Filomat 30:7 (2016), 1987–1995
DOI 10.2298/FIL1607987K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Finite Derivation Type for Graph Products of Monoids

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Abstract. The aim of this paper is to show that the class of monoids of finite derivation type is closed under graph products.

1. Introduction and Preliminaries

In recent years string-rewriting systems have played a major role in the theoretical computer science and mathematics. If a monoid can be presented by a finite and complete (that is, noetherian and confluent) string-rewriting systems ([2]), then the word problem for this monoid is solvable. The property of having finite and complete string-rewriting system is not invariable under monoid presentations (see [9]). For finitely presented monoids, there exists another finiteness condition, namely *finite derivation type* (FDT) which is actually a combinatorial condition of string-rewriting systems. (In some papers, FDT is also called *finite homotopy type*). This property was introduced by Squier in [16] who worked on some relations, namely *homotopy relations*, between paths in the graph associated with a finite monoid presentation. In the same reference, it has been also proved that if a monoid M is presented by a finite complete system, then it has FDT. Again in [16], the author showed that this finiteness condition is independent on the choice of finite presentations of the given monoid.

At this point we should first mention that the property FDT has a completely same role with Gröbner bases (GB) over special structures. (We may refer [10] for the meaning of GB and its applications). Both FDT and GB mainly characterize the study of algebraic structures in the meaning of ordering the elements or subgroups. In fact, by considering the orders of elements in a group, a different classification other than FDT (or GB) has been recently applied in [1]. We should secondly mention that the terminology graph product in this paper will not be the same meaning as in the product of simple graphs (that we may also refer [20] for an example of products of (simple) graphs).

In the literature there are some important results concerning FDT property of some monoid and semigroup constructions. In a joint paper [15], Pride et al. depicted that a submonoid whose complement is

2010 *Mathematics Subject Classification*. Primary 20M05; Secondary20M30; 20M50; 68Q42.
Keywords. Graph Products, Finite Derivation Type, Monoids, Rewriting systems.
Received: 17 November 2014; Accepted: 20 June 2015
Communicated by Hari M. Srivastava
Presented in the Conference 22nd ICFIDCAA-South Korea
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an ideal of a monoid having FDT also has FDT. Newertheless, for finitely presented (fp) monoids A and B, Otto proved that A and B have FDT if and only if the free product $A * B$ has also FDT ([13]). Again for fp A and B, Wang showed that the semi-direct product $A \rtimes_\theta B$ has FDT if both A and B have FDT (see [17]). Later on, the same author in another paper ([18]) presented that small extensions of monoids having FDT also have FDT. Moreover, it is shown that if a congruence ρ has FDT as a subsemigroup of the direct product $S \times S$, then S has FDT (cf. [19]). In addition to these results, in [11], Malherio stated and proved that if a Rees matris semigroup $M[S; I, J; P]$ has FDT, then the semigroup S has also FDT. It has been recently studied FDT for semilattices of semigroups by the same author in [12]. As a next step of these important results, in this paper, we will consider *graph products of monoids*. We remind that graph products of groups were introduced by E. R. Green in [6] (which was used to solved the word problem).

The following theorem is one of the key point in the approximation of our study.

Theorem 1.1. [7] *The graph product of finitely many groups (or monoids) which admit a complete rewriting system admits a canonical complete rewriting system. If the rewriting systems for the vertex groups (or monoids) are finite or regular, then the system for the graph product is also.*

Although Theorem 1.1 does not imply the FDT property, it suggests that it may be possible to show the FDT property in general without any restrictions over monoids. So, in this paper, our aim is to prove that the graph product of monoids (without any restrictions on them) having FDT has also FDT.

It is well known that a graph $\Gamma' = (V, E)$ is a set V of *vertices* together with an irreflexive, symmetric relation $E \subseteq V \times V$ whose elements are called *edges*. We say that u and v are adjacent in Γ' if $(u, v) \in E$. The *graph product of monoids* (*groups*) is a product mixing direct and free products. Whether the product between two monoids is free or direct can be determined by a simplicial graph, that is, a graph with no loops. Considering a monoid attached to each vertex of the graph, the associated graph product is the monoid generated by each of vertex monoids with the added relations that elements of adjacent vertex monoids commute. Some results relative to the graph product of monoids can be found in [4, 5, 8].

Definition 1.2. *Let M_j $(1 \leq j \leq n)$ be monoids presented by $\mathcal{P}_{M_j} = \left[\mathbf{x}_j; \mathbf{s}_j\right]$ such that the generating sets $\mathbf{x}_j$ are all disjoint. Also let Γ' be a simplicial graph with vertices labeled by M_j. Then the associated graph product of monoids M_j is a monoid M with presentation $\mathcal{P}_M = [\, X\, ;\, R\,]$, where $X = \bigcup_{j=1}^{n} \mathbf{x}_j$ and $R = \bigcup_{j=1}^{n} \mathbf{s}_j \cup S_{\Gamma'}$ such that*

$$S_{\Gamma'} = \{(ab, ba) \mid a \in \mathbf{x}_j, b \in \mathbf{x}_k, j \neq k \text{ and } M_j, M_k \text{ are adjacent vertices of } \Gamma'\}.$$

For a particular case, one can consider free monoids having rank 1. In fact the associated graph product of these monoids is called *trace monoid* or *free partially commutative monoid* and it has solvable word problem ([3]).

2. The Main Theorem and its Proof

By considering the monoids M_j $(1 \leq j \leq n)$ with their presentations as in Definition 1.2, the main result of this paper is the following.

Theorem 2.1. *The graph product of monoids M_j $(1 \leq j \leq n)$ has FDT if each M_j has FDT.*

Let us first give some backround material about monoid presentations, associated graphs and the property of finite derivation type. So suppose that $[\mathbf{x}; \mathbf{s}]$ is a monoid presentation, where $S \in \mathbf{s}$ is the form $S^{+1} = S^{-1}$ and S^{+1}, S^{-1} are words on $\mathbf{x}^*$. The monoid defined by $[\mathbf{x}; \mathbf{s}]$ is the quotient of $\mathbf{x}^*$ by the smallest congruence generated by $\mathbf{s}$. In fact we have a graph $\Gamma = \Gamma(\mathbf{x}; \mathbf{s})$ associated with $[\mathbf{x}; \mathbf{s}]$, where the vertices are the elements of $\mathbf{x}^*$ and the edges are the 4-tuples $e = (U, S, \varepsilon, V)$ with $U, V \in \mathbf{x}^*$, $S \in \mathbf{s}$ and $\varepsilon = \pm 1$. The initial, terminal and the inversion functions for an edge e as above are given by $\iota(e) = US^{\varepsilon}V$, $\tau(e) = US^{-\varepsilon}V$ and $e^{-1} = (U, S, -\varepsilon, V)$, respectively. In fact there is a two-sided action of $\mathbf{x}^*$ on Γ as follows. If

$W, W' \in \mathbf{x}^*$, then for any vertex V of Γ, $W.V.W' = WVW'$ (product in $\mathbf{x}^*$), and for any edge $e = (U, S, \varepsilon, V)$ of Γ, $W.e.W' = (WU, S, \varepsilon, VW')$. This action can be extended to the paths in Γ. Now let $P(\Gamma)$ denote the set of all paths in Γ, and let

$$P^2(\Gamma) := \{(p, q) : p, q \in P(\Gamma), \iota(p) = \iota(q), \tau(p) = \tau(q)\}. \tag{1}$$

Definition 2.2. *An equivalence relation $\simeq \subset P^2(\Gamma)$ is called a homotopy relation if it satisfies the following conditions:*
(a) *If e_1, e_2 are edges of Γ, then $(e_1.\iota(e_2))(\tau(e_1).e_2) \simeq (\iota(e_1).e_2)(e_1.\tau(e_2))$.*
(b) *If $p \simeq q$ ($p, q \in P(\Gamma)$), then $U.p.V \simeq U.q.V$ for all $U, V \in \mathbf{x}^*$.*
(c) *If $p, q_1, q_2, r \in P(\Gamma)$ satisfy $\tau(p) = \iota(q_1) = \iota(q_2)$, $\tau(q_1) = \tau(q_2) = \iota(r)$ and $q_1 \simeq q_2$, then $pq_1r \simeq pq_2r$.*
(d) *If $q \in P(\Gamma)$, then $pp^{-1} \simeq 1_{\iota(p)}$.*

We note that, in [14], Pride introduced a geometric configuration, called *spherical monoid pictures*, to represent paths in a graph Γ. (In Remark 2.16 of this paper, we present an example of using these pictures).

It is seen that the collection of all homotopy relations on $P(\Gamma)$ is closed under arbitrary intersection, and so $P^{(2)}(\Gamma)$ itself is a homotopy relation. Hence, if $C \subset P^{(2)}(\Gamma)$, then there is a unique smallest homotopy relation $\simeq_C$ on $P(\Gamma)$ that contains C.

Definition 2.3. *Let $[\mathbf{x}; \mathbf{s}]$ be a finite monoid presentation and Γ be the associated graph. We say that $[\mathbf{x}; \mathbf{s}]$ has finite derivation type (FDT) if there is a finite subset $C \subset P^{(2)}(\Gamma)$ which generates $P^{(2)}(\Gamma)$ as a homotopy relation, that is $\simeq_C = P^{(2)}(\Gamma)$. A finitely presented monoid S has FDT if some (and hence any [16]) finite presentation of S has FDT.*

2.1. *Proof of Theorem 2.1*

Let us consider the presentations $\mathcal{P}_{M_j}$ and $\mathcal{P}_M$ as in Definition 1.2. Also let Γ_{M_j} and Γ_M be graphs associated with presentations $\mathcal{P}_{M_j}$ and $\mathcal{P}_M$, respectively. In fact each Γ_{M_j} can be considered as a subgraph of Γ_M.

Let M_j, M_k and M_l be monoids presented by $\mathcal{P}_{M_j} = \left[\mathbf{x}_j; \mathbf{s}_j\right]$, $\mathcal{P}_{M_k} = [\mathbf{x}_k; \mathbf{s}_k]$ and $\mathcal{P}_{M_l} = [\mathbf{x}_l; \mathbf{s}_l]$, respectively. Let $\overline{\Gamma}$ denote the subgraph of Γ_M which has the same set of vertices as Γ_M but which contains only those edges (U, T, ϵ, V) of Γ_M with $T \in S_{\Gamma_M}$, $U, V \in (\mathbf{x}_j \cup \mathbf{x}_k \cup \mathbf{x}_l)^*$, $\epsilon = \pm 1$. By $P_+(\overline{\Gamma})$ (respectively, $P_-(\overline{\Gamma})$) we denote the set of paths in $\overline{\Gamma}$ that only contain edges of the form $(U, T, +1, V)$ (respectively, $(U, T, -1, V)$). Then we have the following lemmas for adjacent vertices M_j, M_k and M_l of Γ_M.

Lemma 2.4. *Let $p \in P(\overline{\Gamma})$. Then there exist paths $p_+ \in P_+(\overline{\Gamma})$ and $p_- \in P_-(\overline{\Gamma})$ such that $p \simeq p_+p_-$.*

Proof. Let $p = e_1e_2 \dots e_m$ a path in $\overline{\Gamma}$, where $e_1, e_2, \dots, e_m$ are edges of $\overline{\Gamma}$. Then we have $T : ab = ba$ where $a \in \mathbf{x}_j$, $b \in \mathbf{x}_k$. Suppose there is an index i such that $e_i \in P_-(\overline{\Gamma})$ and $e_{i+1} \in P_+(\overline{\Gamma})$. Then let us choose i is minimal, and for $a_i, a_{i+1} \in \mathbf{x}_j$, $b_i, b_{i+1} \in \mathbf{x}_k$, let

$$\begin{aligned} e_i &= (U_i, T_i, -1, V_i), \quad T_i : a_ib_i = b_ia_i, \\ e_{i+1} &= (U_{i+1}, T_{i+1}, +1, V_{i+1}), \quad T_{i+1} : a_{i+1}b_{i+1} = b_{i+1}a_{i+1}. \end{aligned}$$

If $U_i = U_{i+1}$, then $a_i = a_{i+1}$, $b_i = b_{i+1}$ and $V_i = V_{i+1}$. So $e_{i+1} = e_i^{-1}$, and hence $p \simeq e_1 \dots e_{i-1}e_{i+2} \dots e_m$. But if $U_i \neq U_{i+1}$, then $U_ia_ib_iV_i = U_{i+1}a_{i+1}b_{i+1}V_{i+1}$ which implies that these edges involve disjoint applications of relations. In fact, if $U_i = U_{i+1}a_{i+1}b_{i+1}W_{i+1}$ and $V_{i+1} = W_{i+1}a_ib_iV_i$, then by Definition 2.2-(a), we have

$$\begin{aligned} e_ie_{i+1} &= (U_{i+1}a_{i+1}b_{i+1}W_{i+1}, T_i, -1, V_i)(U_{i+1}, T_{i+1}, +1, W_{i+1}a_ib_iV_i) \\ &\simeq (U_{i+1}, T_{i+1}, +1, W_{i+1}b_ia_iV_i)(U_{i+1}b_{i+1}a_{i+1}W_{i+1}, T_i, -1, V_i) \\ &= e_i'e_{i+1}', \end{aligned}$$

where $e_i' \in P_+(\overline{\Gamma})$ and $e_{i+1}' \in P_-(\overline{\Gamma})$. Hence $p \simeq e_1 \dots e_{i-1}e_i'e_{i+1}'e_{i+2} \dots e_m$ (by Definition 2.2-(c)). By repeated use of this above procedure, we get $p \simeq p_+p_-$. $\square$

Lemma 2.5. *Let $p \in P(\overline{\Gamma})$. If $\iota(p) = UV$, $\tau(p) = U'V'$, where $U, U' \in \mathbf{x}_j^*$, $V, V' \in \mathbf{x}_k^*$, then $U = U'$, $V = V'$ and $p \simeq 1$.*

Proof. By the previous lemma, there exist paths $p_+ \in P_+(\overline{\Gamma})$ and $p_- \in P_-(\overline{\Gamma})$ such that $p \simeq p_+p_-$. Since $\iota(p_+) = \iota(p) = UV$ and $\tau(p_-) = \tau(p) = U'V'$, we have $p_+ = 1$ and $p_- = 1$, respectively. Hence, $p \simeq 1$ and $U = U'$, $V = V'$. $\square$

Now let us define homomorphisms

$$f_j : (\mathbf{x}_j \cup \mathbf{x}_k \cup \mathbf{x}_l)^* \to \mathbf{x}_j^* \quad \text{by} \quad f_j(x_j) = x_j, f_j(x_k) = 1, f_j(x_l) = 1,$$
$$f_k : (\mathbf{x}_j \cup \mathbf{x}_k \cup \mathbf{x}_l)^* \to \mathbf{x}_k^* \quad \text{by} \quad f_k(x_j) = 1, f_k(x_k) = x_k, f_k(x_l) = 1,$$
$$f_l : (\mathbf{x}_j \cup \mathbf{x}_k \cup \mathbf{x}_l)^* \to \mathbf{x}_l^* \quad \text{by} \quad f_l(x_j) = 1, f_l(x_k) = 1, f_l(x_l) = x_l,$$

where $x_j \in \mathbf{x}_j$, $x_k \in \mathbf{x}_k$ and $x_l \in \mathbf{x}_l$.

Lemma 2.6. *Let $W \in (\mathbf{x}_j \cup \mathbf{x}_k)^*$. Then, for some $V \in \mathbf{x}_k^*$, there is a path $p_W \in P_+(\overline{\Gamma})$ from W to $Vf_j(W)$. If $p \in P_+(\overline{\Gamma})$ is a path from W to $V'f_j(W)$ for some $V' \in \mathbf{x}_k^*$, then $V = V'$ and $p_W \simeq p$.*

Proof. Let $W = W_0b_1W_1b_2 \dots b_mW_m$, where $b_t \in \mathbf{x}_k$, $W_s \in \mathbf{x}_j^*$ $(1 \le t \le m, 0 \le s \le m)$. Then $f_j(W) = W_0W_1 \dots W_m$. Let $W_0 = a_1a_2 \dots a_r$ $(a_i \in \mathbf{x}_j,\ 1 \le i \le r)$,

$$T_i : a_ib_1 = b_1a_i \quad (1 \le i \le r).$$

Let $W' = W_1b_2W_2b_3 \dots b_mW_m$. Then

$$(a_1a_2 \dots a_{r-1}, T_r, +1, W')(a_1a_2 \dots a_{r-2}, T_{r-1}, +1, a_rW') \dots (1, T_1, +1, a_2 \dots a_rW')$$

is a path in $P_+(\overline{\Gamma})$ from $W = W_0b_1W'$ to b_1W_0W'. If we continue in this way, we can get a path $p_W \in P_+(\overline{\Gamma})$ from W to $Vf_j(W)$ for some $V \in \mathbf{x}_k^*$. If $p \in P_+(\overline{\Gamma})$ is a path from W to $V'f_j(W)$ for some $V' \in \mathbf{x}_k^*$, then $p^{-1}p_W \in P(\overline{\Gamma})$ is a path from $V'f_j(W)$ to $Vf_j(W)$. By Lemma 2.5, $p^{-1}p_W \simeq 1$, so $p_W \simeq p$ (by Definition 2.2-(c),(d)) and $V = V'$. $\square$

Let us suppose that Γ_{M_j,M_k} and Γ_{M_j,M_k,M_l} are subgraphs of Γ_M such that the edges are the union of the edges of Γ_{M_j}, Γ_{M_k}, $\overline{\Gamma}$ and Γ_{M_j}, Γ_{M_k}, Γ_{M_l}, $\overline{\Gamma}$, respectively. Let $p, q \in P(\Gamma_{M_j,M_k})$ and let $\simeq$ be a homotopy relation on $P(\Gamma_{M_j,M_k})$. For some $p_+ \in P_+(\overline{\Gamma})$ and $p_- \in P_-(\overline{\Gamma})$, if $p \simeq p_+qp_-$, then we write $p \rightsquigarrow q$. Note that $\rightsquigarrow$ is transitive and it is compatible with the two-sided action of $(\mathbf{x}_j \cup \mathbf{x}_k)^*$. After that, for the proof of the main lemma (see Lemma 2.14), we need to define the rules

$$\left.\begin{array}{l} \iota(p).q \rightsquigarrow \tau(p).q \\ q.\iota(p) \rightsquigarrow q.\tau(p) \end{array}\right\}, \tag{2}$$

where $p \in P_+(\overline{\Gamma})$ and $q \in P(\Gamma_{M_j,M_k})$. These rules can be easily seen by Definition 2.2.

For each $S_j : S_j^{+1} = S_j^{-1} \in \mathbf{s}_j$ and each $b \in \mathbf{x}_k$, there is a path $p_+ \in P_+(\overline{\Gamma})$ from $S_j^{+1}b$ to bS_j^{+1} and a path $p_- \in P_-(\overline{\Gamma})$ from bS_j^{-1} to $S_j^{-1}b$ by Lemma 2.6. Since $[S_j^{+1}]_{M_j} = [S_j^{-1}]_{M_j}$, we have a path p_{S_j} from S_j^{+1} to S_j^{-1}. Hence, we have a path

$$q_{S_j,b} = p_+(b, S_j, +1, 1)p_-$$

from $S_j^{+1}b$ to $S_j^{-1}b$ (see Figure 1-(a)). Let

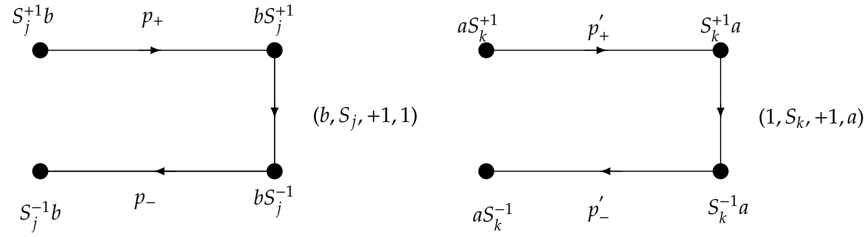
$$C_{j,k} = \left\{\left(\left(1, S_j, +1, b\right), q_{S_j,b}\right) : S_j \in \mathbf{s}_j, b \in \mathbf{x}_k\right\} \subset P^{(2)}(\Gamma_{M_j,M_k}).$$

For each $a \in \mathbf{x}_j$ and each $S_k : S_k^{+1} = S_k^{-1} \in \mathbf{s}_k$, by Lemma 2.6, there are paths $p'_+ \in P_+(\overline{\Gamma})$ and $p'_- \in P_-(\overline{\Gamma})$ from aS_k^{+1} to $S_k^{+1}a$ and from $S_k^{-1}a$ to aS_k^{-1}, respectively. Since $[S_k^{+1}]_{M_k} = [S_k^{-1}]_{M_k}$, we have a path p_{S_k} from S_k^{+1} to S_k^{-1}. Hence there exists a path

$$q'_{a,S_k} = p'_+(1, S_k, +1, a)p'_-$$

from aS_k^{+1} to aS_k^{-1} (see Figure 1-(b)). We then let

$$C'_{j,k} = \left\{((a, S_k, +1, 1), q'_{a,S_k}) : S_k \in \mathbf{s}_k, a \in \mathbf{x}_j\right\} \subset P^{(2)}(\Gamma_{M_j,M_k}).$$



(a) (b)

Figure 1:

For $ab = ba \in S_{\Gamma_M}$ and $c \in \mathbf{x}_l$, where $a \in \mathbf{x}_j$, $b \in \mathbf{x}_k$, there are paths $p''_+ \in P_+(\overline{\Gamma})$ and $p''_- \in P_-(\overline{\Gamma})$ from abc to cab and from cba to bac, respectively. We also have a path from ab to ba. Hence, there exists a path

$$q_{ab,c} = p''_+(c, ab = ba, +1, 1)p''_-$$

from abc to bac (see Figure 2-(a)).

For $bc = cb \in S_{\Gamma_M}$ and $a \in \mathbf{x}_j$, there are paths $p'''_+ \in P_+(\overline{\Gamma})$ and $p'''_- \in P_-(\overline{\Gamma})$ from abc to bca and from cba to acb, respectively. We also have a path from bc to cb. Thus, there exists a path

$$q_{a,bc} = p'''_+(1, bc = cb, +1, a)p'''_-$$

from abc to acb (see Figure 2-(b)). Then, for adjacent vertices M_j, M_k and M_l of Γ_M, let

$$C_{j,k,l} = \left\{((1, ab = ba, +1, c), q_{ab,c}) : ab = ba \in S_{\Gamma_M}, a \in \mathbf{x}_j, b \in \mathbf{x}_k, c \in \mathbf{x}_l\right\}$$
$$\cup \left\{((a, bc = cb, +1, 1), q_{a,bc}) : bc = cb \in S_{\Gamma_M}, a \in \mathbf{x}_j, b \in \mathbf{x}_k, c \in \mathbf{x}_l\right\} \subset P^{(2)}(\Gamma).$$

At the rest of this section we will give more fundamental and important lemmas to state the main lemma (see Lemma 2.14 below).

Lemma 2.7. *Let p, q be paths in Γ_{M_j,M_k} with $\tau(p) = \iota(q)$. If $p \rightsquigarrow p'$, $q \rightsquigarrow q'$ and $\tau(p'), \iota(q') \in (\mathbf{x}_j \cup \mathbf{x}_k)^*$, then $\tau(p') = \iota(q')$ and $pq \rightsquigarrow p'q'$.*

Proof. Since $p \rightsquigarrow p'$ and $q \rightsquigarrow q'$, we have $p \simeq p_+p'p_-$, $q \simeq q_+q'q_-$, where $p_+, q_+ \in P_+(\overline{\Gamma})$ and $p_-, q_- \in P_-(\overline{\Gamma})$. Then

$$pq \simeq p_+p'p_-q_+q'q_-.$$

By Lemma 2.5, we get $p_-q_+ \simeq 1$. Thus, $\tau(p') = \iota(q')$ and $pq \rightsquigarrow p'q'$. □

Lemma 2.8. *Let $e = (U, S_k, \varepsilon, V)$ be an edge of Γ_{M_k}, where $U, V \in \mathbf{x}_k^*$, $S_k \in \mathbf{s}_k$ and $\varepsilon = \pm 1$. Then, for any $a \in \mathbf{x}_j$, there exists a path q in Γ_{M_k} such that*

$$a.e \rightsquigarrow_{C'_{j,k}} q.a.$$

Proof. By Lemma 2.6, there is a path in $P_+(\Gamma_{M_k})$ from aU to $f_k(aU)a$. So by (2), we have

$$a.e \rightsquigarrow_{C'_{j,k}} (h_j(aU)a, S_k, \varepsilon, V) \rightsquigarrow_{C'_{j,k}} q_1.aV,$$

where q_1 is a path in $P(\Gamma_{M_k})$. Now there is also a path in $P_+(\Gamma_{M_k})$ from aV to $f_k(aV)a$. By (2), we have $q_1.aV \rightsquigarrow_{C'_{j,k}} q.a$, where $q = q_1.f_k(aV)$. Hence the result. □

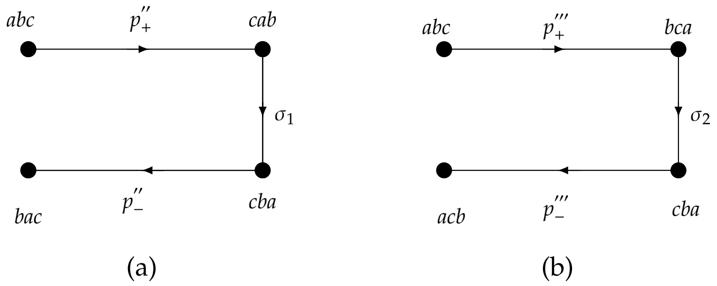


Figure 2: In (a), the edge labelled by σ_1 is actually $(c, ab = ba, +1, 1)$ and, in (b), the edge labelled by σ_2 is $(1, bc = cb, +1, a)$

Lemma 2.9. *Let p be any non-empty path in Γ_{M_k}. Then, for any $W \in \mathbf{x}_j^*$, there exists $q \in P(\Gamma_{M_k})$ such that $W.p \rightsquigarrow_{C'_{j,k}} q.W$.*

Proof. Since the proof can be given easily by applying the induction hypothesis on the length of W, we will just assume that W consist of a single letter $a \in \mathbf{x}_j$. So let $p = e_1e_2\ldots e_m$. Then, by Lemma 2.8, there exists $q_i \in P(\Gamma_{M_k})$ such that $a.e_i \rightsquigarrow_{C'_{j,k}} q_i.a$, where $e_i = (U_i, S_{k_i}, \varepsilon_i, V_i)$ for $1 \leq i \leq m$ and $U_i, V_i \in \mathbf{x}_k^*$. Thus, by Lemma 2.7, we obtain

$$a.p \rightsquigarrow_{C'_{j,k}} (q_1q_2\ldots q_m).a,$$

as required. □

Lemma 2.10. *Let (U, S_k, ε, V) be an edge in Γ_{M_j,M_k}, where $U, V \in (\mathbf{x}_j \cup \mathbf{x}_k)^*, S_k \in \mathbf{s}_k$, $\varepsilon = \pm 1$. Then there exists $q \in P(\Gamma_{M_k})$ such that $(U, S_k, \varepsilon, V) \rightsquigarrow_{C'_{j,k}} q.f_j(UV)$.*

Proof. We have

$$\begin{aligned}
(U, S_k, \varepsilon, V) &\rightsquigarrow_{C'_{j,k}} (f_k(U)f_j(U), S_k, \varepsilon, V), \quad \text{by Lemma 2.6 and (2)},\\
&\rightsquigarrow_{C'_{j,k}} q_1.f_j(U)V, \quad \text{by Lemma 2.9, where } q_1 \in P(\Gamma_{M_k}),\\
&\rightsquigarrow_{C'_{j,k}} q_1f_k(Uf_j(V))f_j(UV), \quad \text{by Lemma 2.6 and (2)},\\
&\rightsquigarrow_{C'_{j,k}} q.f_j(UV), \quad \text{where } q = q_1f_k(Uf_j(V)).
\end{aligned}$$

Hence the result. □

Lemma 2.11. *Let $S_j \in \mathbf{s}_j, W \in \mathbf{x}_k^*$. Then there exists $q \in P(\Gamma_{M_k})$ such that*

$$(1, S_j, \varepsilon, 1).W \rightsquigarrow_{C_{j,k}} (q.S_j^{\varepsilon})(W'.(1, S_j, \varepsilon, 1)),$$

for some $W' \in \mathbf{x}_k^*$.

Proof. For any $U \in (\mathbf{x}_j \cup \mathbf{x}_k)^*$ and $p \in P(\Gamma_{M_k})$, by Lemma 2.6 and (2), we get

$$p.U \rightsquigarrow p'.f_j(U), \tag{3}$$

where $p' \in P(\Gamma_{M_k})$. Additionally, for each $b \in \mathbf{x}_k$, we have

$$(1, S_j, \varepsilon, 1).b \rightsquigarrow_{C_{j,k}} (b.S_j^{\varepsilon})(b.(1, S_j, \varepsilon, 1)), \tag{4}$$

by the definition of $C_{j,k}$. By repeated use of (3), (4) and Lemma 2.7, we get the result, as required. □

The following lemma can be proved similarly by considering the previous lemma.

Lemma 2.12. *Let $T \in S_{\Gamma_M}$, $W \in \mathbf{x}_l^*$. Then there exists $q \in P(\Gamma_{M_j,M_k})$ such that*

$$(1, T, \varepsilon, 1).W \rightsquigarrow_{C_{j,k,l}} (q.T^{\varepsilon})(W'.(1, T, \varepsilon, 1)),$$

for some $W' \in \mathbf{x}_l^*$.

Lemma 2.13. *Let (U, S_j, ε, V) be an edge in Γ_{M_j,M_k}, where $U, V \in (\mathbf{x}_j \cup \mathbf{x}_k)^*$, $S_j \in \mathbf{s}_j$ and $\varepsilon = \pm 1$. Then there is a path $q \in P(\Gamma_{M_k})$ such that*

$$(U, S_j, \varepsilon, V) \rightsquigarrow_{C_{j,k} \cup C'_{j,k}} (q.f_j(US_j^{\varepsilon}V))W(f_j(U), S_j, \varepsilon, f_j(V)).$$

Proof. We have

$$\begin{aligned} (U, S_j, \varepsilon, V) &\rightsquigarrow_{C_{j,k}} (f_k(U)f_j(U), S_j, \varepsilon, f_k(V)f_j(V)), \quad \text{by Lemma 2.6 and (2)}, \\ &\rightsquigarrow_{C_{j,k}} (f_k(U)f_j(U).q_1.S_j^{\varepsilon}f_j(V))(f_k(U)f_j(U)W_1.(1, S_j, \varepsilon, f_j(V))), \\ &\qquad \text{by Lemma 2.11}, \end{aligned}$$

for some $q_1 \in P(\Gamma_{M_k})$ and $W_1 \in \mathbf{x}_k^*$. Also, by Lemma 2.9,

$$f_k(U)f_j(U).q_1.S_j^{\varepsilon}f_j(V) \rightsquigarrow_{C'_{j,k}} q.f_j(US_j^{\varepsilon}V),$$

for some $q \in P(\Gamma_{M_k})$ and, by Lemma 2.6 and (2),

$$h_j(U)f_j(U)W_1.(1, S_j, \varepsilon, f_j(V)) \rightsquigarrow W.(f_j(U), S_j, \varepsilon, f_j(V)),$$

for some $W \in \mathbf{x}_k^*$. Using Lemma 2.7 and the above equivalences, we then have

$$(U, S_j, \varepsilon, V) \rightsquigarrow_{C_{j,k} \cup C'_{j,k}} (q.f_j(US_j^{\varepsilon}V))(W.(f_j(U), S_j, \varepsilon, f_j(V))).$$

In fact, by Lemma 2.6 and the definition of h_j, we have $W = f_k(US_j^{-\varepsilon}V)$. □

Now we present our main lemma.

Lemma 2.14. **(*Principal Lemma*)** *Let $p \in P(\Gamma_{M_j,M_k,M_l})$. Then there exist paths $p_+ \in P_+(\overline{\Gamma})$, $p_- \in P_-(\overline{\Gamma})$, $q = q'.f_l(\iota(p))f_j(\iota(p))$ and $r = f_k(\tau(p))f_l(\tau(p)).r'$, where $q' \in P(\Gamma_{M_k})$ and $r' \in P(\Gamma_{M_j})$ such that*

$$p \simeq_{C_{j,k} \cup C'_{j,k} \cup C_{j,k,l}} p_+qrp_-$$

with $\tau(p_+) = f_k(\iota(p))f_l(\iota(p))f_j(\iota(p))$ and $\iota(p_-) = f_k(\tau(p))f_l(\tau(p))f_j(\tau(p))$.

Proof. For $U, V \in (\mathbf{x}_j \cup \mathbf{x}_k \cup \mathbf{x}_l)^*$, let us suppose that p contains a single edge (U, Q, ε, V). Then the result comes out by

$$\begin{cases} \text{Lemma 2.10;} & \text{if } Q \in \mathbf{s}_k, \\ \text{Lemma 2.13;} & \text{if } Q \in \mathbf{s}_j, \\ \text{Lemma 2.6;} & \text{if } Q \in T. \end{cases}$$

Now suppose $p = p_1e$, where e is an edge and $p_1 \in P(\Gamma_{M_j,M_k,M_l})$. Inductively, we have

$$\begin{aligned} p_1 &\rightsquigarrow_{C_{j,k} \cup C'_{j,k} \cup C_{j,k,l}} (q_1'.f_l(\iota(p))f_j(\iota(p)))(f_k(\tau(p_1))f_l(\tau(p_1).r_1'), \\ e &\rightsquigarrow_{C_{j,k} \cup C'_{j,k} \cup C_{j,k,l}} (q_2'.f_l(\iota(e))f_j(\iota(e)))(f_k(\tau(p))f_l(\tau(p)).r_2'), \end{aligned}$$

where $q_1', q_2' \in P(\Gamma_{M_k})$, $r_1', r_2' \in P(\Gamma_{M_j})$ and

$$\iota(q_1') = f_k(\iota(p)), \ \tau(r_1') = f_j(\tau(p_1)), \ \iota(q_2') = f_k(\iota(e)), \ \tau(r_2') = f_j(\tau(p)).$$

By Lemma 2.7, we have

$$p \rightsquigarrow_{C_{j,k}\cup C'_{j,k}\cup C_{j,k,l}} (q'_1.f_l(\iota(p))f_j(\iota(p)))(f_k(\tau(p_1))f_l(\tau(p_1).r'_1)(q'_2.f_l(\iota(e))f_j(\iota(e)))(f_k(\tau(p))f_l(\tau(p)).r'_2).$$

Since the relations used in the path $f_k(\tau(p_1))f_l(\tau(p_1).r'_1$ and in the path $q'_2.f_l(\iota(e))f_j(\iota(e))$ are disjoint, Definition 2.2-(a) can be applied repeatedly, and so we can get

$$(f_k(\tau(p_1))f_l(\tau(p_1).r'_1)(q'_2.f_l(\iota(e))f_j(\iota(e))) \simeq (q'_2.f_l(\iota(p))f_j(\iota(p)))(f_k(\tau(p))f_l(\tau(p)).r'_1).$$

Assume $q' = q'_1q'_2$ and $r' = r'_1r'_2$. Therefore, for $\iota(q') = f_k(\iota(p))$ and $\tau(r') = f_j(\tau(p))$, we obtain

$$p \rightsquigarrow_{C_{j,k}\cup C'_{j,k}\cup C_{j,k,l}} ((q'.f_l(\iota(p))f_j(\iota(p)))(f_k(\tau(p))f_l(\tau(p)).r')),$$

as required. □

We recall that since each monoid M_j $(1 \le i \le n)$ has *FDT*, there is finite subset $C_{M_j} \subset P^{(2)}(\Gamma_{M_j})$ such that $\simeq_{C_{M_j}} = P^{(2)}(\Gamma_{M_j})$. Now let

$$C = C_{M_j} \cup C_{j,k} \cup C'_{j,k} \cup C_{j,k,l}. \tag{5}$$

Then we have

Corollary 2.15. $\simeq_C = P^{(2)}(\Gamma_M)$.

Proof. Let $(p_1, p_2) \in P^{(2)}(\Gamma_M)$. By the Principal Lemma, we take

$$p \simeq_C p_+q_1r_1p_- \quad \text{and} \quad p_2 \simeq_C p'_+q_2r_2p'_-,$$

where $p_+, p'_+ \in P_+(\overline{\Gamma})$, $p_-, p'_- \in P_-(\overline{\Gamma})$, $q_i = q'_i.f_l(\iota(p_i))f_j(\iota(p_i))$ with $q'_i \in P(\Gamma_{M_k})$ and $r_i = f_k(\tau(p_i))f_l(\tau(p_i)).r'_i$ with $r'_i \in P(\Gamma_{M_j})$ $(i = 1, 2)$. Since $\iota(p_1) = \iota(p_2)$ and $\tau(p_1) = \tau(p_2)$, we have

$$\tau(p_+) = f_k(\iota(p_1))f_l(\iota(p_1))f_j(\iota(p_1)) = f_k(\iota(p_2))f_l(\iota(p_2))f_j(\iota(p_2)) = \tau(p'_+),$$
$$\iota(p_-) = f_k(\tau(p_1))f_l(\tau(p_1))f_j(\tau(p_1)) = f_k(\tau(p_2))f_l(\tau(p_2))f_j(\tau(p_2)) = \iota(p'_-).$$

Therefore, $p_+ \simeq_C p'_+$ and $p_- \simeq_C p'_-$. It is seen that $\iota(q'_i) = f_k(\iota(p_i))$ and $\tau(q'_i) = f_k(\tau(p_i))$ $(i = 1, 2)$. So $\iota(q'_1) = \iota(q'_2)$ and $\tau(q'_1) = \tau(q'_2)$. Thus, $(q'_1, q'_2) \in P^{(2)}(\Gamma_{M_k})$. Since $\simeq_{C_{M_k}} = P^{(2)}(\Gamma_{M_k})$, and $C_{M_k} \subset C$, we have $q'_1 \simeq_C q'_2$ and hence, $q_1 \simeq_C q_2$. Similarly,

$$\iota(r'_1) = f_j(\iota(p_1)) = f_j(\iota(p_2)) = \iota(r'_2)$$

and

$$\tau(r'_1) = f_j(\tau(p_1)) = f_j(\tau(p_2)) = \tau(r'_2),$$

so $(r'_1, r'_2) \in P^{(2)}(\Gamma_{M_j})$. Since $\simeq_{C_{M_j}} = P^{(2)}(\Gamma_{M_j})$, and $C_{M_j} \subset C$, we have $r'_1 \simeq_C r'_2$ and hence, $r_1 \simeq_C r_2$. Thus, $p_1 \simeq_C p_+q_1r_1p_- \simeq_C p'_+q_2r_2p'_- \simeq_C p_2$. Therefore, $\simeq_C = P^{(2)}(\Gamma_M)$. □

Now we can prove the main result (Theorem 2.1) as follows.

Proof of Theorem 2.1. If each monoid M_j $(1 \le j \le n)$ has FDT, then we can assume that all $\mathcal{P}_{M_j}$ are finite presentations and all C_{M_j} are finite sets. So $\mathcal{P}_M$ is a finite presentation and the set C defined in (5) is finite. By Corollary 2.15, we have $\simeq_C = P^{(2)}(\Gamma_M)$. Thus the graph product of monoids M_j has FDT.

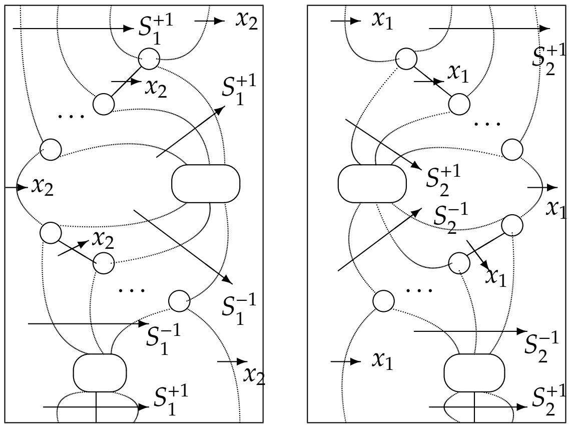


Figure 3: The generating sets $C_{1,2}$ and $C'_{1,2}$

Remark 2.16. *To be an example of spherical monoid pictures, we can draw pictures of the generating sets* $C_{1,2}$ *and* $C'_{1,2}$ *as in Figure* 3.

Acknowledgement 2.17. *The first author is partially supported by TUBITAK with number 113F294, the second author is partially supported by the project office of Balikesir University with numbers 2014/95, 2015/47, and finally the third author is partially supported by the project office of Uludag University with numbers 2013/23, 2014/24, 2015/17.*

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