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On the Constant Term of The Minimal Polynomial of $\cos\left(\frac{2\pi}{n}\right)$ over $\mathbb{Q}$

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Abstract. The algebraic numbers $\cos(2\pi/n)$ and $2\cos(\pi/n)$ play an important role in the theory of discrete groups and has many applications because of their relation with Chebycheff polynomials. There are some partial results in literature for the minimal polynomial of the latter number over rationals until 2012 when a complete solution was given in [5]. In this paper we determine the constant term of the minimal polynomial of $\cos(\frac{2\pi}{n})$ over $\mathbb{Q}$ by a new method.

1. Introduction and Preliminaries

It is a well known result that the n^{th} cyclotomic polynomial $\Phi_n(x)$ is a monic irreducible polynomial of degree $\phi(n)$ with integer coefficients where $\phi(n)$ is the number of integers between 1 and n that are relatively prime to n. Its roots are the primitive n^{th} roots of unity. Thus

$$\Phi_n(x) = \prod_{\substack{k=1 \\ (k,n)=1}}^{n} (x - \zeta_n^k),$$

where $\zeta_n = \exp^{\frac{2\pi i}{n}} = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$. Also

$$x^n - 1 = \prod_{d|n} \Phi_d(x),$$

and this identity can be used to compute $\Phi_n(x)$. For example,

$$\Phi_1(x) = x - 1,\ \Phi_2(x) = x + 1,\ \Phi_3(x) = x^2 + x + 1, \Phi_4(x) = x^2 + 1.$$

In 1933, D. H. Lehmer [3] gave a method for constructing the minimal polynomials of $\cos(\frac{2\pi}{n})$ and $\sin(\frac{2\pi}{n})$ using $\Phi_n(x)$. W. Watkins and J. Zeitlin [4] gave another method for computing the minimal polynomial $\Psi_n(x)$ of $\cos(\frac{2\pi}{n})$ using Chebychev polynomials $T_s(x)$ which are defined by,

$$T_s(\cos\theta) = \cos(s\theta),$$

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for positive integers s and all real θ. We note that

$$deg\ \Psi_n(x) = \begin{cases} 1 & \text{if } n = 1, 2, \\ \phi(n)/2 & \text{if } n \geq 3. \end{cases}$$

In fact they proved the following:

Theorem 1.1. *([4]) Let $\psi_n(x)$ be the minimal polynomial of $\cos(2\pi/n)$ and let $T_s(x)$ denote the sth Chebychev polynomial.*
a) If $n = 2s + 1$ is odd, then

$$T_{s+1}(x) - T_s(x) = 2^s \prod_{d|n} \psi_d(x),$$

and
b) if $n = 2s$ is even, then

$$T_{s+1}(x) - T_{s-1}(x) = 2^s \prod_{d|n} \psi_d(x).$$

Using the above result, I. N. Cangul [1] obtained the formula for $\Psi_n(x)$ in terms of Chebychev polynomials. Recently M. Demirci and I. N. Cangul [2] have determined the constant term of $\Psi_n(x)$ by making use of the behavior of trigonometric functions $\sin nx$ and $\cos nx$. We give here a direct simple proof making use of Theorem 1.1.

2. Constant term of $\Psi_n(x)$

First we find the constant term of the polynomial $T_{s+1}(x) - T_s(x)$.
We have

$$\begin{aligned} T_s(\cos\theta) = \cos(s\theta) &= Re((\cos\theta + i\sin\theta)^s) \\ &= Re\left(\sum_{k=0}^{s} \binom{s}{k} \cos^k\theta\ i^{s-k}\ \sin^{s-k}\theta\right) \end{aligned}$$

$$= \begin{cases} \binom{s}{1}\cos\theta(-1)^{\frac{s-1}{2}}(1-\cos^2\theta)^{\frac{s-1}{2}} + \binom{s}{3}\cos^3\theta(-1)^{\frac{s-3}{2}}(1-cos^2\theta)^{\frac{s-3}{2}} + \\ \cdots + \binom{s}{s}\cos^s\theta, & \text{if } s \text{ is odd}, \\ \binom{s}{0}(-1)^{\frac{s}{2}}(1-\cos^2\theta)^{\frac{s}{2}} + \binom{s}{2}\cos^2\theta(-1)^{\frac{s-2}{2}}(1-\cos^2\theta)^{\frac{s-2}{2}} + \\ \cdots + \binom{s}{s}\cos^s\theta, & \text{if } s \text{ is even}. \end{cases}$$

Hence,

$$T_s(x) = \begin{cases} \sum_{k=1}^{\frac{s+1}{2}} (-1)^{\frac{s-2k+1}{2}} \binom{s}{2k-1} x^{2k-1}(1-x^2)^{\frac{s-2k+1}{2}}, & \text{if } s \text{ is odd}, \\ \sum_{k=0}^{s/2} (-1)^{\frac{s-2k}{2}} \binom{s}{2k} x^{2k}(1-x^2)^{\frac{s-2k}{2}}, & \text{if } s \text{ is even}. \end{cases} \tag{2.1}$$

Thus, we have

$$T_s(0) = \begin{cases} 0, & \text{if } s \text{ is odd}, \\ 1, & \text{if } s \equiv o\ (mod\ 4), \\ -1, & \text{if } s \equiv 2\ (mod\ 4), \end{cases} \tag{2.2}$$

and so

$$T_{s+1}(0) - T_s(0) = \begin{cases} -1, & \text{if } s \equiv 0, 1\ (mod\ 4), \\ 1, & \text{if } s \equiv 2, 3\ (mod\ 4). \end{cases} \tag{2.3}$$

We find the constant term of the minimal polynomial of $\cos(\frac{2\pi}{n})$ by considering the following cases:

1. n odd.
2. $n = 2m, m$ is odd.
3. $n = 2^m$.
4. $n = 2^2m$, where m is odd.
5. $n = 2^\beta m$, where $\beta \geq 3$ and m is odd.

Theorem 2.1. *If n is odd, then the constant term in $2^{\left[\frac{\phi(n)}{2}\right]}\ \psi_n(x)$ is ± 1.*

Proof. For $n = 1$, we have,

$$2^{\left[\frac{\phi(n)}{2}\right]}\ \psi_n(x) = 2^{\left[\frac{1}{2}\right]}\ \psi_1(x) = (x-1).$$

When $n = 3$,

$$2^{\left[\frac{\phi(n)}{2}\right]}\ \psi_n(x) = 2^{\left[\frac{\phi(3)}{2}\right]}\ \psi_3(x) = 2x+1.$$

If n is an odd integer of the form $n = 2s+1$, then the result is true for $s = 0$ and $s = 1$. Suppose the result is true for all odd integers less than $n = 2s+1$. We shall prove the result for $n = 2s+1$. Then $s = \frac{n-1}{2} = \sum_{d|n, d>n} \frac{\phi(d)}{2}$ and so by Theorem 1.1,

$$\begin{aligned} T_{s+1}(x) - T_s(x) &= 2^s \prod_{d|n} \psi_d(x) \\ &= \psi_1(x) \cdot 2^s \prod_{\substack{d|n \\ d>1}} \psi_d(x) \\ &= \psi_1(x) \prod_{\substack{d|n \\ d\ odd \\ d\geq 3}} 2^{\left[\frac{\phi(d)}{2}\right]}\ \psi_d(x). \end{aligned}$$

Hence,

$$T_{s+1}(x) - T_s(x) = g(x)\ 2^{\left[\frac{\phi(n)}{2}\right]}\ \psi_n(x), \tag{2.4}$$

where

$$g(x) = \psi_1(x) \prod_{\substack{d|n \\ d\ odd \\ d\geq 3,\ d\neq n}} 2^{\left[\frac{\phi(d)}{2}\right]}\ \psi_d(x).$$

Now the constant term of $T_{s+1}(x) - T_s(x)$ is ± 1 by (2.3). As n is odd, all its divisors d are also odd and hence the constant term of $g(x)$, by induction hypothesis, is also ± 1. Hence by (2.4), it follows that the constant term of $2^{\left[\frac{\phi(n)}{2}\right]}\psi_n(x)$ is also ± 1. □

Theorem 2.2. *If $n = 2^m$, then the constant term in $2^{\left[\frac{\phi(n)}{2}\right]}\psi_n(x)$ is defined as*

$$\begin{cases} -1, & \text{if } m = 0, \\ +1, & \text{if } m = 1, \\ 0, & \text{if } m = 2, \\ \pm 2, & \text{if } m \geq 3. \end{cases}$$

Proof. It is easy to check the result for $m = 0, 1$ and 2.
Suppose $n = 2^m, m \geq 3$. By induction we show that constant term in $2^{\left[\frac{\phi(n)}{2}\right]}\ \psi_n(x)$ is ± 2. If $n = 2s$, then again by Theorem 1.1,

$$T_{s+1}(x) - T_{s-1}(x) = 2^s \prod_{d|n} \psi_d(x).$$

Hence for $n = 2^m, m \geq 3$,

$$T_{2^{m-1}+1}(x) - T_{2^{m-1}-1}(x) = 2^{2^{m-1}} \prod_{d|2^m} \psi_d(x)$$

$$= 2^2\ \psi_1(x)\ \psi_2(x)\ \psi_4(x) \prod_{\substack{d|2^m \\ d>4}} 2^{[\frac{\phi(d)}{2}]}\ \psi_d(x),$$

since $2^{m-1} - 2 = \frac{1}{2}\ \sum_{d|2^m,\ d>4} \phi(d)$, we have, $2^{2^{m-1}} = 2^2.2^{\sum_{d|2^m, d>4} \frac{\phi(d)}{2}}$.
Hence,

$$T_{2^{m-1}+1}(x) - T_{2^{m-1}-1}(x) = 2^2(x-1)(x+1)x \prod_{\substack{d|2^m \\ d>4}} 2^{[\frac{\phi(d)}{2}]} \psi_d(x).$$

Note that constant term in LHS is 0 by (2.2). Also from (2.1) the coefficient of x in $T_{s+1}(x) - T_{s-1}(x)$ is $2s$ when $s \equiv 0\ (mod\ 4)$. Hence coefficient of x in $T_{2^{m-1}+1}(x) - T_{2^{m-1}-1}(x) = 2^m$
$=$ constant term in $2^2(x-1)(x+1) \left[\prod_{\substack{d|2^{m-1} \\ d>4}} 2^{[\frac{\phi(d)}{2}]} \psi_d(x) \right] 2^{[\frac{\phi(2^m)}{2}]}\ \psi_{2^m}(x)$.
Now by using the induction hypothesis, it follows that, the coefficient of x in $T_{2^{m-1}+1}(x) - T_{2^{m-1}-1}(x) = 2^2(\pm 2)^{m-3} \times$ constant term in $\left(2^{[\frac{\phi(2^m)}{2}]}\ \psi_{2^m}(x) \right)$. Thus constant term in $2^{[\frac{\phi(2^m)}{2}]}\ \psi_{2^m}(x) = \pm 2$. □

Theorem 2.3. *If $n = 2s$ and s is odd, then the constant term in $2^{[\frac{\phi(n)}{2}]}\ \psi_n(x)$ is ± 1.*

Proof. For $s = 1$ and 3 the proof is clear.
Suppose the result is true for all $m < n$ and $m \equiv 2\ (mod\ 4)$. We shall prove the result for n. If $n = 2s$, we have,

$$T_{s+1}(x) - T_{s-1}(x) = 2^s \prod_{d|n} \psi_d(x)$$

$$= 2\ \psi_1(x)\ \psi_2(x)\ 2^{s-1} \prod_{\substack{d|n \\ d>2}} \psi_d(x).$$

Further if $n \equiv 2\ (mod\ 4)$, then, the divisors d of n are either odd or of the form $d \equiv 2\ (mod\ 4)$ and so $\frac{1}{2} \sum_{\substack{d|n \\ d>2}} [\phi(d)] = \frac{1}{2}\ (n - \phi(1) - \phi(2)) = s - 1$. Hence

$$T_{s+1}(x) - T_{s-1}(x) = 2\ \psi_1(x)\ \psi_2(x) \prod_{\substack{d|n \\ d>2}} 2^{\frac{\phi(d)}{2}}\ \psi_d(x)$$

$$= 2\ h(x)\ 2^{[\frac{\phi(n)}{2}]} \psi_n(x), \text{ where}$$

$$h(x) = \psi_1(x)\ \psi_2(x) \prod_{\substack{d|n \\ n>d>2}} 2^{\frac{\phi(d)}{2}}\ \psi_d(x).$$

By induction hypothesis the constant term of $2\ h(x)$ is ± 2. Hence, using (2.1), we get the desired result. □

Theorem 2.4. *Let $n = 2^2.p^\alpha$, where p is an odd prime and $\alpha \geq 1$. Then the constant term in $2^{[\frac{\phi(n)}{2}]} \psi_n(x)$ is $\pm p$.*

Proof. We prove the theorem by induction on α. By Theorem 1.1, we have,

$$T_{2p^\alpha+1}(x) - T_{2p^\alpha-1}(x) = 2^{2p^\alpha} \prod_{d|2^2p^\alpha} \psi_d(x) = 2^2(x-1)(x+1)x \prod_{\substack{d|2^2p^\alpha \\ d\neq 1,2,4}} 2^{[\frac{\phi(d)}{2}]} \psi_d(x).$$

By (2.1), the coefficient of x in LHS is -2^2p^α which is equal to the constant term in

$$2^2(x-1)(x+1)\prod_{\substack{d|2^2p^\alpha\\ d\neq 1,2,4}} 2^{[\frac{\phi(d)}{2}]}\psi_d(x).$$

Now

$$2^2(x-1)(x+1)\prod_{\substack{d|2^2p^\alpha\\ d\neq 1,2,4}} 2^{[\frac{\phi(d)}{2}]}\psi_d(x) = 2^2(x-1)(x+1)\prod_{d=p,p^2,\ldots,p^\alpha} 2^{[\frac{\phi(d)}{2}]}\psi_d(x)\times$$

$$\left\{\prod_{d=2p,2p^2,\ldots,2p^\alpha} 2^{[\frac{\phi(d)}{2}]}\psi_d(x)\right\}\left\{\prod_{d=2^2p,2^2p^2,\ldots,2^2p^{\alpha-1}} 2^{[\frac{\phi(d)}{2}]}\psi_d(x)\right\}\left\{2^{[\frac{\phi(2^2p^\alpha)}{2}]}\psi_{2^2p^\alpha}(x)\right\}.$$

Now by Theorem 2.1, the constant term in

$$\prod_{d=p,p^2,\ldots,p^\alpha} 2^{[\frac{\phi(d)}{2}]}\psi_d(x)$$

is ± 1. From Theorem 2.3, the constant term in

$$\prod_{d=2p,2p^2,\ldots,2p^\alpha} 2^{[\frac{\phi(d)}{2}]}\psi_d(x)$$

is ± 1. By induction hypothesis, the constant term in

$$\prod_{d=2^2p,2^2p^2,\ldots,2^2p^{\alpha-1}} 2^{[\frac{\phi(d)}{2}]}\psi_d(x)$$

is $(\pm p)^{\alpha-1}$. Thus,

$$-2^2p^\alpha = -2^2(\pm 1)\,(\pm 1)\,(\pm p)^{\alpha-1}\times \textit{ constant term in } 2^{[\frac{\phi(2^2p^\alpha)}{2}]}\psi_{2^2p^\alpha}(x)$$

which implies the result. □

Remark 2.5. *Now suppose $n = 2^2m$ where $m = {p_1}^{\alpha_1}\ldots {p_k}^{\alpha_k}$ with $p_1,\ldots,p_k$ distinct odd primes. Then*

$$T_{2p_1^{\alpha_1}\ldots p_k^{\alpha_k}+1}(x) - T_{2p_1^{\alpha_1}\ldots p_k^{\alpha_k}-1}(x) = 2^{2p_1^{\alpha_1}\ldots p_k^{\alpha_k}}\prod_{d|n}\psi_d(x).$$

$$= 2^2(x-1)(x+1)x\prod_{\substack{d|n\\ d\neq 1,2,4}} 2^{[\frac{\phi(d)}{2}]}\psi_d(x).$$

As above, the coefficient of x in LHS is

$$-2^2p_1^{\alpha_1}\ldots p_k^{\alpha_k},$$

which is equal to the constant term in

$$2^2(x-1)(x+1)\prod_{\substack{d|2^2p_1^{\alpha_1}\ldots p_k^{\alpha_k}\\ d\neq 1,2,4}} 2^{[\frac{\phi(d)}{2}]}\psi_d(x).$$

Now the constant term in

$$\prod_{d=2^2p_i,2^2{p_i}^2,\ldots,2^2p_i^{\alpha_i}} 2^{[\frac{\phi(d)}{2}]}\psi_d(x)$$

is equal to $(\pm p_i)^{\alpha_i}$ by Theorem 2.4, for $i = 1,2,\ldots,k$. Hence the constant term in $2^{[\frac{\phi(n)}{2}]}\psi_n(x)$ must be ± 1.

Theorem 2.6. *Suppose $n = 2^\beta p^\alpha$, where p is an odd prime with $\alpha \geq 1$ and $\beta \geq 3$. Then the constant term in $2^{[\frac{\phi(n)}{2}]}\psi_n(d)$ is ± 1.*

Proof. We have, by Theorem 1.1,

$$T_{2^{\beta-1}p^\alpha+1}(x) - T_{2^{\beta-1}p^\alpha-1}(x) = 2^{\frac{n}{2}} \prod_{d|n} \psi_d(x) = 2^2\ (x-1)\ (x+1)\ x\ \times \prod_{\substack{d|2^\beta p^\alpha \\ d\neq 1,2,4}} 2^{[\frac{\phi(d)}{2}]}\psi_d(x)$$

Now by (2.1), the coefficient of x in LHS is $2^\beta p^\alpha$. Now the theorem follows from the fact that the constant terms in the products:

$$\prod_{d=2^3,\ldots,2^\beta} 2^{[\frac{\phi(d)}{2}]}\psi_d(x) \ \text{ and } \prod_{d=2^2p,2^2p^2,\ldots,2^2p^\alpha} 2^{[\frac{\phi(d)}{2}]}\psi_d(x)$$

are respectively $(\pm 2)^{\beta-2}$ and $(\pm p)^\alpha$ by Theorem 2.2 and Theorem 2.4. □

Remark 2.7. *Finally we suppose $n = 2^\beta m$ where $\beta \geq 3$ and m an odd integer. Then the constant term in $2^{[\frac{\phi(n)}{2}]}\psi_n(x)$ is ± 1. We omit the proof as it is similar to the case $n = 2^2 m$, m odd, given in Remark 2.5.*

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