ of all possible values of $k \pmod{o_2}$ for which we might have a solution. Continuing this procedure by taking new primes $p_3, p_4, \dots$ of the form $2in + 1$ with $i \in \mathbb{Z}$, we finished this procedure when the set $K(o_i)$ became empty, proving that equation (1.7) has no solution for the given x and n.

Suppose now that in equation (1.7) we have $x \in \{4, 5, 8, 9, 12, 13\}$. A direct application of Theorem 2.1 to equation (1.7) shows that for each $x \in \{4, 5, 8, 9, 12, 13\}$ we may write $n \le 5$. Finally, for every $x \in \{4, 5, 8, 9, 12, 13\}$ and $n \in \{3, 4, 5\}$ we apply the same procedure as above in the case $y > 10^6$ to conclude that equation (1.7) has no solution for the given x and n. This finishes the proof of our theorem. □

Remark. The algorithms described in the above proof have been implemented in the computer algebra package MAGMA [7]. We mention that the running time of the programme proving that we have no solution for $x = 11$ and $3 \le n \le n_1$ was more than 2 days on an Intel Xeon X5680 (Westmere EP) processor. For $x = 11$ to perform the computation up to the bound $n < n_0$ would have been too long. This is the reason we had to use our bound n_1 proved in Lemma 3.7 under the assumption $y > 10^6$.

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